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SUMMATIONS AND TRANSFORMATIONS FOR VERY WELL-POISED HYPERGEOMETRIC FUNCTIONS ${}_{2q+5}F_{2q+4}(1)$ AND ${}_{2q+7}F_{2q+6}(1)$ WITH ARBITRARY INTEGRAL PARAMETER DIFFERENCES

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Abstract. The present paper aims to derive summation and transformation formulae for the generalized very well-poised hypergeometric functions ${}_{2q+5}F_{2q+4}(1)$ and ${}_{2q+7}F_{2q+6}(1)$ having arbitrary integral parameter differences. These results are derived with the help of Bailey's transform and the extension of Saalschütz summation theorem for the series ${}_{r+3}F_{r+2}(1)$, where r pairs of parameters differ by positive integers. The particularizations of these generalized identities give classical summation theorems due to Dougall, transformation formula due to Whipple and, other related results. Furthermore, the application of ${}_{2q+5}F_{2q+4}(1)$ summation to the limiting case, when $q \to 1$, of one of the Andrews' q-identities gives a Srivastava-Daoust type multiple hypergeometric series.

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1. INTRODUCTION

An extension of terminating ${}_2F_1(1)$ to r pairs of parameters differ by positive integers due to Minton [17], is given by:

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} -n, a, & (b_r+m_r); \\ a+1, & (b_r); \end{matrix}\; 1\right] = \frac{n!}{(a+1)_n}\frac{(b_1-a)_{m_1}\cdots(b_r-a)_{m_r}}{(b_1)_{m_1}\cdots(b_r)_{m_r}}, \tag{1.1}$$

where (m_r) be a non-empty sequence of r positive integers $m_1, \cdots, m_r$ and $n \in \mathbb{Z}$ such that $n \geq m_1 + \cdots + m_r$. The ${}_{r+2}F_{r+1}(1)$ hypergeometric series (1.1) often appears as a solution to the problems in theoretical physics (For example, $r = 2$ case of (1.1) implies Racah coefficients, see [17]), and hence such type of series and their variants are of great importance and worthwhile to study. A reduction formula for generalized hypergeometric functions with certain parameter pairs differ by positive integral values [8, p. 270, Equation (1)] was proved by Karlsson and (in two markedly different simpler ways) by Srivastava [22]. Furthermore, [8] generalized this result to non-terminating ${}_{r+2}F_{r+1}(1)$ under the less restrictive condition that $-n$ may be replaced by the complex number c such that $R(-c) > m_1 + \cdots + m_r - 1$, which guarantees the convergence of ${}_{r+2}F_{r+1}(1)$. Later on, several hypergeometric summation and transformation formulae with integer parameter differences have been investigated, see [9,11–16,18–20,28] and, the papers cited therein. Notably, Miller and Srivastava [16] developed the Karlsson-Minton summation theorems for the generalized hypergeometric series with integer parameter differences. Srivastava et al. [27] were deduced several Srivastava-Daoust double hypergeometric functions with the help of general double series identities, derived using series iteration techniques in conjunction with Kummer and Dixon.

Very recently, Srivastava et al. [26] derived many new summation theorems; where one pair of numerator and denominator parameters differ by negative integers; for the truncated, terminating and non-terminating Gauss hypergeometric series, Kummer's first, second and third summation theorems and, their generalized versions by utilizing a general series identity involving a bounded sequence of complex numbers and its hypergeometric versions asserted and proved in [26, Section 2, pp. 468-469]. In the sequel to [26], Srivastava et al. [25] derived some corollaries [25, Corollaries 3.2-3.4] using the general series identity [26, p. 468] and, utilized them to investigate certain summation theorems with one pair of numerator and denominator parameters differ by negative integers, for the truncated, terminating and non-terminating Clausen's hypergeometric series ${}_3F_2(1)$. The applications of summation theorems, investigated by [25, 26], in deriving the closed-form evaluations of the Mellin transforms of the product of the exponential function $e^{\mu t}$ and Kummer's truncated and non-terminating confluent hypergeometric series and, the product of the exponential function $e^{\mu t}$ and Goursat's non-terminating hypergeometric function ${}_2F_2(1)$, respectively, are also shown, see [25, 26].

Recently, Srivastava et al. [28, Theorems 3.1-3.5] investigated extensions and transformations for the classical summation theorems on very well-poised hypergeometric functions, as listed by Slater [21], namely, Dougall's terminating ${}_5F_4(1)$ summation theorem [21, p. 244, Equation (III.13)], Whipple's ${}_7F_6(1)$ transformation [21, p. 61, Equation (2.4.1.1)], Dougall's ${}_7F_6(1)$ summation [21, p. 244, Equation (III.14)], Bailey's ${}_9F_8(1)$ transformation [21, p. 71, Equation (2.4.4.1)], and Dougall's non-terminating ${}_5F_4(1)$ summation [21, p. 244, Equation (III.12)], respectively, with two pairs of numerator and denominator parameters having unit difference by utilizing Bailey's transform [21, pp. 58-74], and extended Saalschütz summation theorem given by Rakha and Rathie [19] that contains a pair of parameters differ by unity. Furthermore, they produced a longer list of summation and transformation formulae as particular cases of main theorems and shown the application of generalized Dougall's non-terminating ${}_5F_4(1)$ summation theorem [28, Theorem 3.5] in deriving numerous Ramanujan type series involving π and other constants.

