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The mathematical model for lateral stiffness of variable length conical spring

Muhammad Safa Al-Din Tahir[1*], Shakir Sakran Hassan[2] and Jumaa Salman Chiad[3]

[1] Department of Computer Engineering Technology, Faculty of Information Technology, Imam Ja'afar Al-sadiq University, Iraq

[2] Department of Mechanical Engineering, University of Technology, Baghdad, Iraq

[3] Department of Mechanical Engineering, University of Al-Nahrain, Iraq

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ABSTRACT

One form of energy storage in spring is applying a bending moment and converting it into tilt at the head of the spring as strain energy. The relationship between them is the lateral stiffness of the spring. The aim is to find a mathematical equation for the lateral stiffness of the spring and the effect of the length of the spring on the behavior of stiffness.

The mathematical model is created according to Castigliano's second theorem. A simulated model of a conical spring is built using a Solid Work program. The theoretical results are compared with the mathematical model for the same conical spring.

Results of both theoretical and simulated models evinced a linear behavior of lateral, while an exponential relationship between the length of the spring and the lateral stiffness is indicated. The difference between theoretical and simulated models is not exceeded 3.2%, which indicates the acceptability of results.

KEYWORDS

conical springs, mathematical model, lateral stiffness

1. INTRODUCTION

The spring is a mechanical part commonly used in mechanical devices and systems. An apparatus is almost not devoid of quality, whose main function is energy charging and discharging and in many forms depending on the type of strain: compression, tension, bending, toasting, etc. The relationship of stored energy to strain is very important for any design, in this the study will develop an equation describing the stored energy as a result of the lateral bending of a conical spring.

The behavior of conical springs, when exposed to axial load, was studied by creating an analytical model for this case, where the researchers assumed the existence of two behaviors when exposed to this type of load, which are linear or non-linear. Emphasis was placed on creating a mathematical model describing the two cases using an estimated algorithm to give a mathematical relationship between the load in terms of the length of the spring and the opposite, the method of integrating the initial deviations along the entire length of the spring was relied upon. The suitability of using this equation was confirmed by resorting to practical tests on these springs, as said in this study [1]. The study research develops a computer program that deals with conical springs for analysis its properties and design, and to use as a tool for arriving at the best design for conical springs, is designed to compare this tool with mathematical equations and a practical test by taking a conical spring and performing a pressure test on it to obtain a relationship that relates the length to the applied load and comparing the results obtained from this test with mathematical equations, as well as the tool

*Corresponding author.
E-mail: mohammed.safaaldin@sadiq.edu.iq

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used [2]. There is study on helical and conical springs, where the researcher assumes that they behave like beams, and they used two methods analyzed for measuring mechanical properties of the springs, by a two-dimensional model and finite element [3]. In this research, a mathematical model was evaluated that describes the effect of the vertical loads of springs of rectangular, triangular, and circular shapes and conical-shaped springs and compared them with simulations performed on them with the ANSIS program, as well as the effect of buckling on those springs and the shape of the changes occurring on the mathematical models assigned to them [4]. Here, a detailed study of the helical spring was conducted, the mathematical relationships that describe the behavior of the spring were found when bending moment, torsion, and lateral and vertical forces were applied to it; the accuracy of these relationships was proven by practical tests as well as simulation programs, and the results were characterized by a high degree of accuracy [5]. A simple and uncomplicated mechanical relationship was used to describe the lateral stiffness resulting from lateral forces applied to one end of a spiral spring that is deposited in the tip of a rubber layer, depending on the method of superposition. The validity and effectiveness of this relationship have been confirmed by comparing the results obtained with the practical results, where this relationship can be used in the design of the springs in the suspension system of trains [6].

2. THE THEORETICAL PART

To find a mathematical equation describing the relationship between the moment applied to a conical spring and the angle resulting from this moment, using this equation, the lateral stiffness of the conical spring can be calculated. Therefore, the energy technique (strain energy theory) is used, specifically Castiglione's second theorem [7–9], which depends on describing the strain energy in terms of forces and moment acting to reach the desired equation. The theory is based on the derivation of the strain energy about the force or moment, that is, if to extract the amount of deformation in the place where that force is applied, the partially derive on strain energy equation for that force is done, or partially derive the energy equation for the moment to extract the angle at the location at which that moment was applied according to Eqs (1) and (2),

$$\delta i = \frac{\partial U}{\partial Pi}, \tag{1}$$

$$\phi i = \frac{\partial U}{\partial Mi}, \tag{2}$$

where U is the strain energy; Pi is the force applied somewhere; δi is the distortion occurring in the location where that force is applied; Mi is the moment applied somewhere; ϕi is the angle of tendency at where that moment was applied.

