GLOBALLY RIGID AUGMENTATION OF RIGID GRAPHS*

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Abstract. We consider the following augmentation problem: Given a rigid graph $G = (V, E)$, find a minimum cardinality edge set F such that the graph $G' = (V, E \cup F)$ is globally rigid. We provide a min-max theorem and a polynomial-time algorithm for this problem for several types of rigidity, such as rigidity in the plane or on the cylinder. Rigidity is often characterized by some sparsity properties of the underlying graph and global rigidity is characterized by redundant rigidity (where the graph remains rigid after deleting an arbitrary edge) and 2- or 3-vertex-connectivity. Hence, to solve the above-mentioned problem, we define and solve polynomially a combinatorial optimization problem family based on these sparsity and connectivity properties. This family also includes the problem of augmenting a k-tree-connected graph to a highly k-tree-connected and 2-connected graph. Moreover, as an interesting consequence, we give an optimal solution to the so-called global rigidity pinning problem, where we aim to find a minimum cardinality vertex set X for a rigid graph $G = (V, E)$, such that the graph $G + K_X$ is globally rigid in $\mathbb{R}^2$ where K_X denotes the complete graph on the vertex set X.

Key words. Graph rigidity, global rigidity, augmentation, connectivity

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1. Introduction.

In this paper we consider a graph augmentation problem that fits to a branch of connectivity augmentations where edge-connectivity and vertex-connectivity should be augmented simultaneously [8, 17]. For example, our result provides a polynomial algorithm for the following problem: Given a k**-tree connected** graph $G = (V, E)$ (that is, G contains k edge disjoint spanning trees), find a minimum set of edges F such that the graph $G' = (V, E \cup F)$ is **highly** k**-tree-connected** (that is, $G' - e$ still contains k edge disjoint spanning trees for each $e \in E \cup F$) and 2-connected. Nonetheless, the problem gains much of its importance due to its connection to Rigidity Theory, that we introduce now.

A d-dimensional (bar-joint) **framework** is a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map of the vertices to some given subset of the d-dimensional Euclidean space. We call (G, p) a *realization* of G. Two realizations of G, say (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ for every $uv \in E$. Two realizations are *congruent*, if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for every vertex pair $u, v \in V$, or in other words, when (G, p) is isometric to (G, q). We say that the framework (G, p) is **globally rigid** in $\mathbb{R}^d$, if each of its equivalent realizations is also congruent, that is, the edge lengths of the framework uniquely determine its realization up to the isometries of $\mathbb{R}^d$. The framework (G, p) is **rigid** when the above condition only holds for realizations $q : V \to \mathbb{R}^d$ for which $||p(v) - q(v)|| < \varepsilon$ for some $\varepsilon > 0$. This concept of global rigidity plays an important role in rigidity theory and network localization problems [4, 5, 20].

For example, given some sensors in the plane with known distances between some

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of them, one may consider the following question. At least how many sensor-locations do we need to measure exactly to be able to reconstruct the exact location of each sensor? This is the so-called *global rigidity pinning* (or anchoring) problem. Sometimes measuring the exact sensor-locations is too expensive or even impossible. Instead, one may ask at least how many new distances need to be measured so that the distances uniquely determine the positions of the sensors (up to isometry). This problem is called the *global rigidity augmentation* problem. (We note that reconstructing the position of the sensors is a challenging task, even if they are uniquely determined by the framework, see [2, 25, 34]. In this paper we do not address this problem.)

Determining whether a given bar-joint framework is rigid (or globally rigid, respectively) is NP-hard even in the plane (or on the line, respectively) [1, 33]. The analysis gets more tractable, if we consider **generic frameworks** where the set of coordinates of the points is algebraically independent over the rationals [3, 15]. We call a graph G **rigid** (or **globally rigid**, respectively) in $\mathbb{R}^d$ if each (or equivalently, some) of its generic realizations in $\mathbb{R}^d$ is rigid (or globally rigid, respectively). The characterization of rigid and globally rigid graphs is known for $d = 1, 2$ [19, 28, 32] and is a major open problem of rigidity theory for $d \geq 3$.

There are some other types of frameworks for which both rigidity and global rigidity are characterized as a property of their underlying graphs (with some genericity assumptions), for example for body-bar frameworks [6, 36, 38], for body-hinge and body-bar-hinge frameworks [18, 24, 37, 39, 41], and for bar-joint frameworks which are restricted to lie (and move) on some given surface in $\mathbb{R}^3$ such as a sphere [7, 40] or a cylinder [21, 31].

In this paper, we consider the following meta-problem related to the above-mentioned versions of rigidity and global rigidity.

PROBLEM 1. *Given a graph $G = (V, E)$, find an edge set F of minimum cardinality on the same vertex set, such that $G + F = (V, E \cup F)$ is 'globally rigid'.*

As we noted in the beginning, to solve the problem for 'rigid' inputs, we give a common combinatorial generalization of this problem for all the above-mentioned types of rigidity in Section 2. The common point is that (k, ℓ)-sparse graphs are used for the characterization of rigidity, while redundant rigidity (where $G-e$ remains rigid after the deletion of an arbitrary edge) and 2- or 3-vertex-connectivity is usually used for the characterization of global rigidity. The problem of augmenting rigid graphs to redundantly rigid was considered in [14, 27], while vertex-connectivity augmentation problems have a quite extensive literature (see [9, 16, 22] for related results and [13] for a survey) of which we only need some basic ones due to the special conditions of our problem.

2. Preliminaries. In this section we collect the basic definitions and results that we shall use, including the formal definition of the combinatorial problem family solved in this paper, and its connection to the problem presented in the introduction. For a detailed introduction to combinatorial rigidity theory, the reader is referred to [23]. Although our goal is to solve a graph augmentation problem, we will need to use hypergraphs (see Section 3) hence some definitions will be for hypergraphs instead of graphs.

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, let $\boldsymbol{d_{\mathcal{H}}(v)}$ denote the number of hyperedges that contain $v \in V$ and let $\boldsymbol{d_{\mathcal{H}}(X, Y)}$ denote the number of hyperedges that are induced by $X \cup Y$ but not induced by neither X nor Y for $X, Y \subseteq V$. The neighbor set of $X \subset V$ is $\boldsymbol{N_{\mathcal{H}}(X)}:= \{v \in V - X : \exists x \in X \text{ and } e \in \mathcal{E} \text{ such that } v, x \in e\}$.

For two integers k and ℓ for which $0 < k$ and $\ell < 2k$ hold, a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called **(k, ℓ)-sparse** if $i_{\mathcal{H}}(X) \leq k|X| - \ell$ holds for all $X \subseteq V$ with $k|X| - \ell \geq 0$, where **$i_{\mathcal{H}}(X)$** denotes the number of edges induced by X in $\mathcal{H}$. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called **(k, ℓ)-tight** if it is sparse and $|\mathcal{E}| = k|V| - \ell$. Due to its usage in rigidity theory, which we present in Section 2.1, we call a hypergraph **(k, ℓ)-rigid** if it contains a spanning (k, ℓ)-tight subhypergraph and has no loop (that is, no hyperedge which is a singleton) if $k < \ell$. (For example, the $(1, 1)$-sparse graphs are the forests, the $(1, 1)$-tight graphs are the trees, and the $(1, 1)$-rigid graphs are the connected graphs.)

(k, ℓ)-tight hypergraphs have some well known properties. For example, any subhypergraph of a (k, ℓ)-sparse hypergraph is always (k, ℓ)-sparse and any (k, ℓ)-tight subhypergraph of a (k, ℓ)-sparse hypergraph is an induced subhypergraph. If $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ both are tight subhypergraphs of a (k, ℓ)-sparse hypergraph $\mathcal{H}$, then $\mathcal{H}_1 \cap \mathcal{H}_2 = (V_1 \cap V_2, \mathcal{E}_1 \cap \mathcal{E}_2)$ is an induced subhypergraph of $\mathcal{H}$ (by the submodularity of $i_{\mathcal{H}}$).

The hyperedge sets of the (k, ℓ)-tight subhypergraphs of a hypergraph $\mathcal{H}$ correspond to the independent sets of the so-called **(k, ℓ)-sparsity matroid** (or **count matroid**) of $\mathcal{H}$ (see [12, Section 13.5], [30] and [42, Appendix A]). (This matroid family generalizes the graphic matroid as the graphic matroid on the edge set of a graph G is isomorphic to the $(1, 1)$-sparsity matroid of G.) The spanning (k, ℓ)-tight subhypergraphs form a basis of this matroid, while a hypergraph which forms a circuit in this matroid is called a **(k, ℓ)-M-circuit**. In particular, if $\mathcal{H}$ is (k, ℓ)-tight and $e = ij$ is a new (graph) edge, then $G + e$ has a unique (k, ℓ)-M-circuit, denoted by $\mathcal{C}_{\mathcal{H}}(ij)$ or $\mathcal{C}_{\mathcal{H}}(e)$. This circuit contains e. $(V(\mathcal{C}_{\mathcal{H}}(e)), \mathcal{E}(\mathcal{C}_{\mathcal{H}}(e)) - e)$ forms a (k, ℓ)-tight subhypergraph of $\mathcal{H}$, that we call **$\mathcal{T}_{\mathcal{H}}(e)$** or $\mathcal{T}_{\mathcal{H}}(ij)$. (Note that this definition may also be extended to the case where we add a new hyperedge to a (k, ℓ)-tight hypergraph, however, in this paper we only consider additional graph edges.) For the sake of convenience, we do not distinguish a hypergraph from its edge set, that is, $\mathcal{T}_{\mathcal{H}}(e) = \mathcal{E}(\mathcal{C}_{\mathcal{H}}(e)) - e$. When the hypergraph $\mathcal{H}$ is clear from the context, we shall omit the subscript $\mathcal{H}$ from $\mathcal{T}_{\mathcal{H}}(e)$. The next lemma is folklore and follows easily from basic matroid properties.

LEMMA 2.1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a (k, ℓ)-tight graph and let $e = ij$ be an edge for some $i, j \in V$. If $\mathcal{H}'$ is a (k, ℓ)-tight subhypergraph of $\mathcal{H}$ with $\{i, j\} \subseteq V(\mathcal{H}')$, then $\mathcal{T}_{\mathcal{H}}(ij)$ is a subhypergraph of $\mathcal{H}'$. Thus $\mathcal{T}_{\mathcal{H}}(ij)$ is equal to the intersection of all tight subhypergraphs $\mathcal{T}_h$ of $\mathcal{H}$ with $\{i, j\} \subseteq V(\mathcal{T}_h)$.*

A hyperedge e of a rigid hypergraph $\mathcal{H}$ is called **(k, ℓ)-redundant** if $\mathcal{H} - e$ is (k, ℓ)-rigid. A hypergraph is **(k, ℓ)-redundant** if all of its hyperedges are redundant. (For example, the $(1, 1)$-redundant graphs are the 2-edge-connected graphs.)

There are some differences in the properties of (k, ℓ)-rigid hypergraphs depending on the relation of k and ℓ, as the following two results show. To simplify the presentation of our results, let $\boldsymbol{c_{k,\ell}} := \max\left\{\left\lceil \frac{\ell}{k} \right\rceil, 0\right\}$, that is, $c_{k,\ell}$ is zero if $\ell \leq 0$, one if $0 < \ell \leq k$, and two if $k < \ell < 2k$. With standard submodular techniques one can prove the following (see [23, 27]).

LEMMA 2.2. *Let $\mathcal{H} = (V, \mathcal{E})$ be a (k, ℓ)-sparse hypergraph on at least three vertices, and let $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ be (k, ℓ)-tight subhypergraphs of $\mathcal{H}$. If $|V_1 \cap V_2| \geq c_{k,\ell}$, then $\mathcal{H}_1 \cup \mathcal{H}_2$ is a (k, ℓ)-tight subhypergraph of $\mathcal{H}$.*

A graph $G = (V, E)$ is called **k-connected** if $|V| > k$ and $G - X$ is connected for any vertex set $X \subset V$ of cardinality at most $k - 1$. For the sake of convenience, a graph which is not necessarily connected will be called **0-connected** in this paper.

Connectivity has several connections to rigidity. An often used folklore result is the following.

Proposition 2.3. *If $G = (V, E)$ is a (k, ℓ)-rigid graph for which $|V| \geq 3$, then G is $c_{k,\ell}$-connected.*

Based on Proposition 2.3, one may ask the following problem as an extension of the problem which was considered in [27] (see Section 2.2 for more details on this problem).

Problem 2. *Given a (k, ℓ)-rigid graph $G = (V, E)$ with $|V| > 3$, find a graph $H = (V, F)$ with a minimum cardinality edge set F, such that $G \cup H = (V, E \cup F)$ is (k, ℓ)-redundant and $(c_{k,\ell} + 1)$-connected.*

In this paper, we give a min-max theorem and a polynomial algorithm for Problem 2 for all integer pairs of (k, ℓ) where $\max(0, \ell) \leq k$ and also for $0 < k < \ell \leq \frac{3}{2}k$ with the extra assumption that the input is a **simple** graph (that is, it contains no parallel edges and no loops). In all cases, the output edge set F can be provided in such a way that $F \cap E = \emptyset$ if such an augmentation is possible (that is, if the complete graph on V is (k, ℓ)-redundant).

