Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 537–549

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3702

NEW EXISTENCE AND UNIQUENESS RESULT FOR FRACTIONAL BAGLEY–TORVIK DIFFERENTIAL EQUATION

HAMID BAGHANI, MICHAL FEČKAN, JAVAD FAROKHI-OSTAD,
AND JEHAD ALZABUT

Received 05 February, 2021

Abstract. Inspired by the paper of Fazli and Nieto in [*Open Math.* 17 (2019) 499–512], we establish new existence and uniqueness result for a type of fractional Bagley–Torvik differential equation. Reported result not only generalizes previous results but also adopts different technique. We finish this study by concluding remarks which discuss the preference of our theorem compared to previous results. An example is constructed with specific parameters that requires weaker conditions for the existence of a unique solution. Meanwhile, we construct an iterative sequence that converges to the unique solution and can not be commented via the results of Fazli and Nieto.

2010 *Mathematics Subject Classification:* 26A33; 34A08; 34A12.

Keywords: Fractional Bagley–Torvik equation; Coupled fractional equations; Banach fixed point theorem.

1. INTRODUCTION

Guided by their popular applications, fractional differential equations have attracted the awareness of many researchers working in fields of science and engineering. The existing literature on this topic not only covers its theoretical manifestation but also describes its extensive applications in describing reallistic phenomena. The apparatus of fractional calculus, in particular, are efficiently used and applied in modeling many engineering and scientific processes; we propose the popular monographs [13, 19, 23] for readers' consciousness.

Fractional differential equations have proved significant applications in many real life aspects. For this, these type of equations has attracted the attention of many researchers who produced several remarkable results in this regard. A particular emphasize of initial and boundary value problems which have been treated within fractional order settings such as the Langevin equation, the Basset equation and Bagley–Torvik equation which are popular in fluctuating environments, a incompressible viscous fluid and viscoelasticity, respectively. Fractional Langevin equations have been systematically studied [1–5, 11, 16–18]. The latter two equations, however, have comparably gained less attention amongst researchers [6–10, 20, 25].

The Bagley–Torvik equation, which is our concern herein, of the form

$$\xi u''(\varsigma)+\zeta\mathcal{D}^{\frac{3}{2}}u(\varsigma)+\eta u(\varsigma)=\omega(\varsigma),\ \ 0<\varsigma\leq 1,\ 1<\alpha<2, \tag{1.1}$$

appears in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. Here the prime denotes the classical derivative, $\mathcal{D}^{\frac{3}{2}}$ is the Caputo fractional derivative of order $\frac{3}{2}$, $\omega:[0,1]\to\mathbb{R}$ is a given function and $\xi,\zeta,\eta\in\mathbb{R}$, $\xi\neq 0$. Equation (1.1) was initially introduced in [25] and is thoroughly discussed in [23]. Particularly, the investigators have studied the analytical and numerical solutions of equation (1.1); see for instance [12, 14, 21, 22]. In [7], the equivalence between the Caputo and Riemann derivatives are discussed and pointed out that they are identical in describing the linear viscoelastic material just under two minimal restrictions.

Recently in [15], Fazli and Nieto studied fractional Bagley–Torvik equation which is in the form

$$\begin{cases}\left(\mathcal{D}^2+\Lambda_1\mathcal{D}^{\alpha}+\Lambda_2\right)u(\varsigma)=\omega(\varsigma),\ \ 0<\varsigma\leq 1,\ \ 1<\alpha<2,\\ u(0)=\mu, u'(0)=\nu,\end{cases} \tag{1.2}$$

under the same assumptions, μ,ν,Λ_1 and Λ_2 are real numbers and $\mathcal{D}^2:=\frac{d^2}{d\varsigma^2}$. It is clear that equation (1.2) is a generalization of (1.1) to an arbitrary order in fractional derivative settings. The existence and approximations of solutions are proved for equation (1.2) admitting only the existence of a lower (coupled lower and upper) solution. The main results are obtained under certain assumptions and by use of an appropriate fixed point theorem in partially ordered normed linear spaces. Equation (1.2) is refereed to in the text as FBTE.

In this work, we extend the main results of [15] and establish an improved existence-uniqueness theorem. Our result complements and generalizes the result of Fazli and Nieto and also proves the main theorem using different technique. We end the paper by a concluding remark that shows the preference of our theorem over the results of [15]. An example is constructed with specific parameters that requires less restrictive conditions for the existence of a unique solution. Meanwhile, we construct an iterative sequence that converges to the unique solution and can not be commented via the results of [15].

2. Essential background and results of [15]

This section assembles some basic definitions and concepts concerning with theory of fractional calculus. For the terms and terminologies, the reader can consult one of the monographs [13, 19, 23].