To describe the spring mathematically, there is needed an equation describing the shape of spring, for that go to the description of the wire path line mathematically to be able to include it in the equation to describe the strain energy. Since the spring consists of a wire traveling in a circular path, the radius of this circuit has been decreased as the number of coils is increased, as it can be seen in Fig. 1, Eq. (3) [10] will describe the line of the spring wire, and Eq. (4) describes the length of the spring wire.

$$R = R_2 - \frac{R_2 - R_1}{2\pi N}\,\theta, \tag{3}$$

$$L = R\,\theta, \tag{4}$$

where R is the radius of the circular path; R_1 is the small radius; R_2 is the large radius; N is the number of spring coils; L is the length of wire; θ is the angle of tendency at where that moment was applied.

In Fig. 2, the mechanical effect on the cross-section of the spring wire is explained when applying a moment on the head of spring, where the equation of the strain energy will be, as it is given in Eq. (5), which consisted of the energy resulting from the effect of torque and energy resulting from the moment effect shown in Eqs (6) and (7), respectively.

$$U = U_T + U_M, \tag{5}$$

$$U_T = \frac{2}{GJ}\int_0^L T^2 dL, \tag{6}$$

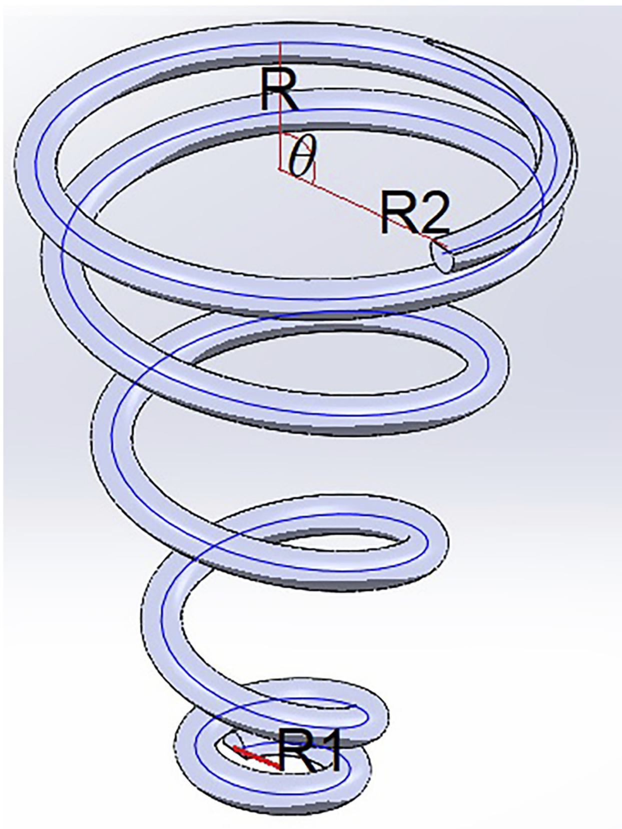

Fig. 1. The effect of the shape of the conical spring on the spring radius

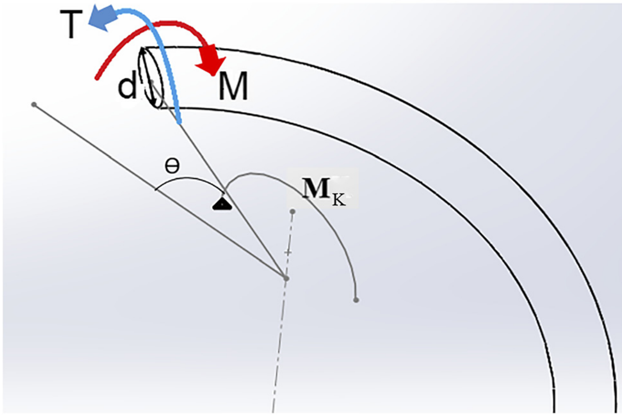

Fig. 2. The mechanical effect on the cross-section of the spring wire

$$U_T = \frac{2}{EI}\int_0^L M^2 dL, \tag{7}$$

where U_T is the torsion strain energy; U_M is the moment strain energy; E is Young's modulus of elasticity; G is the modulus of rigidity; I is the moment of inertia of wire cross-sectional area; J is the polar moment of inertia of wire cross-sectional area; d is the wire diameter.