2.1. Connection to rigidity theory. In this subsection we show how Problem 2 is connected to the problems from rigidity theory presented in Problem 1. We start with the characterization of rigidity and global rigidity of graphs in $\mathbb{R}^2$ and on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ given by Pollaczeck-Geiringer [32], Laman [28], Jackson and Jordán [19], Whiteley [40], and Connelly and Whiteley [7].

Theorem 2.4 ([28, 32, 40]). *The following three statements are equivalent for a graph G. (i) G is rigid in $\mathbb{R}^2$, (ii) G is rigid on $\mathbb{S}^2 \subset \mathbb{R}^3$, (iii) G is $(2, 3)$-rigid.*

Note that parallel edges give no extra condition to a bar-joint framework hence in the characterization of global rigidity we may assume that G is simple.

Theorem 2.5 ([7, 19]). *The following three statements are equivalent for a simple graph G on at least three vertices. (i) G is globally rigid in $\mathbb{R}^2$, (ii) G is globally rigid on $\mathbb{S}^2 \subset \mathbb{R}^3$, (iii) G is $(2, 3)$-redundant and 3-connected.*

Theorems 2.4 and 2.5 imply that the solution of Problem 2 – with the extra condition that both the input and the output graph should be simple – solves the global rigidity augmentation problem in $\mathbb{R}^2$ and on $\mathbb{S}^2 \subset \mathbb{R}^3$ on rigid inputs.

The rigidity and global rigidity of graphs on a cylinder $C^2 \subset \mathbb{R}^3$ has been characterized by Nixon, Owen and Power [31] and Jackson and Nixon [21]. In this case the characterization uses simple $(2, 2)$-rigid (and $(2, 2)$-redundant) graphs. Note that without the simplicity condition a $(2, 2)$-tight graph may have parallel edges (which is meaningless from a rigidity point of view).

Theorem 2.6 ([31]). *A simple graph is rigid on the cylinder $C \subset \mathbb{R}^3$ if and only if it is $(2, 2)$-rigid.*

Theorem 2.7 ([21]). *A simple graph is globally rigid on the cylinder $C \subset \mathbb{R}^3$ if and only if it is $(2, 2)$-redundant and 2-connected.*

Theorems 2.6 and 2.7 imply that the solution of Problem 2 – with the extra condition that we may only use non-graph edges for the augmentation – solves the global rigidity augmentation problem on the cylinder $C \subset \mathbb{R}^3$ on rigid inputs.

Finally we note that the generic rigidity (and generic global rigidity, respectively) of body-bar and body-hinge frameworks in $\mathbb{R}^d$ have been characterized by

$(\binom{d+1}{2}, \binom{d+1}{2})$-rigidity (and $(\binom{d+1}{2}, \binom{d+1}{2})$-redundancy, respectively) of a corresponding graph in [18, 24, 37, 39, 41]. Hence in these cases the global rigidity augmentation problem can be solved optimally in polynomial time by the results of [27] that we summarize in the following section.

2.2. Augmentation to a (k,ℓ)-redundant hypergraph. Let us now investigate the problem of augmenting a (k,ℓ)-tight hypergraph $\mathcal{H} = (V,\mathcal{E})$ to a (k,ℓ)-redundant hypergraph by a minimum number of graph edges. This problem was considered and solved previously in [27]. In this subsection we list some notions and results from [27] that we shall use in this paper.

If we add the edges $e_1,\dots,e_k$ to $\mathcal{H}$, we make some hyperedges of $\mathcal{H}$ redundant. Let us denote the set of these hyperedges by $\mathcal{R}_\mathcal{H}(e_1,\dots,e_k)$. Note that $\mathcal{R}_\mathcal{H}(e_1) = \mathcal{T}_\mathcal{H}(e_1)$. The following statement generalizes this simple fact.

LEMMA 2.8 ([27]). *Let $\mathcal{H} = (V,\mathcal{E})$ be a tight hypergraph. Then $\mathcal{R}_\mathcal{H}(e_1,\dots,e_k) = \mathcal{T}_\mathcal{H}(e_1) \cup \dots \cup \mathcal{T}_\mathcal{H}(e_k)$ for arbitrary edges $e_1,\dots,e_k$.*

Given a tight hypergraph $\mathcal{H} = (V,\mathcal{E})$, a set $C \subsetneq V$ is called **(k,ℓ)-co-tight** if $V - C$ induces a tight subhypergraph. This is equivalent to the following: C is (k,ℓ)-co-tight in $\mathcal{H}$ if $k|V - C| \geq \ell$ and $|\widehat{\mathcal{E}}_\mathcal{H}(C)| = k|C|$ where $\boldsymbol{\widehat{\mathcal{E}}_\mathcal{H}(C)}$ denotes the set of hyperedges of $\mathcal{H}$ for which at least one of its vertices is in C. Notice, that $|\widehat{\mathcal{E}}_\mathcal{H}(X)| = i_\mathcal{H}(X) + d_\mathcal{H}(X, V - X)$ and $|\mathcal{E}| = |\widehat{\mathcal{E}}_\mathcal{H}(X)| + i_\mathcal{H}(V - X)$ holds for every $X \subseteq V$. Hence $|\widehat{\mathcal{E}}_\mathcal{H}(X)| \geq k|X|$ for every $X \subsetneq V$ where $|X| \leq |V| - c_{k,\ell}$ by $|\mathcal{E}| = k|V| - \ell$ and the sparsity of $\mathcal{H} - X$. By Lemma 2.1, the following property follows easily:

PROPOSITION 2.9 ([27]). *Let C be a (k,ℓ)-co-tight set of a (k,ℓ)-tight hypergraph $\mathcal{H}$. If $\{u,v\} \cap C = \emptyset$, then $\mathcal{T}(uv) \cap \widehat{\mathcal{E}}_\mathcal{H}(C) = \emptyset$.*

Let us abbreviate the name of minimal (k,ℓ)-co-tight sets by **(k,ℓ)-MCT** sets and let $\boldsymbol{\mathcal{C}^*_\mathcal{H}}$ denote the family of all (k,ℓ)-MCT sets of $\mathcal{H}$. We shall use the following results.

LEMMA 2.10 ([27]). *Let C_1 and C_2 be two intersecting (k,ℓ)-MCT sets of a (k,ℓ)-tight hypergraph $\mathcal{H} = (V,\mathcal{E})$. Then $|C_1 \cup C_2| \geq |V| - 1$, moreover $C_1 \cup C_2 = V$ if $k \geq \ell$.*

LEMMA 2.11 ([27]). *Let $\mathcal{H}$ be a (k,ℓ)-tight hypergraph. The members of $\mathcal{C}^*_\mathcal{H}$ are pairwise disjoint or there are two vertices $v,w \in V$ such that $\{v,w\} \cap C \neq \emptyset$ for all $C \in \mathcal{C}^*_\mathcal{H}$.*

LEMMA 2.12 ([27, Lemma 5.4]). *Let $\mathcal{H} = (V,\mathcal{E})$ be a (k,ℓ)-tight hypergraph and let $P \subset V$ be a set which intersects each member of $\mathcal{C}^*_\mathcal{H}$. Suppose that $\mathcal{H}' = (V',\mathcal{E}')$ is a (k,ℓ)-tight subhypergraph of $\mathcal{H}$ such that $P \subset V'$. Then $\mathcal{H}' = \mathcal{H}$.*

Lemmas 2.11 and 2.12 imply that if there are at least two intersecting (k,ℓ)-MCT sets, then there exists an edge e such that $\mathcal{T}_\mathcal{H}(e) = \mathcal{H}$. If we consider the other case, then the (k,ℓ)-MCT sets are disjoint. This motivates us to investigate the disjoint (k,ℓ)-MCT sets. The following lemma slightly extends the statement of [27, Lemma 5.6].

LEMMA 2.13. *Let $\mathcal{H} = (V,\mathcal{E})$ be a (k,ℓ)-tight hypergraph and let C, K be two disjoint (k,ℓ)-MCT sets of $\mathcal{H}$. If $k|V - (C \cup K)| \geq \ell$, then $\widehat{\mathcal{E}}_\mathcal{H}(C) \cap \widehat{\mathcal{E}}_\mathcal{H}(K) = \emptyset$.*

Proof. By counting the hyperedges induced by $V - (C \cup K)$, we get that

$$i_\mathcal{H}(V - (C \cup K)) \leq k|V - (C \cup K)| - \ell = k|V| - |\widehat{\mathcal{E}}_\mathcal{H}(C)| - |\widehat{\mathcal{E}}_\mathcal{H}(K)| - \ell$$

where the first inequality comes from the sparsity of $\mathcal{H}$ and the property $k|V-(C\cup K)|\geq \ell$, while the equalities hold because C and K are disjoint (k,ℓ)-MCT sets.

Counting the same hyperedges with their complements implies

$$i_{\mathcal{H}}(V-(C\cup K)) = |\mathcal{E}| - |\widehat{\mathcal{E}}_{\mathcal{H}}(C)\cup\widehat{\mathcal{E}}_{\mathcal{H}}(K)| \geq k|V|-\ell-|\widehat{\mathcal{E}}_{\mathcal{H}}(C)|-|\widehat{\mathcal{E}}_{\mathcal{H}}(K)|.$$

Thus equality must hold throughout. This is only possible if $\widehat{\mathcal{E}}_{\mathcal{H}}(C)\cap\widehat{\mathcal{E}}_{\mathcal{H}}(K)=\emptyset$. □

LEMMA 2.14 ([27, Lemma 5.7]). *Let $\mathcal{H}=(V,\mathcal{E})$ be a (k,ℓ)-tight hypergraph on at least 4 vertices. Let A be a (k,ℓ)-MCT set, $u\in A$ and $v\in V-(A\cup N_{\mathcal{H}}(A))$. Then $A\cup N_{\mathcal{H}}(A)\subset V(\mathcal{T}_{\mathcal{H}}(uv))$.*

THEOREM 2.15 ([27]). *Let $\mathcal{H}=(V,\mathcal{E})$ be a (k,ℓ)-tight hypergraph on at least k^2+3 vertices. If there exists any (k,ℓ)-co-tight set in $\mathcal{H}$, then*

$$\min\{|F| : H=(V,F) \text{ is a graph for which } \mathcal{H}\cup H \text{ is } (k,\ell)\text{-redundant}\}$$
$$= \max\left\{\left\lceil\frac{|\mathcal{C}|}{2}\right\rceil : \mathcal{C} \text{ is a family of disjoint } (k,\ell)\text{-co-tight sets}\right\}.$$

Otherwise, $\mathcal{H}+uv$ is (k,ℓ)-redundant for every pair $u,v\in V$.

2.3. Connectivity augmentation. By Proposition 2.3, every (k,ℓ)-tight graph G is $c_{k,\ell}$-connected and thus we augment a $c_{k,\ell}$-connected graph to a $(c_{k,\ell}+1)$-connected graph where $c_{k,\ell}$ is 0, 1 or 2. There exist several methods to deal with these particular problems, even linear time algorithms [9, 16]. However, we also need to augment G to a (k,ℓ)-redundant graph hence we follow simpler ideas from [9, 22].

Let $G=(V,E)$ be a c-connected graph. Let us call a set $X\subset V$ of cardinality c a **min-cut** of G, if $G-X$ is not connected. For a min-cut X of G, let $\boldsymbol{b^c_X(G)}$ denote the number of components of $G-X$. Let $\boldsymbol{b^c(G)}$ denote the maximum value of $b^c_X(G)$ over all min-cuts X of G if there exist any, and let $b^c(G):=1$ otherwise. Clearly, any edge set F that augments G to a $(c+1)$-connected graph needs to induce a connected graph on the components of $G-X$ for every min-cut X. Thus $|F|\geq b^c(G)-1$. A set $P\subsetneq V$ is called a $\boldsymbol{(c+1)}$**-fragment** of a c-connected graph G which is not $(c+1)$-connected if $N_G(P)$ is a min-cut of G and P induces a connected subgraph of G. Let us denote the maximum number of pairwise disjoint $(c+1)$-fragments by $\boldsymbol{t^c(G)}$. Increasing the connectivity of a c-connected graph G which is not $(c+1)$-connected is equivalent to increasing the number of neighbors of each $(c+1)$-fragment of G. Hence, for any edge set F that augments G to a $(c+1)$-connected graph, $|V(F)|\geq t^c(G)$ must hold. These with Proposition 2.3 imply the following statement.