Motivated from [28], in this paper, the unique families of summation and transformation formulae for the generalized very well-poised hypergeometric functions, namely, Dougall's terminating ${}_{2q+5}F_{2q+4}(1)$ summation (§3, Theorem 2) and Whipple's ${}_{2q+7}F_{2q+6}(1)$ transformation (§4, Theorem 3), involving the arbitrary number of pairs of parameters with arbitrary differences, are investigated. It is accomplished by utilizing Bailey's transform [21, pp. 58-74] and extended Saalschütz summation theorem for the series ${}_{r+3}F_{r+2}(1)$ due to Kim et al. [9]. Theorem 2, proved in this paper by induction, provides an elegant method to obtain the summation s of high-order hypergeometric series when $q \in \mathbb{Z}^+$ and its particular cases are discussed in §7. The study of such type of summations is important, as whenever a generalized hypergeometric function reduces to the quotient of the products of the gamma function, the results are very important from the application point of view in various fields of science and engineering.

The generalized Dougall's terminating ${}_{2p+7}F_{2p+6}(1)$ summation (§5, Theorem 4) and Dougall's non-terminating ${}_{2p+5}F_{2p+4}(1)$ summation (§6, Theorem 5) follow from Theorem 3, which is the most generalized version of Whipple's transformation in the form of ${}_{2p+7}F_{2p+6}(1)$ into ${}_{m+4}F_{m+3}(1); m = \sum_{i=1}^{p} m_i$. Recently, Maier [14] has obtained the generalization of Whipple's transformation in the form of ${}_{2k+7}F_{2k+6}(1)$ into ${}_4F_3(1)$ and ${}_{2k+7}F_{2k+6}(1)$ into ${}_5F_4(1)$ using a different method.

The limiting case, when $q \to 1$, of a forty five year old q-identity due to Andrews [1, Theorem 4] (see also [10, Equation (3.1)]) that relates a terminating very well-poised basic hypergeometric series to a terminating multiple basic hypergeometric series is given by:

$$
{}_{2s+5}F_{2s+4}\left[\begin{matrix} a, 1+\frac{a}{2}, b_1, c_1, \cdots, b_{s+1}, c_{s+1}, -N; \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \cdots, 1+a-b_{s+1}, 1+a-c_{s+1}, 1+a+N; \end{matrix} \; 1\right]
$$

$$
= \frac{(1+a)_N\,(1+a-b_{s+1}-c_{s+1})_N}{(1+a-b_{s+1})_N\,(1+a-c_{s+1})_N} \sum_{k_1,\cdots,k_s=0}^{\infty} \frac{(-N)_{k_1+\cdots+k_s}}{(b_{s+1}+c_{s+1}-a-N)_{k_1+\cdots+k_s}}
$$

$$
\cdot \prod_{j=1}^{s} \frac{(1+a-b_j-c_j)_{k_j}\,(b_{j+1})_{k_1+\cdots+k_j}\,(c_{j+1})_{k_1+\cdots+k_j}}{k_j!\,(1+a-b_j)_{k_1+\cdots+k_j}\,(1+a-c_j)_{k_1+\cdots+k_j}}, \tag{1.2}
$$

where N and s are positive integers and other involved parameters are complex numbers with the restriction that none of the denominator parameters are negative integers. In §8, the generalized Dougall's ${}_{2q+5}F_{2q+4}(1)$ summation formula (Theorem 2) is applied to the left side of identity (1.2) to produce the sum of a terminating Srivastava-Daoust type multiple hypergeometric series.

2. Preliminaries

The generalized Gauss hypergeometric function [23,24] for any integer $r \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ being the set of natural numbers, and parameter vector

$$(\boldsymbol{a};\boldsymbol{b}) = (a_1,\cdots,a_{r+1};b_1,\cdots,b_r) \in \mathbb{C}^{r+1} \times \mathbb{C}^r$$

in which $b_j \notin \mathbb{Z}_0^- \ (j=1,\cdots,r)$; $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ and $\mathbb{C}$ denotes the set of complex numbers, is defined as follows:

$$\begin{aligned} {}_{r+1}F_r\left[\begin{matrix} a_1,a_2,\cdots,a_{r+1}; \\ \\ b_1,b_2,\cdots,b_r; \end{matrix} z\right] &= \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{r+1}(a_j)_n}{\prod\limits_{j=1}^{r}(b_j)_n} \frac{z^n}{n!} \\ &= {}_{r+1}F_r\left(a_1,\cdots,a_{r+1};b_1,\cdots,b_r;z\right), \qquad (2.1) \end{aligned}$$

where $(a)_n \ (n \in \mathbb{N}_0)$ is Pochhammer's symbol (also known as the *shifted* factorial) defined by

$$(a)_0 := 1 \qquad \text{and} \qquad (a)_n := a(a+1)\cdots(a+n-1) \qquad (n \in \mathbb{N}).$$

The compact notation $(a_1,\cdots,a_r)_n = (a_1)_n \cdots (a_r)_n$ will be used throughout the presentation. The series in (2.1) is absolutely convergent when $|z|<1$, and when $|z|=1$ if the parametric excess

$$\omega = \sum_{j=1}^{r} b_j - \sum_{j=1}^{r+1} a_j$$

has positive real part. Furthermore, at the point $z=-1$, the series in (2.1) converges conditionally if $\Re(\omega) > -1$ ([23, p. 20]).