According to Fig. 2, it has been concluded that the torque and moment value is expressed in Eqs (8) and (9),

$$T = M_K \cos\theta, \tag{8}$$

$$M = M_K \sin\theta. \tag{9}$$

For extraction of the magnitude of the angle of inclination at the location of the moment applied on the head spring, Castigliano's second theorem argument is used in Eqs (6) and (7) to obtain Eq. (10):

$$\varnothing = \frac{\partial U}{\partial M_k} = \frac{1}{GJ}\int_0^L T\frac{\partial T}{\partial M_K}dl + \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial M_K}dl. \tag{10}$$

The values of dl, $\frac{\partial T}{\partial M_K}$, and $\frac{\partial M}{\partial M_K}$ are obtained from the partial derivation of Eqs (4), (8), and (6), respectively as it is shown in Eqs (11), (12), and (13). Equation (10) can be written after applying the above Eqs (8) and (9), as it is shown in Eq. (14),

$$dl = R\,d\theta, \tag{11}$$

$$\frac{\partial T}{\partial M_K} = \cos\theta, \tag{12}$$

$$\frac{\partial M}{\partial M_K} = \sin\theta, \tag{13}$$

$$\varnothing = \frac{\partial U}{\partial M_k} = \frac{1}{GJ}\int_0^L M_K \cos\theta\cos\theta R\,d\theta + \frac{1}{EI}\int_0^L M_K \sin\theta\sin\theta R\,d\theta. \tag{14}$$

Using Eq. (3) in Eq. (15) gives

$$\varnothing = = \frac{1}{GJ}\int_0^{2\pi N} M_K\cos^2\theta\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)d\theta + +\frac{1}{EI}\int_0^{2\pi N} M_K\sin^2\theta\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)d\theta, \tag{15}$$

$$\varnothing = M_K\int_0^{2\pi N}\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)\left(\frac{\cos^2\theta}{GJ} + \frac{\sin^2\theta}{EI}\right)d\theta, \tag{16}$$

$$\varnothing = M_K\int_0^{2\pi N}\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)\left(\frac{1-\sin^2\theta}{GJ} + \frac{\sin^2\theta}{EI}\right)d\theta, \tag{17}$$

$$\varnothing = M_K\int_0^{2\pi N}\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)\left(\frac{1}{GJ} + \sin^2\theta\left(\frac{1}{GJ} + \frac{1}{EI}\right)\right)d\theta, \tag{18}$$

Let $\frac{1}{GJ} = A$ and $\frac{1}{GJ} + \frac{1}{EI} = B$,

$$\varnothing = M_K\int_0^{2\pi N}\left(R_2 - \frac{R_2-R_1}{2\pi N}\theta\right)\left(A + B\sin^2\theta\right)d\theta, \tag{19}$$

$$\frac{1}{K_L} = \frac{\partial\varnothing}{\partial M_K} = \frac{1}{16\pi N}(8\pi^2N^2(2A+B)(R_2+R_1) + +B(\cos(4\pi N) - 1)(R_2 - R_1) - B\sin(4\pi N)4\pi N R_1), \tag{20}$$

where K_L is the lateral stiffness of conical spring.

Some mechanical applications require variable lateral stiffness values for conical springs. The variable lateral stiffness property is difficult to apply when changing the type of spring material or the diameter of the wire made from it. Instead of that the resorting to reducing the length of the spring (shortening its length by pressure), leads to reducing the number of active coils. To explain this effect theoretically, an equation was used to describe the relationship between the lengths with the lateral stiffness, as shown in Eq. (21). This equation was obtained by converting the signification of the lateral stiffness equation from length (Ls) to the number of active coils [10, 11]. Since the dimension (diameter of the wire, pitch of coil, and coils radius) and properties of the material (Young's modulus and modulus of rigidity) of conical spring do not affect the general behavior of the relationship, based on that the chosen the ground end type conical spring has a constant diameter of the wire is 4 mm also the pitch amount (P) of the spring is 29.48 mm, and variable in coils radius from $R_2 = 25.5$ mm to $R_1 = 7.06$ mm. Eqs (20) and (21) were applied on the springs made of high carbon steel wire [12]. High carbon steel mechanical properties include yield stress of 535 MPa, ultimate stress of 810 MPa, Young's modulus of 200 GPa, and modulus of rigidity 78.7 GPa [13, 14],