The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $u : [0,1] \to \mathbb{R}$ is defined as

$$I^{\alpha}u(\varsigma) = \int_0^{\varsigma} \frac{(\varsigma - s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds, \qquad 0 \le \varsigma \le 1, \tag{2.1}$$

where Γ is the Gamma function. For a function $u : [0,1] \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha \in \mathbb{R}^+$ is defined as

$$\begin{aligned}\mathcal{D}^{\alpha}u(\varsigma) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\varsigma^n} \int_0^{\varsigma} (\varsigma - s)^{n-\alpha-1} \Big(u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} s^k \Big) ds \\ &= \mathcal{D}^n I^{n-\alpha} \Big(u(\varsigma) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} \varsigma^k \Big),\end{aligned} \tag{2.2}$$

where $n - 1 < \alpha < n$, provided that the right side is pointwise defined on $[0,1]$. Note that if $n - 1 < \alpha < n$ and $u \in AC^{n-1}[0,1]$, then

$$\begin{aligned}\mathcal{D}^{\alpha}u(\varsigma) &= \frac{1}{\Gamma(n-\alpha)} \int_0^{\varsigma} (\varsigma - s)^{n-\alpha-1} u^{(n)}(s)ds \\ &= I^{n-\alpha}u^{(n)}(\varsigma).\end{aligned}$$

For the relationship between (2.1) and (2.2) and other properties of these notions, we refer the reader to [13, 19, 23].

Lemma 1. [15] *u is a solution of FBTE if and only if it is a solution of*

$$u(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) - \Lambda_1 I^{3-\alpha}u'(\varsigma) + I^2\Big[\omega(\varsigma) - \Lambda_2 u(\varsigma)\Big], \tag{2.3}$$

in the set $\mathcal{C} := C^1[0,1]$.

The main results of [15] are proved by the help of some fixed point theorems in partially ordered metric spaces. Because the definition of coupled solutions becomes different based on the sign of the product $\Lambda_1 \cdot \Lambda_2$, the proofs are presented in two folds. We review the herein the two cases for the sake of completeness.

2.1. *The case* $\Lambda_1 \cdot \Lambda_2 < 0$

Definition 1. *[15]* An element $(u_0, v_0) \in \mathcal{C} \times \mathcal{C}$ is said to be a coupled lower and upper solution of FBTE if

$$\begin{cases} u_0'(\varsigma) \le \nu\Big(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)}\varsigma^{2-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(2-\alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{1-\alpha} v_0'(\tau)d\tau + \int_0^{\varsigma} [\omega(\tau) + |\Lambda_2| u_0(\tau)]d\tau, \\ u_0(0) \le \mu \end{cases}$$

and

$$\begin{cases} v_0'(\varsigma) \ge \nu\Big(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)}\varsigma^{2-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(2-\alpha)} \int_0^{\varsigma} (\varsigma - \tau)^{1-\alpha} u_0'(\tau)d\tau + \int_0^{\varsigma} [\omega(\tau) + |\Lambda_2| v_0(\tau)]d\tau, \\ v_0(0) \ge \mu \end{cases}$$

for all $\varsigma \in [0,1]$.

Theorem 1. [15] *Let $(u_0, v_0) \in \mathcal{C} \times \mathcal{C}$ be a coupled lower and upper solution of FBTE and $k_1 := \max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$.*

I. *Then FBTE has a unique solution $u^* \in \mathcal{C}$.*

II. *Moreover, there exist two monotone sequence $\{u_n\}$ and $\{v_n\}$ such that both sequences converge to u^* in $\mathcal{C}$.*

III. *Further, the following error estimates hold,*

$$\begin{aligned}
\|u_n - u^*\|_{\mathcal{C}} &\le \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|u_1 - u_0\|_{\mathcal{C}} + \|v_1 - v_0\|_{\mathcal{C}}\Big), \\
\|v_n - u^*\|_{\mathcal{C}} &\le \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|u_1 - u_0\|_{\mathcal{C}} + \|v_1 - v_0\|_{\mathcal{C}}\Big), \\
\|u_n - v_n\|_{\mathcal{C}} &\le \frac{k_1^n}{1-k_1}\Big(\|u_1 - u_0\|_{\mathcal{C}} + \|v_1 - v_0\|_{\mathcal{C}}\Big).
\end{aligned} \tag{2.4}$$

2.2. *The case $\Lambda_1 \cdot \Lambda_2 > 0$*

Definition 2. *[15]* An element $u_0 \in \mathcal{C}$ is said to be a lower solution of FBTE if

$$\begin{aligned}
u_0'(\varsigma) \le H(\varsigma) &:= \nu\Big(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)}\varsigma^{2-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(2-\alpha)}\int_0^{\varsigma}(\varsigma-\tau)^{1-\alpha}u_0'(\tau)d\tau \\
&\quad + \int_0^{\varsigma}[\omega(\tau) - \Lambda_2 u_0(\tau)]d\tau, \\
u_0(0) &\le \mu
\end{aligned} \tag{2.5}$$

for all $\varsigma \in [0,1]$.