LEMMA 2.16. *Given a (k,ℓ)-rigid graph G. The minimum number of edges that augment G to a $(c_{k,\ell}+1)$-connected graph is at least* $\max\left\{b^{c_{k,\ell}}(G)-1, \left\lceil\frac{t^{c_{k,\ell}}(G)}{2}\right\rceil\right\}$.

Let us call an inclusion-wise minimal $(c+1)$-fragment a $\boldsymbol{(c+1)}$**-end**. As every $(c+1)$-fragment contains at least one $(c+1)$-end, $t^c(G)$ is equal to the number of pairwise disjoint $(c+1)$-ends. It is easy to see that, for $c=1$, the $(c+1)$-ends are pairwise disjoint. As we will see in the following lemma, this statement is also true for $c=2$, even though in this case the structure is slightly more difficult as there are two types of min-cuts. A min-cut $\{u,v\}$ of a 2- but not 3-connected connected graph G is called a **weak min-cut** if it separates another min-cut $\{u',v'\}$ of G, that is, u' and v' are in different connected components of $G-\{u,v\}$. Note that in this case the min-cut $\{u',v'\}$ is also weak and $b^2_G(\{u,v\})=b^2_G(\{u',v'\})=2$. If a min-cut is not weak then it is called a **strong min-cut**. (For example, in a cycle of length four, the two neighbors of a vertex form a weak min-cut as the complement of this two element set form also a min-cut. On the other hand, if we add a diagonal to the cycle, the resulting graph has only one min-cut, the two endpoints of of the diagonal edge.) When $(k, \ell) = (2, 3)$, the structure of G is much simpler by the following result of Jackson and Jordán [19].

LEMMA 2.17 ([19]). *Let G be a $(2, 3)$-rigid graph. Then G contains no weak min-cuts.*

Lemma 2.17 immediately implies the following statement when $(k, \ell) = (2, 3)$. However, it holds for general pairs of k and ℓ, too.

LEMMA 2.18. *Let G be a $c_{k,\ell}$-connected graph. Then the $(c_{k,\ell} + 1)$-ends of G are pairwise disjoint.*

Proof. If G is $(c_{k,\ell} + 1)$-connected, the statement holds obviously. Also, if $k \geq \ell$ thus $c_{k,\ell} \leq 1$, then the $(c_{k,\ell} + 1)$-ends of G are clearly pairwise disjoint.

Now suppose that $k < \ell$ hence $c_{k,\ell} = 2$. Let C_1 and C_2 be two intersecting 3-ends and let $N(C_1) = \{u_1, v_1\}$ and $N(C_2) = \{u_2, v_2\}$ be the two (weak) min-cuts defining C_1 and C_2. We may suppose that $u_1 \in C_2$ and $u_2 \in C_1$. If we consider $N(C_1 \cap C_2)$ we can conclude that $N(C_1 \cap C_2) = \{u_1, u_2\}$ that contradicts the minimality of the 3-ends C_1 and C_2. ∎

3. The (k, ℓ)-M-component hypergraph.

In our main theorem we shall combine the results presented in the previous two subsections. However, it was shown in [27] that the problem of augmenting a (k, ℓ)-rigid graph to a (k, ℓ)-redundant graph with the minimum number of edges is NP-hard. In this section, we show how this issue can be bypassed by using an auxiliary (k, ℓ)-tight hypergraph which is constructed by using an extra property of $(c_{(k,\ell)} + 1)$-connected (k, ℓ)-redundant graphs, namely, their (k, ℓ)-M-connectivity.

First, we list some basic definitions concerning the sparsity matroid. We refer to [23, 42] for more details. As we have noted before, the edge sets of spanning (k, ℓ)-tight subgraphs of a graph G correspond to the bases of the (k, ℓ)-sparsity matroid of G. It is well-known, that an equivalence relation can be defined on the ground set S of an arbitrary matroid $\mathcal{M}$ (by using the circuit axioms of a matroid), as follows. Two elements $x, y \in S$ are equivalent if there exists a circuit C of $\mathcal{M}$ such that $x, y \in C$. The equivalence classes of this matroid are called *components* of $\mathcal{M}$. The components of the 2-dimensional rigidity matroid of G are often called the *M-components* of G (see e.g. in [19]). By extending this notion to other sparsity matroids, we will call a component of the (k, ℓ)-sparsity matroid of G a **(k, ℓ)-M-component**. Note that if an edge e of G is not redundant, then $\{e\}$ is a (k, ℓ)-M-component of G and it is called a **trivial** (k, ℓ)-M-component of G. (See Fig. 1 (later) for an illustration of non-trivial $(2, 3)$-M-components in a $(2, 3)$-rigid graph.) Let us also show the following easy properties of the (k, ℓ)-M-components.

OBSERVATION 1. *Let G be a (k, ℓ)-rigid graph and C a (k, ℓ)-M-component of G. Then C is an induced subgraph of G.*

Proof. Suppose that $i, j \in V(C)$. Then there exists a circuit $C' \subseteq C$ for which $i, j \in V(C')$. However, this means that there exists a (k, ℓ)-tight subgraph $T \subset C'$ for which $i, j \in V(T)$ and hence $\mathcal{T}_{C'}(ij) \subset C'$ by Lemma 2.2. If ij is an edge of G, then $\mathcal{T}_T(ij) + ij$ is a circuit that intersects C', thus the equivalence relation on the matroid circuits shows that $ij \in C$. ∎

Lemma 3.1. *Let $G = (V, E)$ be a (k, ℓ)-rigid graph and let $G^* = (V, E^*)$ be an arbitrary (k, ℓ)-tight spanning subgraph of G. Then every trivial (k, ℓ)-M-component is contained in E^*, and, for any non-trivial (k, ℓ)-M-component C of G, $i_{G^*}(V(C)) = k|V(C)| - \ell$.*

Proof. If C is a trivial (k, ℓ)-M-component of G, then C consists of a single non-redundant edge e of G. Thus e must also be an edge of G^* since G^* is (k, ℓ)-rigid while $G - e$ is not (k, ℓ)-rigid.

Suppose now that C is non-trivial. Let $B = E^* \cap C$ that is $i_{G^*}(V(C)) = |B|$. Now B must be a base of C in the (k, ℓ)-sparsity matroid since otherwise we may add edges from C to G^* by maintaining its sparsity (as the edges in C are only contained in (k, ℓ)-circuits of G consisting of the edges of C by the definition of a (k, ℓ)-M-component). This shows that $|B| = k|V(C)| - \ell$. ◻

If G has only one (k, ℓ)-M-component, then it is called **(k, ℓ)-M-connected**. Note that each non-trivial (k, ℓ)-M-component is (k, ℓ)-M-connected. It is obvious that the (k, ℓ)-M-connectivity of a graph implies that it is (k, ℓ)-redundant (see [19] for $(k, \ell) = (2, 3)$). The converse implication is not always true. However, for our purpose, the following extension of a result from Jackson and Jordán [19] is enough.

Lemma 3.2. *Let k be a positive integer and ℓ be an integer such that $\ell \le \frac{3}{2}k$ and let G be a $(c_{k,\ell} + 1)$-connected and (k, ℓ)-redundant graph. If $k < \ell$, then suppose also that G has no two vertices which are connected by more than $2k - \ell$ edges. Then G is (k, ℓ)-M-connected.*

Proof. Suppose that G is not (k, ℓ)-M-connected and let $H_1, \dots, H_q$ be its (k, ℓ)-M-components. Notice that $|H_i| \neq 1$ for $i = 1, \dots q$, because G is (k, ℓ)-redundant. Let $X_i = V(H_i) - \bigcup_{j \neq i} V(H_j)$ denote the set of vertices that do not belong to any (k, ℓ)-M-component other than H_i. Let $Y_i = V(H_i) - X_i$. Clearly $|V| = \sum_{i=1}^{q} |X_i| + |\bigcup_{i=1}^{q} Y_i|$ and $\sum_{i=1}^{q} |Y_i| \ge 2|\bigcup_{i=1}^{q} Y_i|$ hence $|V| \le \sum_{i=1}^{q} |X_i| + \frac{1}{2}\sum_{i=1}^{q} |Y_i|$. Moreover, notice that by the $(c_{k,\ell} + 1)$-connectivity of G $|Y_i| \ge c_{k,\ell} + 1$. (More precisely we can only claim that $|Y_i| \ge c_{k,\ell} + 1$ when $|V(H_i)| \ge c_{k,\ell} + 1$, however, this is obvious if $c_{k,\ell} \le 1$ and follows from our assumption on the the number of parallel edges in G if $k < \ell$ and thus $c_{k,\ell} = 2$.)

Let us now choose a (k, ℓ)-tight subgraph $G^* = (V, E^*)$ of G. Let $B_i = H_i \cap E^*$ for $i = 1, \dots, q$. Note that $\bigcup_{i=1}^{q} B_i = E^*$. Hence, by using the above inequalities and Lemma 3.1, we get $k|V| - \ell = |\bigcup_{i=1}^{q} B_i| = \sum_{i=1}^{q} |B_i| = \sum_{i=1}^{q} k|V(H_i)| - \ell = k\sum_{i=1}^{q} |X_i| + k\sum_{i=1}^{q} |Y_i| - q\ell = k(\sum_{i=1}^{q} |X_i| + \frac{1}{2}\sum_{i=1}^{q} |Y_i|) + \frac{k}{2}\sum_{i=1}^{q} |Y_i| - q\ell \ge k|V| + \frac{k}{2}\sum_{i=1}^{q} |Y_i| - q\ell \ge k|V| + \frac{k(c_{k,\ell}+1)q}{2} - q\ell$. If $0 < \ell \le k$, then the previous inequality gives $k|V| - \ell \ge k|V| + q\frac{2}{2}k - q\ell > k|V| - \ell$, a contradiction. If $k < \ell \le \frac{3}{2}k$, then it gives $k|V| - \ell \ge k|V| + q\frac{3}{2}k - q\ell > k|V| - \ell$, also a contradiction. ◻

Notice that, for example, if G is simple, then G has no two vertices which are connected by more than $2k - \ell$ edges.

For a (k, ℓ)-rigid graph $G = (V, E)$, let $\boldsymbol{\mathcal{H}_G = (V, \mathcal{E})}$ be a hypergraph, called the **(k, ℓ)-M-component hypergraph** of G, such that $\mathcal{E}$ consists of the non-redundant edges of E and $k|V(C)| - \ell$ parallel copies of the hyperedge formed on $V(C)$ for each non-trivial (k,ℓ)-M-component C of G. (For example, the $(1,1)$-M-component hypergraph of G contains $|X|-1$ parallel copies of the hyperedge on the vertex set X for each 2-connected component X of G.) The $(2,3)$-M-component hypergraph was defined previously by Fekete and Jordán [11].

LEMMA 3.3. *Let $G=(V,E)$ be a (k,ℓ)-rigid graph, let $G^*=(V,E^*)$ be a spanning (k,ℓ)-tight subgraph of G, and let $\mathcal{H}_G$ be the (k,ℓ)-M-component hypergraph of G. Then $i_{\mathcal{H}_G}(X)\le i_{G^*}(X)$ holds for each $X\subseteq V$. Furthermore, equality holds exactly when X induces either all or none of the edges of each (k,ℓ)-M-component of G.*

Proof. Let E' denote the set of non-redundant edges of G and $H_1,\dots,H_t$ denote the non-trivial (k,ℓ)-M-components of G.

Note that $|G^*\cap H_i| = k|V(H_i)|-\ell = i_{\mathcal{H}_G}(V(H_i))$ holds for every $i=1,\dots,t$ by Lemma 3.1. Notice that, for each $e\in E'$, $e\in E^*$ and $e\in\mathcal{H}_G$ must also hold. Recall that the (k,ℓ)-M-components partition the edge set of G and the non-trivial ones are induced subgraphs by Observation 1. Observe also that, for $X\subseteq V$ and $i\in\{1,\dots,t\}$, either $X\cap V(H_i)$ induces no hyperedge in $\mathcal{H}_G$ or $V(H_i)\subseteq X$. Hence, we have $i_{G^*}(X)=i_{E'}(X)+\sum_{i=1}^{t} i_{G^*}(X\cap V(H_i)) \ge i_{E'}(X)+\sum_{i=1}^{t} i_{\mathcal{H}_G}(X\cap V(H_i)) = i_{\mathcal{H}_G}(X)$ for each $X\subseteq V$ where equality holds exactly when for all $i=1,\dots,t$ either $X\cap V(H_i)$ induces no edge in G^* or $V(H_i)\subseteq X$. ∎

Lemma 3.3 has the following corollary.

OBSERVATION 2. *If G is a (k,ℓ)-rigid graph, then the (k,ℓ)-M-component hypergraph $\mathcal{H}_G$ of G is a (k,ℓ)-tight hypergraph. Furthermore, if X induces a (k,ℓ)-tight subhypergraph of $\mathcal{H}_G$, then $G[X]$ is a (k,ℓ)-rigid subgraph of G.*

The following lemma may be understood as the converse of Lemma 3.1.