The generalized hypergeometric series ${}_{r+1}F_r(1)$ has several summation and transformation formulae that have great importance in the theory of special functions (For example, Equation (1.1)). Many of the ${}_{r+1}F_r(1)$ type series like Saalschütz summation theorem are enriched with balancing property. A generalized hypergeometric function is called ω-balanced if the sum of denominator parameters exceed by ω to the sum of numerator parameters. The classical summation theorems like ${}_5F_4(1)$, ${}_7F_6(1)$, ${}_9F_8(1)$, etc. (see [21]), are not only balanced but also posses characteristic of well-poisedness. The well-poisedness of a generalized hypergeometric function lies in the fact that $1+a_1 = a_2+b_1 = a_3+b_2 = \cdots = a_{r+1}+b_r$. A generalized hypergeometric series is called very well-poised if it is well-poised and, $a_2 = 1+\frac{a_1}{2}$ and $b_1 = \frac{a_1}{2}$.

The following classical very well-poised transformation due to Whipple, the extension of Saalschütz summation theorem given by [9] and, well-known Bailey's transform will be used in our investigation:

$$ {}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, e, -N; \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+N; \end{matrix} 1\right] $$

$$ = \frac{(1+a)_N\,(1+a-d-e)_N}{(1+a-e)_N\,(1+a-d)_N}\;{}_4F_3\left[\begin{matrix} 1+a-b-c, d, e, -N; \\ 1+a-b, 1+a-c, d+e-a-N; \end{matrix} 1\right]. \tag{2.2} $$

The Bailey-type proof of (2.2), using the Saalschütz summation theorem [21, p. 243, Equation (III.2)] and a ${}_5F_4(1)$ summation [21, p. 244, Equation (III.13)], is outlined in [21, p. 61]. The extension of Saalschütz summation theorem [9] when r pairs of parameters differ by (m_r), a non-empty sequence of positive integers, is as follows:

$$ {}_{r+3}F_{r+2}\left[\begin{matrix} a, b, -n, & (f_r+m_r); \\ c, 1+a+b-c+m-n, & (f_r); \end{matrix} 1\right] $$

$$ = \frac{(c-a-m)_n\,(c-b-m)_n}{(c)_n\,(c-a-b-m)_n}\frac{((\eta_m+1))_n}{((\eta_m))_n}, \tag{2.3} $$

where $n \in \mathbb{Z}^+$, $f_i \neq a$ or b $(1 \le i \le r)$ and, the lower parameters (η_m) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m $(= \sum_{i=1}^r m_i)$ given by

$$ Q_m(t) = \sum_{k=0}^{m} B_k (a)_k (b)_k (t)_k G_{m,k}(t) \tag{2.4} $$

with

$$ B_k = (-1)^k A_k\,(c-a-m+k)_{m-k}\,(c-b-m+k)_{m-k} \tag{2.5} $$

and

$$ G_{m,k}(t) = {}_3F_2\left[\begin{matrix} -(m-k), t+k, c-a-b-m; \\ c-a-m+k, c-b-m+k; \end{matrix} 1\right]. \tag{2.6} $$

The function $G_{m,k}(t)$ is a polynomial in t of the degree $m-k$ where $0 \le k \le m$. The coefficients A_k are defined by

$$ A_k = \sum_{j=k}^{m} S_j^{(k)} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad A_m = 1, \tag{2.7} $$

where $S_j^{(k)}$ are the Stirling number of second kind defined as

$$ S_j^{(k)} = \sum_{l=0}^{k} \frac{(-1)^{k-l}}{k!} \binom{k}{l} l^j. $$

Since $S_j^{(k)}$ represents the number of ways to partition j objects into k non-empty equivalence classes, it is evident that $S_0^{(0)} = S_j^{(j)} = S_j^{(1)} = 1$, and $S_j^{(0)} = S_0^{(j)} = 0$. For further details on the Stirling number of second kind, please refer to Graham et al. [5, Chapter 6]. The coefficients σ_j $(0 \le j \le m)$ can be obtained from the following generating relation:

$$(f_1+x)_{m_1} \cdots (f_r+x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j. \tag{2.8}$$

The case of m being zero implies that (m_r) is empty.

W. N. Bailey [2, 3] developed a series transform known as Bailey's transform and gave a mechanism to derive ordinary and q-hypergeometric identities and Rogers-Ramanujan type identities. Its extensions and other types of Bailey's transform and their applications in deriving many new summation and transformation formulae can be found in [6, 7]. It states that if

$$\beta_n = \sum_{r=0}^{n} \alpha_r \, u_{n-r} \, v_{n+r} \tag{2.9}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \, u_{r-n} \, v_{r+n}, \tag{2.10}$$

then, subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.11}$$

3. Generalization of Dougall's Summation

Theorem 1. *For any $m \in \mathbb{Z}_0^+$, it is asserted that*

$$ {}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, a-f+1, c, f+m, -N; \\ \frac{a}{2}, 1+a-b, f, 1+a-c, 1+a-f-m, 1+a+N; \end{matrix} \; 1\right]$$
$$= \frac{(1+a)_N \, (1+a-b-c-m)_N \, ((g'_m+1))_N}{(1+a-b)_N \, (1+a-c)_N \, ((g'_m))_N}, \tag{3.1}$$

where the lower parameters (g'_m) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m $\left(= \sum_{i=1}^{1} m_i\right)$ defined as:

$$Q_m(t) = \sum_{k=0}^{m} B_k (f-b)_k (c)_k (t)_k G_{m,k}(t) \tag{3.2}$$

with

$$B_k = (-1)^k A_k \, (1+a-f-m+k)_{m-k} \, (1+a-b-c-m+k)_{m-k} \tag{3.3}$$

and

$$G_{m,k}(t) = {}_3F_2\left[\begin{matrix} -(m-k), t+k, 1+a-f-c-m; \\ 1+a-f-m+k, 1+a-b-c-m+k; \end{matrix} 1\right]. \tag{3.4}$$