$$N = \frac{L_s - 2d}{P}. \quad (21)$$

3. SIMULATION PART

Finding out the accuracy of the previous theoretical equation using analytic simulation, the spring was drawn by SolidWorks program after that a simulation of a conical spring by using the same program to avoid error is performed applying. The information used on simulation requirements was similar to that used in the theoretical part (spring design characteristics, wire material, and boundary conditions). The spring design characteristics are coil diameter, step amount, number of active coils, and free length of spring. The boundary conditions applied include: The spring is fixed in the small end, the centerline of the spring's large end diameter is fixed with a pin method that allows rotation only around this line, and The moment is applied on the head of a large end as shown in Fig. 3 where the green arrows represent the consolidation areas, and the purple arrows represent the applied moment. The material mechanical properties used in SolidWorks were the same properties adopted in the mathematical equations, as mentioned previously. The created mesh is the next step and applied on spring by curvature-based mesh method with dimensions (4.71–0.94 mm) consisting of 68,650 elements and 126,403 nodes, which gives appropriate results due to the high degrees of freedom it provides. To apply the case of reducing length spring and that affects lateral stiffness use reduction technique in the amount of the pitch for the first and then the second as consecutive until the appropriate length is reached as shown in Fig. 4.

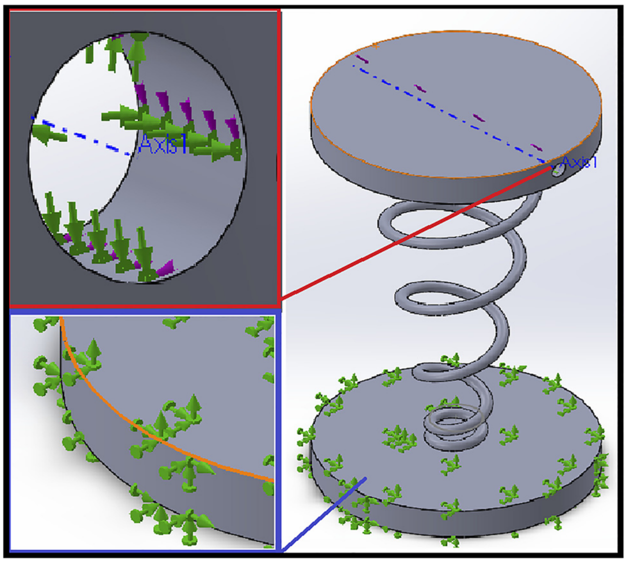

Fig. 3. The boundary condition of simulation conical spring

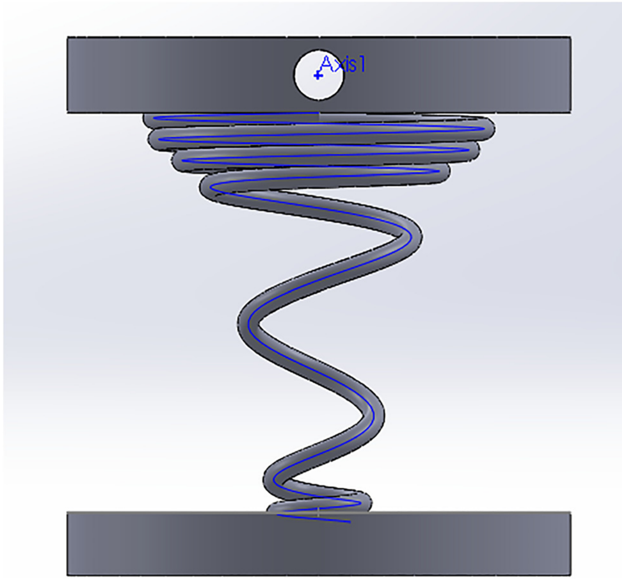

Fig. 4. The length reduction of spring

4. THE RESULTS

After applying the proposed theoretical equations to describe the lateral stiffness of the conical spring on the metal and the aforementioned dimensions, the results elucidated the relationship between the amount of applied moment measured by Nm and the amount of inclination measured by radian, as shown in Fig. 5. As for the results obtained from the simulation process, they were the amount of deformation at the tip of the disk fixed on the head of the large diameter of the conical spring, so the deformation must convert to an angle value in units of radian depending on the amount of distance of that point from the center of rotation, as shown in Fig. 6 to obtain the relationship between moment and angle as in Fig. 4. The results depicted that the theoretical lateral stiffness was 3.26 Nm/rad, but by simulation, it was 3.365 Nm/rad. The effect of changing the length in both the theoretical and simulated cases was an