LEMMA 3.4. *Let $\mathcal{H}=(V,\mathcal{E})$ be a (k,ℓ)-tight hypergraph. Suppose, for a hyperedge $e\in\mathcal{E}$, that e has exactly $k|V(e)|-\ell$ parallel copies in $\mathcal{E}$. Let $\mathcal{H}'$ be the hypergraph we get by deleting all the $k|V(e)|-\ell$ parallel copies of e from $\mathcal{E}$ and inserting an arbitrary (k,ℓ)-tight spanning subgraph on $V(e)$. Then $\mathcal{H}'$ is also (k,ℓ)-tight.*

Proof. As the number of (hyper)edges does not change we only need to show the (k,ℓ)-sparsity of $\mathcal{H}'$. For the sake of contradiction suppose that $\mathcal{H}'$ is not (k,ℓ)-sparse. Let Y denote the vertex set of a circuit in $\mathcal{H}'$. By the (k,ℓ)-sparsity of $\mathcal{H}$, $|V(e)\cap Y|\ge 2$. Hence Lemma 2.2 may be used on the (k,ℓ)-tight subgraph of $\mathcal{H}'$ induced by $V(e)$ and on Y minus one edge which is not induced by $V(e)$. This shows that $V(e)\cup Y$ induces a (k,ℓ)-rigid subgraph in $\mathcal{H}'$ that is not (k,ℓ)-tight which contradicts $i_{\mathcal{H}}(V(e)\cup Y)=i_{\mathcal{H}'}(V(e)\cup Y)$. ∎

The key observation which will imply that the global rigidity augmentation problem is polynomially solvable for all rigid inputs (contrary to the case if we want to augment G to a (k,ℓ)-redundant graph, see in [27]) is the following.

LEMMA 3.5. *Let $G=(V,E)$ be a (k,ℓ)-rigid graph, let $\mathcal{H}_G=(V,\mathcal{E})$ be the (k,ℓ)-M-component hypergraph of G, and let F be an edge set on V.*

(i) If $G+F$ is (k,ℓ)-M-connected, then $\mathcal{H}_G+F$ is (k,ℓ)-redundant.

(ii) If $\mathcal{H}_G+F$ is (k,ℓ)-redundant, then $G+F$ is (k,ℓ)-redundant.

Proof. (i) As $\mathcal{H}_G$ is a (k,ℓ)-tight hypergraph by Observation 2, each $f\in F$ is redundant in $\mathcal{H}_G+F$. Let us take now a hyperedge $e'\in\mathcal{E}$. Let $e\in E$ be any edge from the (k,ℓ)-M-component corresponding to e'. As $G+F$ is (k,ℓ)-M-connected, for any $f\in F$, there exists an M-circuit C of $G+F$ such that $e,f\in C$. Let us choose a (k,ℓ)-tight spanning subgraph $G^* = (V, E^*)$ of G such that $C - f \subset E^*$. Clearly, $e \in \mathcal{T}_{G^*}(f)$. Now $i_{\mathcal{H}_G}(X) \le i_{G^*}(X)$ for all $X \subseteq V(\mathcal{T}_{G^*}(f))$ holds by Lemma 3.3, which results that $V(\mathcal{T}_{G^*}(f)) \subseteq V(\mathcal{T}_{\mathcal{H}_G}(f))$ by Lemma 2.1. This shows that $e' \in \mathcal{T}_{\mathcal{H}_G}(f)$ implying that e' is redundant in $\mathcal{H}_G + F$.

(ii) As G is a (k,ℓ)-rigid graph, each $f \in F$ is redundant in $G+F$. It is also obvious that every edge that is contained by a non-trivial (k,ℓ)-M-component is redundant. Now let us consider an edge e that is not redundant in G. That is, $e \in E \cap \mathcal{E}$. Now, as $\mathcal{H}_G$ is (k,ℓ)-tight and $\mathcal{H}_G + F$ is (k,ℓ)-redundant, there is an $f \in F$, such that $e \in \mathcal{T}_{\mathcal{H}_G}(f)$ thus $\mathcal{H}_G - e + f$ is (k,ℓ)-tight. Now by using Lemma 3.4 sequentially on the non-trivial hyperedges starting with $\mathcal{H}_G - e + f$ we can get a (k,ℓ)-tight graph G^*, as the conditions of Lemma 3.4 are met after every step we made. In every step an arbitrary (k,ℓ)-tight subgraph can be inserted, hence we may insert the one from G provided by Lemma 3.1. Thus $G^* \subset G$, G^* is (k,ℓ)-tight and $e \notin G^*$. This shows that e is (k,ℓ)-redundant in G. ◻

Note that Lemma 3.2 implies that if F is a feasible solution of Problem 2 for a (k,ℓ)-rigid graph G (and $G+F$ is simple when $k < \ell \le \frac{3}{2}k$), then $G+F$ is (k,ℓ)-M-connected. Now, Lemma 3.5 implies that $\mathcal{H}_G + F$ is (k,ℓ)-redundant. On the other hand, if $\mathcal{H}_G + F$ is (k,ℓ)-redundant, then $G+F$ is also (k,ℓ)-redundant by Lemma 3.5. Hence, to solve Problem 2, it is enough to find a minimal edge set F for which $G+F$ is $(c_{k,\ell}+1)$-connected and $\mathcal{H}_G + F$ is (k,ℓ)-redundant. As $\mathcal{H}_G$ is (k,ℓ)-tight by Observation 2, the results on (k,ℓ)-redundant augmentations can be applied this way. (Note that, when we seek for a (k,ℓ)-redundant augmentation of a (k,ℓ)-rigid graph, the (k,ℓ)-M-connectivity of $G+F$ is not guaranteed. It was shown in [27] that the problem of finding a minimum cardinality edge set that makes a (k,ℓ)-rigid (hyper)graph (k,ℓ)-redundant is NP-hard whenever $\ell > k$.)

4. The min-max theorem. In this section we shall merge the results on the problem of augmenting a (k,ℓ)-tight hypergraph to a (k,ℓ)-redundant hypergraph and on the $(c_{k,\ell}+1)$-connectivity augmentation problem to a new min-max theorem for Problem 2 by mixing the statements of Theorem 2.15 and Lemma 2.16, as follows.

THEOREM 4.1. *Let $k > 0$ and ℓ be two integers such that $\ell \le \frac{3}{2}k$. Let $G = (V, E)$ be a (k,ℓ)-rigid graph on at least $k^2 + 3$ vertices. Suppose also that G is simple if $k < \ell$. Let $\mathcal{H}_G = (V, \mathcal{E})$ be the M-component hypergraph of G. If G is $(c_{k,\ell}+1)$-connected, (k,ℓ)-tight and there is no (k,ℓ)-co-tight set in $\mathcal{H}_G$, then any new edge makes G (k,ℓ)-redundant. Otherwise,* $\min\{|F| : G+F = (V, E\cup F)$ *is (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected*$\} = \max\left\{b^{c_{k,\ell}}(G) - 1, \max\left\{\left\lceil \frac{|\mathcal{A}|}{2} \right\rceil : \mathcal{A}\right.\right.$ *is a family of disjoint (k,ℓ)-co-tight sets of $\mathcal{H}_G$ and $(c_{k,\ell}+1)$-fragments of* $\left.\left. G\right\}\right\}$.

Note that, for a non-tight (k,ℓ)-rigid graph G which is not (k,ℓ)-M-connected, $\mathcal{H}_G$ always has a (k,ℓ)-co-tight set since the vertex set of a hyperedge corresponding to a non-trivial M-component is (k,ℓ)-tight and hence the complement of its vertex set is (k,ℓ)-co-tight. This statement is also true for $(2,3)$-tight graphs as any edge of G forms a $(2,3)$-tight subgraph of G. Also, if G is already (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected (and hence (k,ℓ)-M-connected by Lemma 3.2), then both sides in Theorem 4.1 are 0. Nonetheless, if G is (k,ℓ)-tight for $(k,\ell) \ne (2,3)$, it can happen that G has no (k,ℓ)-co-tight sets (see [27]).

Our main tool to prove Theorem 4.1 for (k,ℓ)-rigid (and not for only (k,ℓ)-tight) inputs is the usage of the M-component hypergraph. If $G+F$ is (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected, then Lemma 3.2 can be used to prove that it is (k,ℓ)-M-connected and hence $\mathcal{H}_G + F$ is (k,ℓ)-redundant (by Lemma 3.5) except when $\ell > k$ and $G+F$ has more than $2k-\ell$ parallel edges between two vertices. The following statement implies that this exceptional case can be avoided.

LEMMA 4.2. *Let $k > 0$ and ℓ be two integers such that $\ell \le \frac{3}{2}k$, and let $G = (V,E)$ be a (k,ℓ)-rigid graph on at least k^2+3 vertices. Then there exists an edge set F with $\min\{|F'| : G+F' = (V, E\cup F')$ is (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected$\}$ edges for which $G+F$ is (k,ℓ)-redundant, $(c_{k,\ell}+1)$-connected and no edge in F is parallel to any edge in G.*

Proof. Let F be a minimum cardinality edge set for which $G+F$ is (k,ℓ)-redundant, $(c_{k,\ell}+1)$-connected and F has the minimal number of parallel edges with G. Assume that an edge $e \in F$ is parallel to some edge e' of G. As the omission of e from F does not affect the $(c_{k,\ell}+1)$-connectivity of $G+F$, we only need to deal with the (k,ℓ)-redundancy of $G+F$.

Let $G' = (V,E')$ be a (k,ℓ)-tight spanning subgraph of G with $e' \in E'$. It is easy to check that a simple complete graph K_V on V is (k,ℓ)-redundant if $|V| \ge k^2+3$. Hence, by Lemma 2.8, $E' = \bigcup_{f\in K_V - E'} \mathcal{T}_{G'}(f)$, that is, for each edge e_i in E' (in particular, for e') there exists an edge $f_{e_i} \in K_V - E'$ such that $e_i \in \mathcal{T}_{G'}(f_{e_i})$. Thus $\mathcal{T}_{G'}(e) = \mathcal{T}_{G'}(e') \subseteq \mathcal{T}_{G'}(f_{e'})$ by Lemma 2.1. This combined with the fact that $E' = \bigcup_{f\in F\cup(E-E')} \mathcal{T}_{G'}(f)$ by Lemma 2.8 results that $E' = \bigcup_{f\in(F-e)\cup(E-E'-e)\cup f'} \mathcal{T}_{G'}(f)$ also holds, that is, $F' = F - e \cup f'$ is also a minimal edge set for which $G+F'$ is (k,ℓ)-redundant, $(c_{k,\ell}+1)$-connected and has less edges parallel to the edges of G than F (since, if f' would be parallel to an edge $e^* \in E - E' - e$, $\mathcal{T}_{G'}(e) \subseteq \mathcal{T}_{G'}(e^*)$ would contradict the minimality of F), a contradiction. Thus F contains no parallel edge to G. ∎

We start this section by proving Theorem 4.1 for $(k,\ell) = (2,3)$, because of its importance in rigidity theory. As it is mentioned in Section 2.1 this is the global rigidity augmentation problem in $\mathbb{R}^2$. Later in this section we sketch how the presented method can be generalized to solve the cases where $k < \ell \le \frac{3}{2}k$ but $(k,\ell) \ne (2,3)$ and in the end for $\ell \le k$.

4.1. Proof of Theorem 4.1 for $(k,\ell) = (2,3)$. For the sake of simplicity, we shall omit the prefix $(2,3)$ from all the notions in this subsection such as $(2,3)$-tight graph or set, $(2,3)$-co-tight set, $(2,3)$-MCT set or $(2,3)$-M-component, and use the term of **rigid** and **redundantly rigid** graph instead of *simple* $(2,3)$-rigid and $(2,3)$-redundant graph, respectively, to match the terminology of rigidity theory. When we are talking about hypergraphs, we keep the notions $(2,3)$-rigid and $(2,3)$-redundant. We may call graphs that are redundantly rigid and 3-connected **globally rigid**. As in this case $c_{k,\ell} = 2$ we may omit it from the superscript of $b_X^2(G)$ and $b^2(G)$. When a graph is 2-connected but not 3-connected all its min-cuts have cardinality two. A min-cut of size two will be called a **cut-pair**.

Notice that, if G is 3-connected, then Theorem 4.1 follows directly by Theorem 2.15 and Lemmas 3.2, 3.5 and 4.2. For a non-3-connected graph G the min $\ge$ max implication in Theorem 4.1 is obvious by Proposition 2.9 and Lemmas 2.16, 3.2, 3.5 and 4.2. To prove the min $\le$ max part, let us consider the family which consists of all MCT sets of $\mathcal{H}_G$ and all 3-ends of G. Let us call the inclusion-wise minimal elements of this family the **atoms** of G. (In Fig. 1, these are the three sets formed by the highlighted vertices: the big (blue) disks form an MCT set of $\mathcal{H}_G$, the (gray) square vertex forms an MCT set of $\mathcal{H}_G$ which is also a 3-end of G, and the (red) triangle vertices form a 3-end of G. At the end of Section 4.1, we present other examples.)