The coefficients A_k are defined by

$$A_k = \sum_{j=k}^{m} S_j^{(k)} \sigma_{m-j}, \quad A_0 = (f)_m, \quad A_m = 1, \tag{3.5}$$

The coefficients σ_j $(0 \le j \le m)$ can be obtained from the following generating relation:

$$(f+x)_m = \sum_{j=0}^{m} \sigma_{m-j} x^j. \tag{3.6}$$

Remark 1. For $m = 1$, the Theorem 1 gives a theorem due to [28, Theorem 3.1]. Further, by letting $f \to \infty$ or $f = b$ or $m = 0$ in Theorem 1, a terminating classical ${}_5F_4(1)$ summation theorem [21, p. 244, Eq. (III.13)] follows.

Remark 2. For $m = 2$, the equation (3.5) gives the following parameters as:

$$A_0 = f(f+1), \quad A_1 = 2(f+1), \quad A_2 = 1.$$

The associated parametric polynomial $Q_2(t)$ (see (3.2)) (with zeros g_1' and g_2') is given by

$$\begin{aligned} Q_2(t) &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{2\sigma t}{\lambda\lambda'} + \frac{(\sigma)_2 (t)_2}{(\lambda)_2 (\lambda')_2}\right) - A_1 (f-b) ct (\lambda+1)(\lambda'+1) \\ &\quad \cdot \left(1 - \frac{\sigma(1+t)}{(\lambda+1)(\lambda'+1)}\right) + A_2 (f-b)_2 (c)_2 (t)_2 \\ &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{2\Upsilon_1 t}{\lambda\lambda'} + \frac{\Upsilon_2 (t)_2}{(\lambda)_2 (\lambda')_2}\right) \end{aligned} \tag{3.7}$$

where

$$\Upsilon_1 = \sigma + \frac{(f-b)c}{f}, \quad \Upsilon_2 = (\sigma)_2 + \frac{(f-b)c}{f}\left(2\sigma + \frac{(f-b+1)(c+1)}{f+1}\right)$$

and

$$\lambda = (a-f-1),\ \lambda \ne 0, \quad \lambda' = (a-b-c-1),\ \lambda' \ne 0, \quad \sigma = (a-f-c-1).$$

Then, the Theorem 1 gives a ${}_7F_6(1)$ summation formula, as written below:

$${}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, a-f+1, c, f+2, -N; \\ \frac{a}{2}, 1+a-b, f, 1+a-c, a-f-1, 1+a+N; \end{matrix} 1\right]$$

$$= \frac{(1+a)_N\,(a-b-c-1)_N\,(g_1'+1)_N\,(g_2'+1)_N}{(1+a-b)_N\,(1+a-c)_N\,(g_1')_N\,(g_2')_N}. \tag{3.8}$$

Similarly, the following ${}_7F_6(1)$ summation formula for $m=3$ holds true, where the lower parameters g_1', g_2', and g_3' on the right side are non-vanishing zeros of the associated parametric polynomial $Q_3(t)$ evaluated by manipulating the equations (3.2)-(3.6) accordingly.

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, a-f+1, c, f+3, -N; \\ \frac{a}{2}, 1+a-b, f, 1+a-c, a-f-2, 1+a+N; \end{matrix}\, 1\right] \\
&= \frac{(1+a)_N\,(a-b-c-2)_N\,(g_1'+1)_N\,(g_2'+1)_N\,(g_3'+1)_N}{(1+a-b)_N\,(1+a-c)_N\,(g_1')_N\,(g_2')_N\,(g_3')_N}.
\end{aligned} \tag{3.9}$$

Proof of Theorem 1. Selecting $e=f+m, c=a-f+1$ and, $d=c$ in the Whipple's transformation (2.2) and summing the ${}_4F_3(1)$ on the right side by using the extended Saalschütz theorem (2.3) ($r=1$ case), followed by some simplifications, would lead to the Theorem 1. □

Theorem 2. *For q, $m \in \mathbb{Z}_0^+$, it is asserted that*

$$\begin{aligned}
&{}_{2q+5}F_{2q+4}\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, a-f_1+1, \cdots, a-f_q+1, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-f_1-m_1, \cdots, 1+a-f_q-m_q, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} f_1+m_1, \cdots, f_q+m_q, -N; \\ f_1, \cdots, f_q, 1+a+N; \end{matrix}\, 1\right] \\
&= \frac{(1+a)_N\,(1+a-b-c-m)_N\,\left((g_m^q+1)\right)_N}{(1+a-b)_N\,(1+a-c)_N\,\left((g_m^q)\right)_N},
\end{aligned}$$

where the lower parameters $\left(g_m^q\right)$ on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m $\left(=\sum_{i=1}^{q} m_i\right)$ defined as:

$$Q_m(t) = \sum_{k=0}^{m} B_k (f_q-b-M)_k (c)_k (t)_k G_{m,k}(t); \quad M = \sum_{i=1}^{q-1} m_i$$

with

$$B_k = (-1)^k A_k \,(1+a-f_q-m_q+k)_{m-k}\,(1+a-b-c-m+k)_{m-k}$$

and

$$G_{m,k}(t) = {}_3F_2\left[\begin{matrix} -(m-k), t+k, 1+a-f_q-c-m_q; \\ 1+a-f_q-m_q+k, 1+a-b-c-m+k; \end{matrix} 1\right].$$