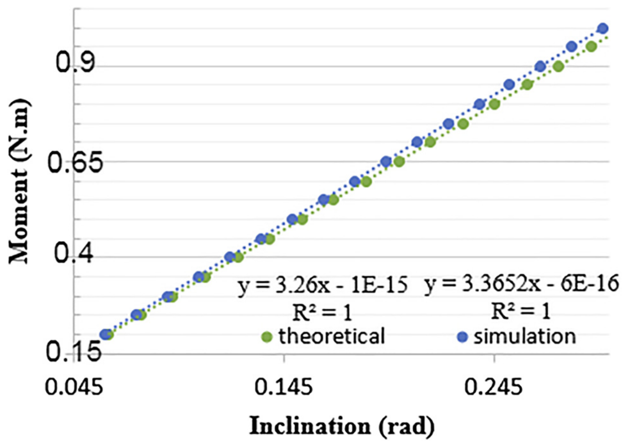

Fig. 5. The relation of moment applied with an inclination

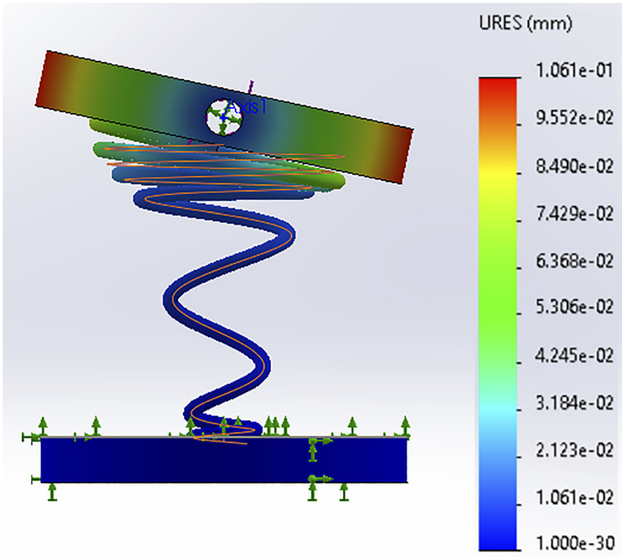

Fig. 6. The deformation at the tip of the disk of the conical spring

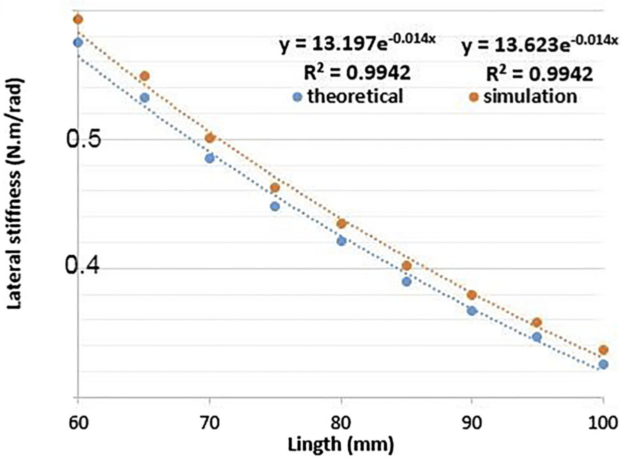

Fig. 7. The relationship between spring length and lateral stiffness

exponential relationship, as shown in Fig. 7, which describes the relationship between the length and the amount of lateral stiffness.

5. CONCLUSION

1. The mathematical model gives an acceptable result in compression with the numerical model during which the difference between this model is not exceeding 3.3%;
2. The relationship obtained between the length of the spring and the lateral stiffness is indicated an exponential behavior;
3. Results indicated that the increase in the lateral stiffness is less than the length of the conical spring.

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