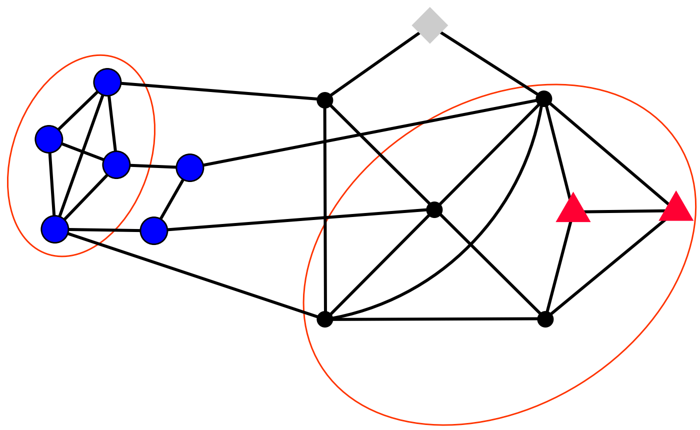


Fig. 1: A rigid graph with its M-components (encircled). It has two 3-ends: the one formed by the (red) triangles and the other one formed by the (gray) square. The M-component hypergraph has two MCT sets: the one formed by the big (blue) disks and the other one formed by the (gray) square. Adding an edge between the (gray) square and one (red) triangle augments the graph to a 3-connected graph. Adding one edge between the (gray) square and one (blue) disk augments the M-component hypergraph to a redundantly rigid hypergraph. Hence the addition of these two edges to the graph results in a globally rigid graph.

Let us denote the family of atoms by $\boldsymbol{\mathcal{A}^*}$. We shall show that the atoms are pairwise disjoint and there exists a set of $\max\left\{b(G)-1, \left\lceil\frac{|\mathcal{A}^*|}{2}\right\rceil\right\}$ edges that augments G to a globally rigid graph. Hence we first need to prove the following.

Lemma 4.3. *Let $G=(V,E)$ be a rigid graph which is not 3-connected. Then the atoms of G are pairwise disjoint.*

To prove Lemma 4.3, we need the following three statements.

Observation 3. *Suppose that C is a co-tight set in the tight hypergraph $\mathcal{H}_G=(V,\mathcal{E})$, and $C' \subsetneq C$ such that $d_{\mathcal{H}_G}(C', C-C')=0$. Then C' is also co-tight.*

Proof. Recall that $d_{\mathcal{H}_G}(C', C-C')=0$ means that no hyperedge of $\mathcal{H}_G$ has vertices in both C' and $C-C'$. This implies that $|\widehat{\mathcal{E}}(C)|=|\widehat{\mathcal{E}}(C')|+|\widehat{\mathcal{E}}(C-C')|$. Recall that a set X is co-tight if and only if $k|V-X|\geq \ell$ and $\widehat{\mathcal{E}}(X)=k|X|$. Furthermore, for any set Y with $k|V-Y|\geq\ell$, $\widehat{\mathcal{E}}(Y)\geq k|Y|$ always holds. Thus if C' is not $(2,3)$-co-tight, then $|\widehat{\mathcal{E}}(C')|\geq 2|C'|+1$ and hence $|\widehat{\mathcal{E}}(C-C')|\leq 2|C-C'|-1$, a contradiction. ∎

Lemma 4.4. *Let $G=(V,E)$ be a rigid graph which is not 3-connected and let $a\in A\in\mathcal{A}^*$ be a vertex from an atom of G. Then there is no $v\in V$ such that a and v forms a cut-pair.*

Proof. If A is a 3-end, then the statement follows immediately by Lemmas 2.17 and 2.18.

Now let A be an MCT set of $\mathcal{H}_G$. Then $\mathcal{H}_G[V-A]$ is tight and hence Observation 2 implies that $G[V-A]$ is rigid. Suppose that a, v forms a cut-pair for $a\in A$ and

$v \in V$.

Suppose first that $|V - A| > 2$. Then $G[V - A]$ is 2-connected by Proposition 2.3. Thus $V - A$ intersects only one component of $G - \{a, v\}$, otherwise v would be a cut-vertex in $G[V - A]$. Now $A - a$ contains at least one component of $G - \{a, v\}$ (which contains a 3-end of G), contradicting the minimality of A.

Now assume that $|V - A| \leq 2$. By the minimality of A, it cannot contain any components of $G - \{a, v\}$. Thus $V - A$ consists of two vertices from the two component of $G - \{a, v\}$. However, this contradicts the fact that $\mathcal{H}_G[V - A]$ is tight, because every trivial component of $\mathcal{H}_G$ is also an edge of G. □

Lemma 4.5. *Let $G = (V, E)$ be a rigid graph which is not 3-connected and let $\mathcal{H}_G = (V, \mathcal{E})$ be its M-component hypergraph. Let C and L be two distinct atoms of G such that C is an MCT set of $\mathcal{H}_G$ and L is a 3-end of G. Then there is no M-component of G which has a vertex set intersecting both $C - L$ and L.*

Proof. For the sake of a contradiction, suppose that there exists an M-component of G with vertex set M such that $M \cap L \neq \emptyset$ and $M \cap (C - L) \neq \emptyset$. By Lemma 4.4, $|C \cap N_G(L)| = 0$ thus this M-component cannot be trivial. Conseqently, $G[M]$ is M-connected and hence redundantly rigid and thus 2-connected. Therefore, $N_G(L) \subset M$. $|\widehat{\mathcal{E}}(C - M)| \leq |\widehat{\mathcal{E}}(C)| - (2|M| - 3) = 2|C| - (2|M| - 3) \leq 2|C| - (2|C \cap M| + 2|N_G(L)| - 3) < 2|C - M|$, where the second inequality comes from $|C \cap N_G(L)| = 0$ by Lemma 4.4. As $|C - M| < |C| \leq |V| - 2$, $|\widehat{\mathcal{E}}(C - M)| < 2|C - M|$ is a contradiction by our previous observation that $|\widehat{\mathcal{E}}(X)| \geq 2|X|$ holds for each $X \subset V$ with $|X| \leq |V| - 2$. □

Proof of Lemma 4.3. Let $\mathcal{C}^*$ denote the family of MCT sets of $\mathcal{H}_G$ and let $\mathcal{L}^*$ denote the family of 3-ends of G. By Lemma 2.18, the members of $\mathcal{L}^*$ are pairwise disjoint.

Suppose that $C \in \mathcal{C}^* \cap \mathcal{A}^*$ and $L \in \mathcal{L}^* \cap \mathcal{A}^*$. By Lemma 4.5, $d_{\mathcal{H}_G}(C \cap L, C - L) = 0$. Then, by Observation 3, either $C \cap L = \emptyset$ or $C \cap L$ is co-tight in $\mathcal{H}_G$ contradicting the minimality of C.

Suppose now that there exist two distinct intersecting sets $C_1, C_2 \in \mathcal{C}^* \cap \mathcal{A}^*$. By Lemma 2.10, $|C_1 \cup C_2| \geq |V| - 1$ contradicting Lemma 4.4 as G is not 3-connected. □

Now, we turn to prove that there exists a set of $\max\left\{b(G) - 1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$ edges that augments $\mathcal{H}_G$ to a $(2, 3)$-redundant hypergraph and G to a 3-connected graph. A set X is called a **transversal** of a family $\mathcal{S}$ if $|X \cap S| = 1$ for each $S \in \mathcal{S}$ and $|X| = |\mathcal{S}|$. Let P be a transversal of $\mathcal{A}^*$. As the members of $\mathcal{A}^*$ are pairwise disjoint if G is not 3-connected by Lemma 4.3, choosing one arbitrary vertex from every $A \in \mathcal{A}^*$ obtains a transversal. Observe that P is a minimum cardinality vertex set that intersects all MCT sets and 3-ends, and consequently all co-tight sets and 3-fragments. Hence $|\mathcal{A}| \leq |P|$ holds for an arbitrary family $\mathcal{A}$ of disjoint co-tight sets and 3-fragments. We shall show now that a connected graph on P augments G to a 3-connected graph and $\mathcal{H}_G$ to a $(2, 3)$-redundant hypergraph. Later, we will reduce the number of edges needed for this augmentation to the optimum value.

Lemma 4.6. *Suppose that G is a rigid graph which is not 3-connected. Let P be a transversal of $\mathcal{A}^*$. Then, for any connected graph $H = (P, F)$ on P, $G + F$ is 3-connected.*

Proof. G is 2-connected by Proposition 2.3. Also, P contains no member of any cut-pair by Lemma 4.4. If there exists a cut-pair in $G + F$, then in one of its components there is no vertex from P, but P intersects all 3-ends and this component is the union of some 3-fragments of G which must contain a 3-end and hence an atom, a contradiction to the choice of P. ∎

To show that $\mathcal{H}_G$ and a connected graph on P results a $(2,3)$-redundant hypergraph, we extend the ideas of the proof of Theorem 2.15 from [27].

LEMMA 4.7. *Let $G = (V,E)$ be a rigid graph which is not 3-connected and let $\mathcal{H}_G = (V,\mathcal{E})$ be its M-component hypergraph. Let A, B be two atoms such that A is an MCT set of $\mathcal{H}_G$. Then $A \cap N_{\mathcal{H}_G}(B) = \emptyset$.*

Proof. Recall that A and B are disjoint by Lemma 4.3. Since G is not 3-connected, $|V-(A\cup B)| \geq 2$ by Lemma 4.4. Thus if both of A and B are MCT sets, then the statement follows by Lemma 2.13.

Suppose that B is a 3-end. By Lemma 4.3 $A - B = A$ hence Lemma 4.5 implies $A \cap N_{\mathcal{H}_G}(B) = \emptyset$. ∎

Lemma 4.7 and the fact that 3-ends are not connected in G immediately imply the following.

OBSERVATION 4. *The vertex set P induces no edge in G.*

Recall that $\mathcal{R}_{\mathcal{H}_G}(F)$ denotes the set of redundant hyperedges of $\mathcal{H}_G$ in $\mathcal{H}_G + F$. The following lemma and its proof is a direct extension of [27, Lemma 5.8].

LEMMA 4.8. *Suppose that G is a rigid graph which is not 3-connected and $\mathcal{H}_G$ is its M-component hypergraph. Let $\mathcal{A}^*$ be the set of atoms of G and let P be a transversal of $\mathcal{A}^*$. Let F be an edge set of a connected graph on $P' \subseteq P$. Then $\mathcal{R}_{\mathcal{H}_G}(F)$ is the minimal tight subhypergraph inducing all elements of P'. In particular, if F is the edge set of a star $K_{1,|P|-1}$ on the vertex set P, then $\mathcal{H}_G + F$ is $(2,3)$-redundant.*

Proof. Recall that $\mathcal{R}_{\mathcal{H}_G}(F) = \bigcup_{f\in F} \mathcal{T}_{\mathcal{H}_G}(f)$ by Lemma 2.8. Let us use induction on $|F|$. If $F = \{ij\}$, then $\mathcal{R}_{\mathcal{H}_G}(F) = \mathcal{T}_{\mathcal{H}_G}(ij)$ which is the minimal tight subhypergraph of $\mathcal{H}_G$ containing both of i and j by Lemma 2.1.