The coefficients A_k are defined by

$$A_k = \sum_{j=k}^{m} S_j^{(k)} \sigma_{m-j}, \quad A_0 = \left((g_M^{q-1})\right)_1 (f_q)_{m_q}, \quad A_m = 1,$$

The coefficients $\sigma_j (0 \leq j \leq m)$ are generated by

$$\left((g_M^{q-1}+x)\right)_1 (f_q+x)_{m_q} = \sum_{j=0}^{m} \sigma_{m-j} x^j,$$

where $\left(g_M^{q-1}\right)$ themselves are non-vanishing zeros of associated parametric polynomial $Q_M(t)$ of degree M $\left(=\sum_{i=1}^{q-1} m_i\right)$. Also,

$$\left((g_M^{q-1})\right)_1 = (g_1^{q-1})(g_2^{q-1})\cdots(g_M^{q-1})$$

$$\left((g_M^{q-1}+x)\right)_1 = (g_1^{q-1}+x)(g_2^{q-1}+x)\cdots(g_M^{q-1}+x)$$

The above-mentioned products vanishes when $M = 0$.

Remark 3. The classical Dougall's terminating ${}_5F_4(1)$ summation theorem [21, p. 244, Equation (III.13)] follows from Theorem 2 when $m = 0$.

Proof of Theorem 2. In view of Theorem 1, it can be observed that the Theorem 2 holds true for $q = 1$. Next, assume that it holds true for an arbitrary fixed positive integer p, i.e.

$$\begin{aligned} &{}_{2p+5}F_{2p+4}\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, a-f_1+1, \cdots, a-f_p+1, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-f_1-m_1, \cdots, 1+a-f_p-m_p, \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} f_1+m_1, \cdots, f_p+m_p, -N; \\ f_1, \cdots, f_p, 1+a+N; \end{matrix} 1\right] \\ &= \frac{(1+a)_N (1+a-b-c-m)_N \left((g_m^p+1)\right)_N}{(1+a-b)_N (1+a-c)_N \left((g_m^p)\right)_N}, \end{aligned} \tag{3.10}$$

where the lower parameters $(g^p{}_m)$ on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m $\left(=\sum_{i=1}^{p} m_i\right)$, which is similar to one obtained in Theorem 2. Finally, it remains to verify that the Theorem 2 is also true for $q = p+1$ (see (3.11)).

$$
\begin{aligned}
&{}_{2(p+1)+5}F_{2(p+1)+4}\left[\begin{array}{l} a, 1+\frac{a}{2}, b, c, a-f_1+1, \cdots, a-f_{p+1}+1, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-f_1-m_1, \cdots, \end{array}\right. \\
&\qquad\left.\begin{array}{r} f_1+m_1, \cdots, f_{p+1}+m_{p+1}, -N; \\ 1+a-f_{p+1}-m_{p+1}, f_1, \cdots, f_{p+1}, 1+a+N; \end{array} 1\right] \\
&\qquad = \frac{(1+a)_N\,(1+a-b-c-m)_N\left((g_m^{p+1}+1)\right)_N}{(1+a-b)_N\,(1+a-c)_N\left((g_m^{p+1})\right)_N},
\end{aligned} \tag{3.11}
$$

where the lower parameters $(g^{p+1}{}_m)$ on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m\left(=\sum_{i=1}^{p+1} m_i\right)$ and, can be obtained by manipulating the equations (2.4)-(2.8) according to the substitution $a = f_{p+1} - b - M; M = \sum_{i=1}^{p} m_i$, $b = c$ and $c = 1+a-b$. The modified equation (2.8) has the parameters $\left(g_M^p\right)$, which themselves are non-vanishing zeros of a parametric polynomial $Q_M(t)$ of degree $M\ \left(=\sum_{i=1}^{p} m_i\right)$ associated with equation (3.10). Selecting

$$
\alpha_r = \frac{\left(a, 1+\frac{a}{2}, b, a-f_1+1, \cdots, a-f_p+1, f_1+m_1, \cdots, f_p+m_p, d\right)_r (-1)^r}{\left(\frac{a}{2}, 1+a-b, f_1, \cdots, f_p, 1+a-d, 1+a-f_1-m_1, \cdots, 1+a-f_p-m_p\right)_r r!},
$$

$$
u_r = \frac{1}{r!}, \quad v_r = \frac{1}{(1+a)_r} \quad \text{and} \quad \delta_r = \frac{(c, e, -N)_r}{(c+e-a-N)_r}
$$

in (2.9) and (2.10) and making use of the summation theorem given in equation (3.10) and Saalschütz summation theorem [21, p. 243, Equation (III.2)] to obtain the following values of β_n and γ_n:

$$
\beta_n = \frac{(1+a-b-d-m)_n}{(1+a-d, 1+a-b)_n} \frac{\left((g_m^p+1)\right)_n}{\left((g_m^p)\right)_n n!},
$$

where the parameters (g_m^p) are as defined with (3.10) and

$$
\gamma_n = \frac{(1+a-c, 1+a-e)_N}{(1+a, 1+a-c-e)_N} \frac{(c, e, -N)_n (-1)^n}{(1+a-c, 1+a-e, 1+a+N)_n}.
$$