CLAIM 4.9. *For each $p \in P$ there exists a set D_p such that $D_p \subset V(\mathcal{T}_{\mathcal{H}_G}(pq))$ with $|D_p| \geq 2$ for all $q \in P - p$.*

Proof. Let $A, B \in \mathcal{A}^*$ such that $p \in A$ and $q \in B$. We claim that $D_p := N_G(A)$ is a suitable set. By Proposition 2.3, $|D_p| \geq 2$. If A is an MCT set of $\mathcal{H}_G$, then Lemmas 4.3 and 4.7 imply that $(A \cup N_{\mathcal{H}_G}(A)) \cap B = \emptyset$. Hence, by the definition of $\mathcal{H}_G$ and Lemma 2.14, $A \cup N_G(A) \subseteq A \cup N_{\mathcal{H}_G}(A) \subset V(\mathcal{T}_{\mathcal{H}_G}(pq))$, and thus $D_p \subset V(\mathcal{T}_{\mathcal{H}_G}(pq))$. If A is a 3-end, then each $q \in P-p$ is an element of $V-(A\cup N_G(A))$ by Lemmas 4.3 and 4.4. Now the tightness of $\mathcal{T}_{\mathcal{H}_G}(pq)$ and the definition of $\mathcal{H}_G$ imply that $G[V(\mathcal{T}_{\mathcal{H}_G}(pq))]$ is rigid and hence 2-connected by Proposition 2.3. Since p and q are from different connected components of $G - N_G(A)$, $D_p = N_G(A) \subset V(\mathcal{T}_{\mathcal{H}_G}(pq))$ follows. ∎

Let $ij \in F$ such that $F - ij$ is connected. By induction, $\mathcal{R}_{\mathcal{H}_G}(F-ij)$ is a tight subhypergraph of $\mathcal{H}_G$ which induces each element of $V(\mathcal{R}_{\mathcal{H}_G}(F-ij))$, in particular, we may assume (by possibly switching the role of i and j) that $i \in V(\mathcal{R}_{\mathcal{H}_G}(F-ij))$. If $j \in V(\mathcal{R}_{\mathcal{H}_G}(F-ij))$ also holds, then $\mathcal{T}_{\mathcal{H}_G}(ij) \subseteq \mathcal{R}_{\mathcal{H}_G}(F-ij)$ by Lemma 2.1. Hence we may assume that $j \notin V(\mathcal{R}_{\mathcal{H}_G}(F-ij))$. The connectivity of $F - ij$ implies that there exists an edge $ij' \in F - ij$. Note that $\mathcal{T}_{\mathcal{H}_G}(ij') \subseteq \mathcal{R}_{\mathcal{H}_G}(F-ij)$ by Lemma 2.8. Hence $D_i \subset V(\mathcal{T}_{\mathcal{H}_G}(ij')) \subseteq V(\mathcal{R}_{\mathcal{H}_G}(F-ij))$ and $D_i \subset V(\mathcal{T}_{\mathcal{H}_G}(ij))$ by Claim 4.9. Thus we may use Lemmas 2.2 and 2.8 to conclude that $\mathcal{R}_{\mathcal{H}_G}(F) = \mathcal{R}_{\mathcal{H}_G}(F-ij) \cup \mathcal{T}_{\mathcal{H}_G}(ij)$ is tight.

Let now $\mathcal{T}$ be the minimal tight subhypergraph of $\mathcal{H}_G$ which induces all elements of P'. Lemma 2.1 imply that $\mathcal{T}_{\mathcal{H}_G}(f) \subseteq \mathcal{T}$ for each $f \in F$. Hence it follows by

Lemma 2.8 that $\mathcal{R}_{\mathcal{H}_G}(F) = \bigcup_{f\in F} \mathcal{T}_{\mathcal{H}_G}(f) \subseteq \mathcal{T}$, that is, $\mathcal{R}_{\mathcal{H}_G}(F) = \mathcal{T}$.

Finally, if $P' = P$, then $P \subset V(\mathcal{R}_{\mathcal{H}_G}(F))$ and thus $\mathcal{R}_{\mathcal{H}_G}(F) = \mathcal{H}_G$ by Lemma 2.12 since P intersects every MCT set. ∎

Now we show how the cardinality of the augmenting edge set provided by the above lemmas can be reduced to the optimum. By a direct extension of [27, Lemma 5.9] and its proof, we get the following.

LEMMA 4.10. *Let $G = (V, E)$ be a not 3-connected rigid graph with M-component hypergraph $\mathcal{H}_G$. Let $\mathcal{A}^*$ be the set of atoms of G and let P be a transversal of $\mathcal{A}^*$. Suppose that $x_1, x_2, x_3, y \in P$ are distinct vertices. Let $\mathcal{T}^* = \mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2y) \cup \mathcal{T}_{\mathcal{H}_G}(x_3y)$. Then $\mathcal{T}^* = \mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3)$ or $\mathcal{T}^* = \mathcal{T}_{\mathcal{H}_G}(x_2y) \cup \mathcal{T}_{\mathcal{H}_G}(x_1x_3)$ holds.*

Proof. Let $\mathcal{T}^* = (V^*, \mathcal{E}^*)$. Let us suppose that $\mathcal{T}^* \neq \mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3)$. Thus there exists a hyperedge e, for which $e \in \mathcal{E}^*$ and $e \notin \mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3)$.

Lemmas 2.8 and 4.8 imply that $\mathcal{T}^*$ is the minimal tight subhypergraph of G inducing all of $x_1,\ x_2,\ x_3$ and y. However, they similarly imply that this statement also holds for $\mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3) \cup \mathcal{T}_{\mathcal{H}_G}(x_3y)$ and $\mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3) \cup \mathcal{T}_{\mathcal{H}_G}(x_1x_2)$, that is, these two hypergraphs both are equal to $\mathcal{T}^*$. Since $e \in \mathcal{T}^*$ and $e \notin \mathcal{T}_{\mathcal{H}_G}(x_1y) \cup \mathcal{T}_{\mathcal{H}_G}(x_2x_3)$, we get $e \in \mathcal{T}_{\mathcal{H}_G}(x_3y)$ and $e \in \mathcal{T}_{\mathcal{H}_G}(x_1x_2)$.

Now Lemma 2.2 implies that $\mathcal{T}_{\mathcal{H}_G}(x_3y) \cup \mathcal{T}_{\mathcal{H}_G}(x_1x_2)$ is a tight subhypergraph of G (and also of $\mathcal{T}^*$) inducing all of $x_1,\ x_2,\ x_3$ and y, hence it must be equal to $\mathcal{T}^*$. ∎

Observe that the operation in Lemma 4.10 allows us to reduce the cardinality of the edge set used for the augmentation by maintaining the property that it augments $\mathcal{H}_G$ to a $(2,3)$-redundant hypergraph (and hence G to a redundantly rigid graph by Lemma 3.5). However, we also need to maintain the 3-connectivity of $G + F$ to complete the proof of Theorem 4.1.

Proof of Theorem 4.1 for $(k, \ell) = (2, 3)$. As we have seen at the beginning of this section, we only need to prove the min $\leq$ max part of Theorem 4.1 and only for the case where G is not 3-connected. In this case, the atoms of G (denoted by $\mathcal{A}^*$) are pairwise disjoint by Lemma 4.3 and a tree on a transversal P of $\mathcal{A}^*$ augments G to a globally rigid graph with $|\mathcal{A}^*| - 1$ edges by Lemmas 3.5, 4.6 and 4.8. Note that, as $\mathcal{A}^*$ consists of pairwise disjoint MCT sets of the M-component hypergraph$\mathcal{H}_G$ of G and 3-ends of G, the maximum in Theorem 4.1 is at least max $\left\{b(G) - 1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$, furthermore, this latter value equals to $|\mathcal{A}^*| - 1$ when $|\mathcal{A}^*| \leq 3$ completing our proof for this case.

To reduce the number of edges needed for the augmentation, we do the following procedure. Let us define a vertex set $N \subseteq P$. The set N stands for "not fixed" vertices while vertices in $P - N$ are the "fixed" vertices. We can **fix** an edge xy by removing x and y from N and adding xy to F.

We shall keep some properties during the whole procedure:

1. For an arbitrary star S_N on the vertex set N, $\mathcal{H}_G{+}F{+}S_N$ is a $(2,3)$-redundant hypergraph.
2. In every 3-end of $G + F$, there is at least one vertex from N.
3. $\max\left\{b(G+F) - 1, \left\lceil \frac{|N|}{2} \right\rceil\right\} + |F| = \max\left\{b(G) - 1, \left\lceil \frac{|P|}{2} \right\rceil\right\}$.

Notice that Properties 1–3 hold for $N = P$ and $F = \emptyset$ by Lemmas 4.6 and 4.8.

Remark 4.11. Properties 2 and 1 ensure that $G + F + S_N$ is 3-connected and $\mathcal{H}_G + F + S_N$ is $(2,3)$-redundant and thus $G + F + S_N$ is redundantly rigid by Lemma 3.5.

Remark 4.12. If $|N| \geq 4$, then from any two edges chosen on $x_1, x_2, x_3 \in N$ one may fix at least one of them (by Lemma 4.10) in such a way that this fixing maintains Property 1.

By Remark 4.12 we always aim to find at least two possibilities to fix such that Property 2 is maintained. Also, if it can be done in such a way that $\max\left\{b(G + F) - 1, \left\lceil \frac{|N|}{2} \right\rceil\right\}$ decreases by one, then we can maintain Properties 1–3. Roughly, we distinguish 4 different possibilities in each of which we find 3 vertices from N such that we can apply Remark 4.12 and hence we can fix one edge while maintaining Properties 1–3.

Lemma 4.13. *Let G be a not 3-connected rigid graph with M-component hypergraph $\mathcal{H}_G$. Let $\mathcal{A}^*$ denote the atoms of G. Assume that $|\mathcal{A}^*| \geq 4$. Let P be a transversal on $\mathcal{A}^*$. Let $N \subseteq P$ be a vertex set and F be an edge set on P such that G, N and P satisfy Properties 1–3. If $|N| \geq \max\{4, b(G + F) + 1\}$, then we can choose $x, y \in N$, such that for $N - \{x, y\}$ and $F + \{xy\}$ (that is, for fixing xy) Properties 1–3 also hold.*

Proof. We use the following method for the proof. Notice, that this can be turned into a polynomial time algorithm.

1 **If** $b(G + F) - 1 \geq \left\lceil \frac{|N|}{2} \right\rceil$, **then**
2   **If** there is only one cut-pair (u, v) such that $b_{(u,v)}(G + F) = b(G + F)$, **then**
        Choose x_1, x_2 from a component of $G + F - \{u, v\}$ that contains at least two vertices from N. Let $x_3 \in N$ be a vertex from a component of $G + F - \{u, v\}$ that does not contain x_1 and x_2.
3   **else**
        Let (u_1, v_1) and (u_2, v_2) be two cut-pairs for which $b_{(u_1,v_1)}(G + F) = b(G + F) = b_{(u_2,v_2)}(G + F)$. Choose $x_1, x_2 \in N$ from two different components of $G + F - \{u_1, v_1\}$ that do not contain $\{u_2, v_2\}$. Choose $x_3 \in N$ from a component of $G + F - \{u_2, v_2\}$ that does not contain $\{u_1, v_1\}$.
4 **else**
5   **If** there is a 3-fragment K of G such that $|N \cap K| \geq 2$ and $|N - K| \geq 2$, **then**
        Choose x_1, x_2 from $N \cap K$ and choose x_3 from $N - K$.
6   **else**   (Notice that if $b(G + F) = 1$, then this is the only possible case.)
        Choose $x_1, x_2, x_3 \in N$ arbitrarily.
7 **If** $\mathcal{H}_G + F + S(N - \{x_1, x_3\}) + x_1x_3$ is $(2, 3)$-redundant, **then**
      $x := x_1, y := x_3$.
  else
      $x := x_2, y := x_3$.

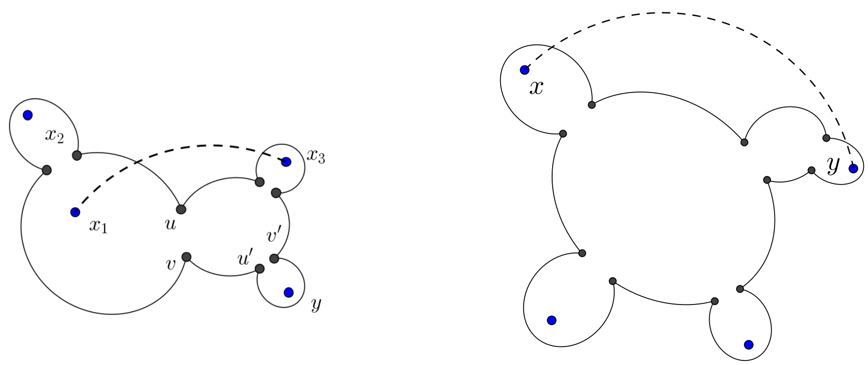
First we prove that the method above is consistent, that is, we can execute each of its steps. As $|N| \geq b(G + F) + 1$ and P contains no vertex from a cut-pair of G by Lemma 4.4, $|N| > b_{(u,v)}(G + F)$ for an arbitrary cut-pair $\{u, v\}$ hence there exists a component of $G + F - \{u, v\}$ that contains at least two vertices from N. This shows that we can choose vertices in Step **2** consistently. In Step **3** there are at least two components of $G + F - \{u_1, v_1\}$ that do not contain $\{u_2, v_2\}$ since $|N| \geq 4$ and thus $b_{(u_1,v_1)}(G + F) \geq 3$. The consistency of Steps **5** and **6** is obvious.

Now let us show that the choice of x and y maintains Property 2.

Claim 4.14. *Suppose that there is a cut-pair $\{u, v\}$ such that for one component of $G - \{u, v\}$, say K, $x_1, x_2 \in N \cap K$ and $x_3, y \in (V - K) \cap N$. Then fixing either*

x_1x_3 or x_2x_3 maintains Property 2.

Proof. Notice that the role of x_1 and x_2 is symmetric thus we might suppose that we fixed the edge x_1x_3. Suppose that we form a new 3-end L with it in $G+F$. Then necessarily $x_1, x_3 \in L$. If $x_2 \in L$ or $y \in L$, then Property 2 holds automatically. On the other hand, if none of them is in L, then, as the cut-pair $\{u, v\}$ is strong (since all the cut-pairs are strong by Lemma 2.17) there is a cut-pair of G in $K \cup \{u\}$ or in $K \cup \{v\}$ which separates x_1 from x_2 (see Fig. 2a). There is another cut-pair $\{u', v'\}$ in $V - K$ (other than $\{u, v\}$) which separates x_3 from y. Both remain cut-pairs after fixing the edge x_1x_3. However, this contradicts the assumption that L is 3-end, as $|N_G(L)| = 2$ must hold for a 3-end. ◻



(a) Illustration of Claim 4.14. Notice, that we need the existence of the vertex y.

(b) In case of Step **6** we cannot form a new 3-end.

Fig. 2: Proofs why the algorithm of Lemma 4.13 maintains Property 2.

Notice, that the conditions of this claim hold in Steps **2**, **3** and **5** thus with our choice of x_1, x_2, and x_3 Property 2 is maintained. If $G+F$ is already 3-connected, then Property 2 is obvious. Otherwise, in Step **6**, every cut-pair cuts $G+F$ into two components one of which contains exactly one vertex from N by the condition of Step **5** (see Fig. 2b). For the sake of a contradiction, assume that $G+F+xy$ contains a 3-end L which contains no element of $N - \{x, y\}$. Let $N_G(L) = \{u, v\}$. Then $N \cap L = \{x, y\}$, $V - L - \{u, v\} \neq \emptyset$, and u, v is a cut pair of $G+F$. By the condition of Step **5**, (u, v) cuts $G+F$ into two component one of which contains exactly one vertex from N. Hence exactly L and $V - L - \{u, v\}$ are these two components. Moreover, as $|L \cap N| = 2$, this implies $|N \cap (V - L - \{u, v\})| = 1$, contradicting $|N| \geq 4$.

Now we show that our method maintains Property 3. Fixing any edge decreases $\left\lceil \frac{|N|}{2} \right\rceil$ by one while increases F by one. When we chose x_1, x_2 and x_3 in Steps **5** or **6**, this fact is enough to keep Property 3 true as in these cases $\max\left\{b(G+F) - 1, \left\lceil \frac{|N|}{2} \right\rceil\right\} > b(G+F) - 1$. We need to show that if the condition in Step **1** is true, then we also decrease $b(G+F)$. By a simple calculation on the number of 3-ends, it can be shown that if $b(G+F) - 1 \geq \left\lceil \frac{|N|}{2} \right\rceil$, then there are at most two cut-pairs of $G+F$ satisfying $b_{(u,v)}(G+F) = b(G+F)$ (see [22]). If there is only one such cut-pair, the pair (u,v) chosen in Step **2**, then we only need to decrease $b_{(u,v)}(G+F)$ to decrease $b(G+F)$. Since x_1x_3 and x_2x_3 both connect two different components of $G+F-\{u,v\}$, $b_{(u,v)}(G+F)$ decreases by one after fixing any of them. If there are at least two such cut-pairs, then there are exactly two of them (see for example [22, Lemma 2.3]). Let now (u_1,v_1) and (u_2,v_2) be chosen in Step **3**, then we need to decrease $b_{(u_1,v_1)}(G+F)$ and $b_{(u_2,v_2)}(G+F)$ simultaneously. Again our choice of x_1x_3 and x_2x_3 guarantees this.

Therefore, by Remark 4.12 applied to Step **7**, fixing xy maintains Properties 1–3. This completes the proof of Lemma 4.13. ∎

We apply Lemma 4.13 recursively until $|N| < \max\{4, b(G+F)+1\}$. To complete the proof of Theorem 4.1, we need to show the following.

Claim 4.15. *Let F, N be sets, such that they satisfy Properties 1–3 with G. If $2 \le |N| \le \max\{3, b(G+F)\}$, then, for an arbitrary star S_N on N, $G+F+S_N$ forms a 3-connected redundantly rigid graph for which $|F|+|S_N| = \max\left\{b(G)-1, \left\lceil \frac{|P|}{2} \right\rceil\right\}$.*

Proof. $G+F+S_N$ is 3-connected and redundantly rigid by Remark 4.11. By Property 3 it is enough to show that $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\} = |S_N| = |N|-1$. If $|N| = b(G+F)$, then $\max\left\{b(G+F)-1, \left\lceil \frac{|N|}{2} \right\rceil\right\} = |N|-1$ as $\left\lceil \frac{|N|}{2} \right\rceil \le |N|-1$. On the other hand, if $|N| < b(G+F)$, then $2 \le |N| \le 3$ thus $\left\lceil \frac{|N|}{2} \right\rceil = |N|-1$. ∎

Recall that $\mathcal{A}^*$ consists of pairwise disjoint MCT sets and 3-ends of G and hence the maximum in Theorem 4.1 is at least $\max\left\{b(G)-1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$. On the other hand, the above claim implies that G can be augmented to a globally rigid graph by an addition of an edge set of cardinality $\max\left\{b(G)-1, \left\lceil \frac{|P|}{2} \right\rceil\right\} = \max\left\{b(G)-1, \left\lceil \frac{|\mathcal{A}^*|}{2} \right\rceil\right\}$. This completes the proof of Theorem 4.1. ∎

Observation 5. *The method in Lemma 4.13 adds edges only between vertices from P. This means that $G+F$ is a simple graph by our assumption on G and Observation 4. Thus $G+F$ is globally rigid in $\mathbb{R}^2$ by Theorem 2.5.*

Before proving Theorem 4.1 for the cases other than $(k,\ell) = (2,3)$, let us follow our proof on the graph G in Fig. 3 to find an optimal solution for Problem 2 when $(k,\ell) = (2,3)$. Note that the 3-ends and the atoms of G do not depend on the form of the inner M-connected graph G_0, however, $b(G)$ and hence the size of the optimal solution of Problem 2 may do. For example, when $G_0 = K_{12}$ is the complete graph on 12 vertices, then $b(G) = 2$. In this case, the optimal solution has four edges by Theorem 4.1. Indeed, we need at least four edges for the augmentation as we need to touch each atom of G by Proposition 2.9 and Lemmas 2.16, 3.2 and 3.5. On the other hand, we know that any connected graph on a transversal of the atoms (for example, on the set N of the vertices represented by (red) triangles) augments G to a globally rigid graph by Lemmas 4.6 and 4.8. We start to run the algorithm of Lemma 4.13. As $b(G) = 2$, the condition of Step **1** does not hold hence the algorithm checks the condition of Step **5** which holds for any 3-fragment of the cut-pair $\{u,v\}$. Hence the algorithm may choose x_1, x_2 and x_3, as drawn in Fig. 3 and after that it adds the edge x_1x_3 in Step **7**. Now, the condition of Step **5** does not hold for $G+x_1x_3$, and hence in the next step the algorithm takes three arbitrary vertices from $N-\{x_1,x_3\}$ and uses Step **7** of the algorithm to find the next augmenting edge, for example, x_2a. This way the number of non-covered elements of N reduces to three, and hence the algorithm stops and extends the augmenting edge set with a star on the remaining

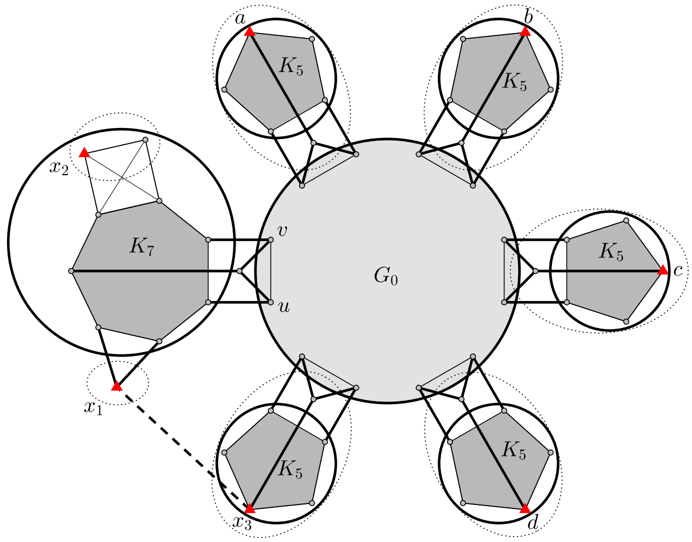


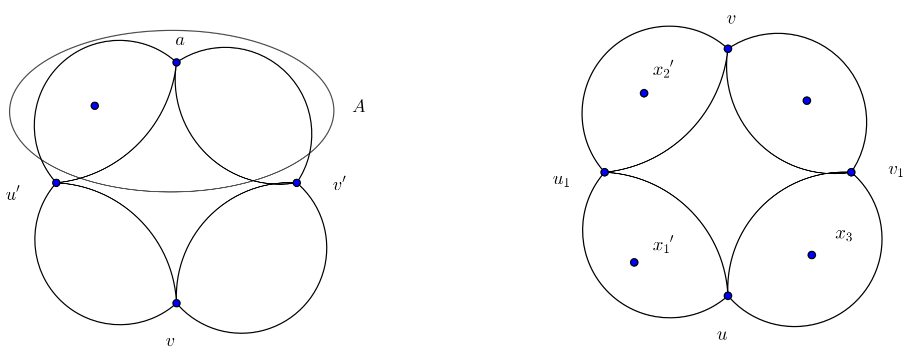
Fig. 3: A $(2,3)$-rigid graph G with its $(2,3)$-M-components (encircled with solid circles) where the graph G_0 in the light gray area is an arbitrary $(2,3)$-M-connected graph on 12 vertices and the dark grey areas are complete graphs on the drawn vertex sets. The 3-ends of this graphs are the dotted sets (since G_0 cannot contain any 3-ends as each of its vertices is contained in a cut-pair of G). The $(2,3)$-MCT sets of the $(2,3)$-M-component hypergraph are the vertex sets of the five K_5 subgraphs and the singleton formed by the vertex x_1 of degree two. Hence the atoms are the vertex sets of the K_5 subgraphs and the two dotted sets which are not containing any K_5 subgraph. These are disjoint as claimed by Lemma 4.3 and no edge of the graph connects them as stated in Lemma 4.7. The vertices, which are represented by (red) triangles, form a transversal of the atoms. The addition of the dashed edge represents the first step of the algorithm of Lemma 4.13 for several choices of G_0.

three vertices by Claim 4.15, for example, it may add bc and cd. Thus, the resulted (optimal) augmenting edge set $\{x_1x_3, x_2a, bc, cd\}$ has cardinality four.

In our second example, let G_0 be the graph which contains the 6 edges drawn in Fig. 3 (between the elements of each cut-pair which separates other parts of G from G_0) and the edges from u and v to each other vertex of G_0, that is, let G_0 be the drawn matching plus the complete bipartite graph $K_{2,10}$ where the two element set of the bipartition is $\{u,v\}$. In this case, $b(G) = b_{(u,v)}(G) = 6$ and hence the optimal solution has five edges by Theorem 4.1. Indeed, we need at least five edges to make G 3-connected, as $G-\{u,v\}$ has six connected components. On the other hand, similarly to the previous example, we know that any connected graph on the set transversal N of the atoms which is formed by the vertices represented by (red) triangles augments G to a globally rigid graph and we may reduce its cardinality (which is at least seven) by running the algorithm of Lemma 4.13. Now, the condition of Step **1** of the algorithm holds and the algorithm may choose x_1, x_2 and x_3 as drawn in Fig. 3 in Step **2**. Next, it takes the augmenting edge x_1x_3 in Step **7**. Now, $b(G + x_1x_3) = 5$ and we have 5 vertices in our transversal set which are not covered by an augmenting edge. Hence the condition of Lemma 4.13 does not hold any more, and the algorithm stops. Now, Claim 4.15 states that x_1x_3 and a star on $N - \{x_1, x_3\}$ form an optimal augmenting edge set (for example, $\{x_1x_3, x_2a, ab, ac, ad\}$) of cardinality five.

4.2. Proof sketch of Theorem 4.1 for $k < \ell \leq \frac{3}{2}k$. In this subsection we briefly sketch how the proof of Theorem 4.1 for $(k, \ell) = (2, 3)$ presented before can be extended for general $(k, \ell) \neq (2, 3)$ where $k < \ell \leq \frac{3}{2}k$. In this case we still want to augment G to a 3-connected graph. The bulk of the proof can be transferred literally, however, there are two main differences caused by the weak cut-pairs. This is due to the fact that Lemma 2.17 does not extend for general (k, ℓ), there may be weak cut-pairs that pose a challenge.