Putting these values of β_n and γ_n in (2.11), a transformation of ${}_{2p+7}F_{2p+6}(1)$ follows, which is being differed in the next section (see Theorem 3). Next, selecting $e =$

$f_{p+1}+m_{p+1}$ and $d=1+a-f_{p+1}$ in Theorem 3 and finally, applying (2.3) followed by some simplifications, lead to the required summation . □

4. Generalization of Whipple's Transformation of ${}_7F_6(1)$ to ${}_4F_3(1)$

Theorem 3. *For $p,\ m\in\mathbb{Z}_0^+$, it is asserted that*

$$\frac{(1+a-c)_N\,(1+a-e)_N}{(1+a)_N\,(1+a-c-e)_N}\;{}_{2p+7}F_{2p+6}\left[\begin{matrix} a,1+\frac{a}{2},b,a-f_1+1,\cdots,a-f_p+1,d \\ \frac{a}{2},1+a-b,1+a-d,1+a-f_1-m_1,\cdots,\end{matrix}\right.$$
$$\left.\begin{matrix} f_1+m_1,\cdots,f_p+m_p,c,e,-N; \\ 1+a-f_p-m_p,f_1,\cdots,f_p,1+a-c,1+a-e,1+a+N;\end{matrix}\;1\right]$$
$$={}_{m+4}F_{m+3}\left[\begin{matrix} c,e,1+a-b-d-m,-N,(g_m^p+1); \\ 1+a-b,1+a-d,c+e-a-N,(g_m^p);\end{matrix}\;1\right] \tag{4.1}$$

Remark 4. The transformation in Theorem 3 is a generalization of Whipple's ${}_7F_6$ transformation (Equation (2.2)), which follows upon letting $m=0$ in Theorem 3. The case $p=m=1$ of Theorem 3 is previously observed by [28, Theorem 3.2], and subsequently by Mishev [18, Proposition 3.1 and 3.6] using a different method given in [4, Chapter 4]. Clearly, for the arbitrary values of p and m in Theorem 3, it is always possible to generate other high-order transformations.

5. Generalization of Dougall's terminating ${}_7F_6(1)$ summation

Theorem 4. *For $p,\ m\in\mathbb{Z}_0^+$, it is asserted that*

$${}_{2p+7}F_{2p+6}\left[\begin{matrix} a,1+\frac{a}{2},b,a-f_1+1,\cdots,a-f_p+1,1+2a-b-c-d+N-m, \\ \frac{a}{2},1+a-b,f_1,\cdots,f_p,b+c+d-a-N+m,1+a-f_1-m_1,\cdots,\end{matrix}\right.$$
$$\left.\begin{matrix} f_1+m_1,\cdots,f_p+m_p,c,d,-N; \\ 1+a-f_p-m_p,1+a-c,1+a-d,1+a+N;\end{matrix}\;1\right]=\frac{(1+a)_N}{(1+a-b)_N}\cdot$$
$$\frac{(1+a-c-d)_N\,(1+a-b-c-m)_N\,(1+a-b-d-m)_N\,((\eta_m+1))_N}{(1+a-c)_N\,(1+a-d)_N\,(1+a-b-c-d-m)_N\,((\eta_m))_N}, \tag{5.1}$$

where the lower parameters (η_m) on the right side are non-vanishing zeros of associated parametric polynomial $Q_m(t)$ of degree m $\left(=\sum_{i=1}^{p} m_i\right)$. The polynomial $Q_m(t)$ can be obtained easily by manipulating the equations (2.4)-(2.8) accordingly.

Remark 5. The case $m=0$ of Theorem 4 is the classical Dougall's theorem [21, p. 244, Equation (III.14)]. Furthermore, by setting $p=m=1$ in Theorem 4, leads to a result due to [28, Theorem 3.3].

Proof of Theorem 4. Choosing $d=1+2a-b-c-e+N-m$ in Theorem 3 and applying (2.3) (for the case $m_1=\cdots=m_r=1$) and then replacing $e\to d$, gives the Theorem 4. □

6. Generalization of Dougall's non-terminating ${}_5F_4(1)$ summation

Theorem 5. *For p, $m\in\mathbb{Z}_0^+$, it is asserted that*

$$
{}_{2p+5}F_{2p+4}\left[\begin{matrix} a,1+\frac{a}{2},b,a-f_1+1,\cdots,a-f_p+1, \\ \frac{a}{2},1+a-b,f_1,\cdots,f_p,1+a-f_1-m_1,\cdots, \end{matrix}\right.
$$
$$
\left.\begin{matrix} f_1+m_1,\cdots,f_p+m_p,c,d; \\ 1+a-f_p-m_p,1+a-c,1+a-d; \end{matrix}\; 1\right]
$$
$$
=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d-m)\Gamma((\eta_m))}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-b-c-m)\Gamma(1+a-b-d-m)\Gamma((\eta_m+1))},
$$
provided $Re(a-b-c-d+(1-m))<0$, where the lower parameters (η_m) on the right side are non-vanishing zeros of associated parametric polynomial $Q_m(t)$ of degree m $\left(=\sum_{i=1}^{p} m_i\right)$. The polynomial $Q_m(t)$ can be obtained easily by manipulating the conditions obtained with Theorem 4.