The first issue is in the proof of the extension of Lemma 4.4 for general (k, ℓ). When the atom A is a 3-end we used Lemmas 2.17 and 2.18 in the proof to conclude that it cannot contain any vertex a which forms a cut-pair with another vertex v. In the general case, $\{a, v\}$ may be a weak cut-pair which separates the two vertices of $N(A) = \{u', v'\}$. In this case a is a cut vertex of $G[A \cup N(A)]$ that separates u' and v'. Moreover, $G[A \cup N(A)] - a$ has exactly two components since otherwise a would be a cut vertex of G (see Fig. 4a for an illustration). Note that $|A| \geq 2$ must hold since G is a simple (k, ℓ)-rigid graph in which each vertex has a degree of at least k that is at least 3 by our assumptions on (k, ℓ). Thus one of the two connected components in $G[A \cup N(A)] - a$, say the component U' containing u' has cardinality at least two. Now $N_G(U' - u') = \{u', a\}$, and hence $U' - u' \subsetneq A$ is a 3-fragment of G, contradicting the fact that A is a 3-end. Hence we proved the statement if A is a 3-end. The rest of the proof (that is, when A is a (k, ℓ)-MCT set) can be generalized easily.



(a) If the 3-end A contains an element a of a cut-pair, then we obtain a smaller 3-end which is a contradiction.

(b) All path from x_2' to u which avoids v must induce u_1 hence $u \notin A_2'$ in the proof of Claim 4.16.

Fig. 4: Extension of the proof in Section 4.1 to the case where $k < \ell \leq \frac{3}{2}k$.

The second issue appears in the proof of Lemma 4.13 since we used Lemma 2.17 for the proof of Claim 4.14. Note that for a weak pair $\{u', v'\}$, $b^2_{(u',v')}(G + F) = 2$ hence weak pairs can occur only in Step **5**. Hence we still can use Claim 4.14 to prove that Property 2 for Steps **2** and **3** as the cut-pair $\{u, v\}$ is strong in those cases. However, our choice in Step **5** may destroy Property 2. Hence we need to modify this step in the general case, as follows.

5' **If** there is a 3-fragment K of G such that $|N \cap K| \geq 2$ and $|N - K| \geq 2$, **then**
Choose x_1', x_2' from $N \cap K$ and choose x_3 from $N - K$.
If every 3-end of $G + F + x_1'x_3$ contains a vertex from $N - \{x_1', x_3\}$,
then let $x_1 = x_2 := x_1'$,
else let $x_1 = x_2 := x_2'$.

CLAIM 4.16. *If $x_1 = x_2$ and x_3 is chosen by* STEP **5'**, *then Properties 1 – 3 are maintained after fixing the edge x_1x_3.*

Proof. Let $\{u, v\}$ be the cut-pair for which K is a component of $G + F - \{u, v\}$. To see that Property 1 holds, observe that $\{u, v\}$ separates $x_1 = x_2$ and x_3 and it also separates the vertices of $N - \{x_1, x_3\}$ by the condition in STEP **5'**. This implies that the star $S_{N-\{x_1,x_3\}}$ has an edge wz connecting two distinct components of $G - \{u, v\}$. Now $\mathcal{T}_{\mathcal{H}_G}(wz)$ and $\mathcal{T}_{\mathcal{H}_G}(x_1x_3)$ are (k, ℓ)-tight subhypergraphs of $\mathcal{H}_G$ (on at least 3 vertices) and hence their vertex sets induce (k, ℓ)-rigid subgraphs of G (by the definition of the M-component hypergraph) which are 2-connected by Proposition 2.3. This implies that they both contain u and v. Hence Lemma 2.2 implies that $\mathcal{T}_{\mathcal{H}_G}(wz) \cup \mathcal{T}_{\mathcal{H}_G}(x_1x_3)$ is $(k, \ell)-$tight and hence $\mathcal{T}_{\mathcal{H}_G}(wx_1) \subseteq \mathcal{T}_{\mathcal{H}_G}(wz) \cup \mathcal{T}_{\mathcal{H}_G}(x_1x_3)$ by Lemma 2.1. This with Lemmas 2.8 and 4.8 implies that $\mathcal{R}_{\mathcal{H}_G}(S_{N-\{x_1,x_3\}} \cup x_1x_3) = \mathcal{R}_{\mathcal{H}_G}(S_{N-\{x_1,x_3\}} \cup \{x_1x_3, wx_1\}) = \mathcal{R}_{\mathcal{H}_G}(S_N)$ and hence Property 1 remains true.

If neither the fixing of $x_1'x_3$ nor the fixing of $x_2'x_3$ maintains Property 2, then it means that there is a 3-end with vertex set A_i in $G + F + x_i'x_3$ such that A_i contains no vertex from $N - \{x_i', x_3\}$ for $i = 1, 2$. Let $N_{G+F}(A_i) = \{u_i, v_i\}$ for $i = 1, 2$. Now, $N \cap A_i = \{x_i', x_3\}$ and $\{u, v\}$ (chosen in STEP **5'**) separates $\{u_i, v_i\}$ in $G + F$, as it separates x_i' and x_3 for $i = 1, 2$. This also means that x_i' is separated from any other vertex of N by, say, $\{u, u_i\}$ or $\{v, u_i\}$ since $K \cup \{u, v\}$ contains either u_i or v_i and this vertex (say, u_i) is a cut vertex in $(G + F)[K \cup \{u, v\}]$. Let us denote the vertex set of the corresponding component of $G - \{u, u_i\}$ or $G - \{v, u_i\}$ that contains only x_i' from N by A_i' for $i = 1, 2$. Without loss of generality, we may assume that x_1' is separated from any other vertex of N by $\{u, u_1\}$. Now, a similar argument and the existence of the 3-end A_1 in $G + F + x_1'x_3$ implies that x_3 is separated from any other vertex of N by $\{u, v_1\}$. Furthermore, all paths in $G[K \cup \{u, v\}]$ from x_2 to u contain u_1 and hence A_2' cannot contain u since otherwise it should also contain u_1 and hence, by the connectivity of $G[K]$, all vertices from A_1' (in particular, x_1') contradicting that it contains only x_2' from N (see Fig. 4b for an illustration). Hence, the the existence of the 3-end A_2 in $G + F + x_2'x_3$ implies that x_3 is separated from any other vertex of N by $\{v, v_2\}$. However, in this case, v_1 and all the components of $G[V - K] - v_1$ other than A_2 must be in the component of $G[V - K] - v_2$ containing x_3 and v, and hence it must contain all the vertices in $N - K$, a contradiction.

After STEP **5'** $\left\lceil \frac{|N|}{2} \right\rceil$ decreased by 1 while $|F|$ increased by 1, and, as the condition in STEP **1** did not hold in this case, this is sufficient to maintain Property 3. ∎

With this modification on STEP **5** we can use the algorithm from Lemma 4.13 so that it results an optimal edge set for any $(k, \ell) \neq (2, 3)$ pair where $k < \ell \leq \frac{3}{2}k$.

4.3. Proof sketch of Theorem 4.1 for $\ell \leq k$. It is easy to see, how the results presented in Section 2 with some elementary observations can be used to prove Theorem 4.1 in the case where $\ell \leq 0$. (Notice that in this case $c_{k,\ell} = 0$, thus we aim to augment G to a (k, ℓ)-redundant and connected graph.) We leave the details of this rather simple special case to the reader and this enables us to assume in what follows that k and ℓ are positive integers. This simplifies the presentation of the results. Let us now briefly sketch, how the proof presented in Subsection 4.1 may be transferred to the values of $0 < \ell \le k$. (We note that similar methods may be used also for the case where $\ell \le 0$.) In this case $c_{k,\ell} = 1$ thus we aim to augment G to a 2-connected and (k,ℓ)-redundant graph. This means, that each 2-end is separated from G by a cut-vertex and thus cut-pairs in the proofs should be changed to cut-vertices. In fact, all our proofs can be extended (almost) literally hence we only reprove the counterpart of Lemma 4.4 as its statement is slightly modified in this case.

LEMMA 4.17. *Let k and ℓ be positive integers with $k \ge \ell$ and let $G = (V,E)$ be a (k,ℓ)-rigid graph which is not 2-connected and let $a \in A \in \mathcal{A}^*$ be a vertex from an atom of G. Then a is not a cut-vertex in A.*

Proof. If A is a 2-end, then the statement follows immediately by Lemma 2.18.

Now let A be a (k,ℓ)-MCT set of $\mathcal{H}_G$. Then $\mathcal{H}_G[V-A]$ is (k,ℓ)-tight and hence Observation 2 implies that $G[V-A]$ is (k,ℓ)-rigid and hence connected. For the sake of a contradiction, suppose that $a \in A$ is a cut-vertex of G. This immediately implies that $|A| \ge 2$ and $A - a$ contains at least one component of $G - a$ (which also contains a 2-end of G), contradicting the minimality of A. ◻

As the M-connected hypergraph of any (k,ℓ)-rigid graph [12, 29, 35], all the (k,ℓ)-MCT sets of a (k,ℓ)-tight hypergraph [27] and all the 2-ends of a connected graph and 3-ends of a 2-connected graph [9, 16, 22] can be computed in polynomial time, it is easy to see that the method presented in the proof of Theorem 4.1 yields a polynomial algorithm for finding the optimal edge set. By developing some further details, the running time of this algorithm can be reduced to $O(|V|^2)$ [26].

5. Concluding remarks. Theorem 4.1 leaves open the natural question, what can we do if G is not rigid. For general inputs, we give a 2-approximation, as follows.

As we saw in Section 2, the (k,ℓ)-sparse edge sets form the independent sets and the (k,ℓ)-tight sets form the bases of a matroid. Thus all the edge sets that optimally augment G to a rigid graph have the same cardinality. Also, such a set can be easily computed in polynomial time [12, 29]. Moreover, such a set can be chosen in such a way that no newly added edge is parallel to any original edge of G (if its vertex set is sufficiently large). Hence our algorithm consists of the following two parts: first we find a minimal cardinality edge set F_1 such that $G' = G + (V, F_1)$ is a (k,ℓ)-rigid graph (which is still simple if $k < \ell$), then using the algorithm presented in Section 4 we augment G' to a (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected graph with a new edge set F_2. We show that this result indeed has the approximation ratio of 2.

Any edge set F that augments G to a (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected graph must also augment G to a (k,ℓ)-rigid graph. Thus $|F| \ge |F_1|$ holds. On the other hand, if $G+F$ is (k,ℓ)-redundant and $(c_{k,\ell}+1)$-connected, then $G+F_1+F$ is also (k,ℓ)-redundant (since for each edge e $G+F+F_1-e$ contains the (k,ℓ)-tight spanning subgraph of $G+F-e$) and, obviously, $(c_{k,\ell}+1)$-connected. Hence $|F| \ge |F_2|$ follows.

Let us recall the global rigidity pinning problem. In this problem, the goal is to anchor a minimum set of points of a framework such that the resulting framework is globally rigid. We note that the complexity of this problem is open, only a 3-approximation algorithm was given by Fekete and Jordán [11] in the generic case for arbitrary input graphs. However, we can show that our method yields an optimal pinning set for rigid graphs and a 2-approximation for general graphs.

It is easy to see that pinning can be modeled by adding a complete graph on the anchored vertices to the graph (see [11]). Let $G = (V, E)$ be a $(2, 3)$-rigid (but not globally rigid) graph that we want to pin down to a globally rigid graph. If G can be augmented to a globally rigid graph by a single edge, then pinning down its endpoints results a globally rigid graph. Hence we may assume that no edge augments G to a globally rigid graph. It is clear that each 3-end of G needs to be pinned down to eliminate its cut-pairs. On the other hand, each (k, ℓ)-MCT set of $\mathcal{H}_G$ needs to be pinned down by Lemmas 2.1, 3.2 and 3.5. However, by Lemmas 2.11 and 4.3 all the atoms of G are pairwise disjoint (if no edge augments it to a globally rigid graph). Hence, we must pin down a vertex from each atom of G. By Lemmas 4.6 and 4.8 this pinning results a globally rigid graph and thus this is an optimal pinning. When G is not rigid, then we can follow the idea of the above approximation algorithm: First, pin G down to a rigid graph (which can be done optimally in polynomial time [10, 23]) and next pin this (already rigid graph) down to a globally rigid one. Similarly to the case of augmentation, it can be shown that the approximation ratio of this algorithm is 2.

Finally, we note that the pinning problem is also solvable in the case where we have some already pinned vertices. In this case the model is the following. We are given a graph $G = (V, E)$ and a set $V' \subseteq V$ of the already pinned vertices. We seek a set $P \subseteq V - V'$ of minimum cardinality for which $G \cup K_{P \cup V'}$ is globally rigid. When $G \cup K_{V'}$ is rigid, then this problem can be solved optimally since we only need to cover the atoms of $G \cup K_{V'}$ which do not contain any vertex from V'. On the other hand, when $G \cup K_{V'}$ is not rigid, we can also give a 2-approximation algorithm as above, since it is not hard to modify the algorithm of Fekete [10] in such a way that it outputs an minimum cardinality set $P_1 \subseteq V - V'$ for which $G \cup K_{P_1 \cup V'}$ is rigid.

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