Remark 6. The case $m=0$ of Theorem 5 is the classical Dougall's non-terminating theorem [21, p. 244, Equation (III.12)]. Furthermore, by selecting $p=m=1$, a result due to [28, Theorem 3.5] follows.

Proof of Theorem 5. The Theorem 5 follows from Theorem 4 by taking the limit when $N\to\infty$. □

7. Particular cases of Theorem 2

Theorem 6. *It is asserted that*

$$
{}_9F_8\left[\begin{matrix} a,1+\frac{a}{2},b,c,a-f_1+1,a-f_2+1,f_1+m_1,f_2+m_2,-N; \\ \frac{a}{2},1+a-b,1+a-c,1+a-f_1-m_1,1+a-f_2-m_2,f_1,f_2,1+a+N; \end{matrix}\; 1\right]
$$
$$
=\frac{(1+a)_N\,(1+a-b-c-m)_N\,((g''_m+1))_N}{(1+a-b)_N\,(1+a-c)_N\,((g''_m))_N},
$$

where the lower parameters (g''_m) *on the right side are non-vanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* $m\left(=\sum_{i=1}^{2} m_i\right)$ *and, can be obtained by manipulating the equations* (2.4)-(2.8) *according to the substitution* $a = f_2 - b - m_1, b = c$ *and* $c = 1 + a - b$.

Proof of Theorem 6. The Theorem 6 follows from Theorem 2 when $q = 2$. □

Theorem 7. *It is asserted that*

$$
{}_9F_8\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, a-f_1+1, a-f_2+1, f_1+1, f_2+1, -N; \\ \\ \frac{a}{2}, 1+a-b, 1+a-c, a-f_1, a-f_2, f_1, f_2, 1+a+N; \end{matrix} 1\right]
$$
$$
= \frac{(1+a)_N\,(a-b-c-1)_N\,((g''_2+1))_N}{(1+a-b)_N\,(1+a-c)_N\,((g''_2))_N}
$$

Proof of Theorem 7. The Theorem 7 follows from Theorem 6 when $m_1 = m_2 = 1$ and the associated parametric polynomial $Q_2(t)$ (with zeros g''_1 and g''_2) is as given below:

$$
\begin{aligned}
Q_2(t) &= A_0\,(\lambda)_2\,(\lambda')_2\left(1 - \frac{2\sigma t}{\lambda\lambda'} + \frac{(\sigma)_2\,(t)_2}{(\lambda)_2\,(\lambda')_2}\right) - A_1(f_2-b-1) \\
&\cdot ct(\lambda+1)(\lambda'+1)\left(1 - \frac{\sigma(1+t)}{(\lambda+1)(\lambda'+1)}\right) + A_2\,(f_2-b-1)_2\,(c)_2\,(t)_2 \\
&= A_0\,(\lambda)_2\,(\lambda')_2\left(1 - \frac{\Upsilon_1 t}{\lambda\lambda'} + \frac{\Upsilon_2\,(t)_2}{(\lambda)_2\,(\lambda')_2}\right),
\end{aligned}
\tag{7.1}
$$

where

$$
\Upsilon_1 = 2\sigma + X\,(f_2-b-1)\,c, \tag{7.2}
$$

$$
\Upsilon_2 = (\sigma)_2 + (f_2-b-1)\,c\left(X\sigma + \frac{(f_2-b)\,(c+1)}{g'_1 f_2}\right) \tag{7.3}
$$

with $X = \dfrac{(g'_1+f_2+1)}{g'_1 f_2}$ and

$$
\lambda = (a-f_2),\ \lambda \neq 0, \quad \lambda' = (a-b-c-1),\ \lambda' \neq 0, \quad \sigma = (a-f_2-c).
$$

The parameter g'_1 is itself a non-vanishing zero of the associated parametric polynomial $Q_1(t)$ given by

$$
g'_1 = \frac{f_1(f_1-a)(b+c-a)}{f_1(a-f_1)-bc}
$$

and can be obtained by setting $f = f_1$, $m = 1$ in the equations (3.2)-(3.6) given with Theorem 1. □

Theorem 8. *It is asserted that*

$$
{}_9F_8\left[\begin{matrix} a,1+\frac{a}{2},b,c,a-f_1+1,a-f_2+1,f_1+2,f_2+1,-N; \\ \frac{a}{2},1+a-b,1+a-c,a-f_1-1,a-f_2,f_1,f_2,1+a+N; \end{matrix} 1\right]
$$

$$
=\frac{(1+a)_N\,(a-b-c-2)_N\,((g_3''+1))_N}{(1+a-b)_N\,(1+a-c)_N\,\left((g_3'')\right)_N}
$$

Proof of Theorem 8. The Theorem 8 follows from Theorem 6 when $m_1=2$ and $m_2=1$ and the associated parametric polynomial $Q_3(t)$ (with zeros g_1'', g_2'' and g_3'') is as given below:

$$
Q_3(t)=A_0\,(\lambda)_3\,(\lambda')_3\left(1-\frac{\Upsilon_1 t}{\lambda\lambda'}+\frac{(t)_2\,\Upsilon_2}{(\lambda)_2\,(\lambda')_2}-\frac{(t)_3\,\Upsilon_3}{(\lambda)_3\,(\lambda')_3}\right) \tag{7.4}
$$

where

$$
\Upsilon_1=3\sigma+X\,(f_2-b-2)\,c, \tag{7.5}
$$

$$
\Upsilon_2=3\,(\sigma)_2+(f_2-b-2)\,c\,(2X\sigma+Y\,(f_2-b-1)\,(c+1)), \tag{7.6}
$$

$$
\Upsilon_3=(\sigma)_3+(f_2-b-2)\,c\left(X\,(\sigma)_2+Y\,(f_2-b-1)\,(c+1)\,\sigma+\frac{(f_2-b-1)_2\,(c+1)_2}{g_1'g_2'f_2}\right) \tag{7.7}
$$

with

$$
X=\frac{g_1'g_2'+(f_2+1)\left(g_1'+g_2'+1\right)}{g_1'g_2'f_2} \qquad Y=\frac{\left(f_2+g_1'+g_2'+3\right)}{g_1'g_2'f_2}
$$

and $\lambda=(a-f_2)$, $\lambda\neq 0$, $\lambda'=(a-b-c-2)$, $\lambda'\neq 0$, $\sigma=(a-f_2-c)$. $\square$

Theorem 9. *It is asserted that*

$$
{}_{11}F_{10}\left[\begin{matrix} a,1+\frac{a}{2},b,c,a-f_1+1,a-f_2+1,a-f_3+1, \\ \frac{a}{2},1+a-b,1+a-c,1+a-f_1-m_1,1+a-f_2-m_2,1+a-f_3-m_3, \end{matrix}\right.
$$

$$
\left.\begin{matrix} f_1+m_1,f_2+m_2,f_3+m_3,-N; \\ f_1,f_2,f_3,1+a+N; \end{matrix} 1\right]
$$

$$
=\frac{(1+a)_N\,(1+a-b-c-m)_N\,((g_m'''+1))_N}{(1+a-b)_N\,(1+a-c)_N\,((g_m'''))_N},
$$

where the lower parameters (g_m''') *on the right side are non-vanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* $m\left(=\sum_{i=1}^{3} m_i\right)$ *and, can be obtained by manipulating the equations* (2.4)-(2.8) *according to the substitution* $a = f_3 - b - M; M = \sum_{i=1}^{2} m_i, b = c$ *and* $c = 1 + a - b$. *The modified equation* (2.8) *has the parameters* $\left(g_M''\right)$, *which themselves are non-vanishing zeros of associated parametric polynomial* $Q_M(t)$ *of degree* $M\ \left(=\sum_{i=1}^{2} m_i\right)$, *see Theorem* 6.

Proof of Theorem 9. The Theorem 9 follows from Theorem 2 when $q = 3$. □

Remark 7. Clearly, for the arbitrary values of q in Theorem 2, it is always possible to generate other high-order summation s like ${}_{13}F_{12}(1)$, ${}_{15}F_{14}(1)$, etc.

8. A Summation for Multiple Srivastava-Daoust Type Series

Theorem 10. *It is asserted that*

$$\sum_{k_1,\cdots,k_s=0}^{\infty} \frac{(b,c,-N)_{k_1+\cdots+k_s}(-m_s)_{k_s}}{k_s!\ (f_s, 1+a-f_s-m_s, b+c-a-N)_{k_1+\cdots+k_s}} \Upsilon$$

$$= \frac{(1+a-b-c-m)_N\,((g_m^s+1))_N}{(1+a-b-c)_N\,((g_m^s))_N},$$

where

$$\Upsilon = \prod_{j=1}^{s-1} \frac{(-m_j)_{k_j}(f_{j+1}+m_{j+1}, 1+a-f_{j+1})_{k_1+\cdots+k_j}}{k_j!\,(f_j)_{k_1+\cdots+k_j}(1+a-f_j-m_j)_{k_1+\cdots+k_j}}$$

and the lower parameters (g_m^s) *on the right side are non-vanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* $m\ \left(=\sum_{j=1}^{s} m_j\right)$, *which is similar to one obtained in Theorem* 2.

Proof of Theorem 10. Choosing

$$b_{s+1} = b, \quad c_{s+1} = c, \quad (b_1,\cdots,b_s) = (a-f_1+1,\cdots,a-f_s+1)$$

$$(c_1,\cdots,c_s) = (f_1+m_1,\cdots,f_s+m_s)$$

in equation (1.2) and applying Theorem 2, followed by some simplifications, would lead to the Theorem 10, which is a sum of very general class of terminating Srivastava-Daoust type multiple hypergeometric series. □

Remark 8. The Dougall's summation theorem for ${}_{2p+7}F_{2p+6}(1)$ (Theorem 4) can be applied to equation (1.2) to obtain a class of another type of Srivastava-Daoust type multiple hypergeometric series.

9. CONCLUSION

In this paper, some classes of summation and transformation theorems are developed, which provide the generalizations of Dougall's terminating summation theorem for ${}_5F_4(1)$ and ${}_7F_6(1)$, Dougall's non-terminating summation theorem for ${}_5F_4(1)$ and Whipple's transformation formula for ${}_7F_6(1)$. Thus, this work has extended existing literature by adding several additional hypergeometric summation s and transformations to the Slater's list [21, p. 243-245, Appendix III]. Furthermore, the generalization of Dougall's terminating summation theorem for ${}_5F_4(1)$ is applied to obtain a summation theorem for a class of Srivastava-Daoust type multiple hypergeometric series by making use of the limiting case, when $q \to 1$, of Andrews' identities.

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