Theorem 2. [15] *Let $u_0 \in \mathcal{C}$ be a lower solution of FBTE and $k_2 := |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$.*

I. *Then FBTE has a unique solution $u^* \in \mathcal{C}$.*

II. *Moreover, the iterative sequence $\{u_n\}$ defined by*

$$\begin{aligned}
u_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) &- \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^{\varsigma}(\varsigma-\tau)^{2-\alpha}u_{n-1}'(\tau)d\tau \\
&+ \int_0^{\varsigma}(\varsigma-\tau)[\omega(\tau) - \Lambda_2 u_{n-1}(\tau)]d\tau,
\end{aligned}$$

converges to u^* in $\mathcal{C}$.

III. *Further, the following error estimates hold,*

$$\begin{aligned}
\|u_n - u^*\|_{\mathcal{C}} &\le \frac{k_2^n}{1-k_2}\|u_1 - u_0\|_{\mathcal{C}}, \\
\|u_{n+1} - u_n\|_{\mathcal{C}} &\le k_2^n\|u_1 - u_0\|_{\mathcal{C}}.
\end{aligned}$$

3. MAIN RESULT

This segment is dedicated to our main theorem.

Theorem 3. *Let* $k := |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$.

I. *Then, FBTE has a unique solution* $u^* \in \mathcal{C}$.

II. *Each sequence* $\{u_n\}$ *along with an initial point* u_0 *(not necessarily a lower solution) given by*

$$u_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^{\varsigma}(\varsigma-\tau)^{2-\alpha}u'_{n-1}(\tau)d\tau + \int_0^{\varsigma}(\varsigma-\tau)[\omega(\tau) - \Lambda_2 u_{n-1}(\tau)]d\tau,$$

converges to u^* *in* $\mathcal{C}$. *In addition, the following error estimates hold,*

$$\|u_n - u^*\|_{\mathcal{C}} \le \frac{k^n}{1-k}\|u_1 - u_0\|_{\mathcal{C}},$$
$$\|u_{n+1} - u_n\|_{\mathcal{C}} \le k^n\|u_1 - u_0\|_{\mathcal{C}}.$$

III. *Further, if* $k_1 = \max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$, *then,* $\exists$ *two sequences* $\{U_n\}$ *and* $\{V_n\}$ *such that both sequences converge to* u^* *in* $\mathcal{C}$. *Also, the following error estimates are valid,*

$$\|U_n - u^*\|_{\mathcal{C}} \le \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|U_1 - U_0\|_{\mathcal{C}} + \|V_1 - V_0\|_{\mathcal{C}}\Big),$$
$$\|V_n - u^*\|_{\mathcal{C}} \le \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|U_1 - U_0\|_{\mathcal{C}} + \|V_1 - V_0\|_{\mathcal{C}}\Big), \tag{3.1}$$
$$\|U_n - V_n\|_{\mathcal{C}} \le \frac{k_1^n}{1-k_1}\Big(\|U_1 - U_0\|_{\mathcal{C}} + \|V_1 - V_0\|_{\mathcal{C}}\Big). \tag{3.2}$$

Proof. Let $U := \mathcal{C}$ be denoted the class of continuously differentiable functions on finite interval $[0,1]$. Then, $(U, \|.\|_{\mathcal{C}})$ is a Banach space with metric $\|u\|_{\mathcal{C}} = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \max_{0\le\varsigma\le1}|u(\varsigma)|$.

Suppose $\ominus : U \to U$ is defined by

$$\ominus u(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^{\varsigma}(\varsigma-\tau)^{2-\alpha}u'(\tau)d\tau + \int_0^{\varsigma}(\varsigma-\tau)[\omega(\tau) - \Lambda_2 u(\tau)]d\tau$$

for each $\varsigma \in [0,1]$.

It is clear that if u is a solution of FBTE, then it is equivalent to a fixed point u of the operator $\ominus$.

Now, $\forall\, u, v \in U$ and $\varsigma \in [0,1]$, we have

$$|\ominus(u)(\varsigma) - \ominus(v)(\varsigma)| \le |\Lambda_1||I^{3-\alpha}u'(\varsigma) - I^{3-\alpha}v'(\varsigma)| + |\Lambda_2||I^2u(\varsigma) - I^2v(\varsigma)|$$

$$\leq \frac{|\Lambda_1|}{\Gamma(4-\alpha)}\|u'-v'\|_\infty + \frac{|\Lambda_2|}{2}\|u-v\|_\infty$$
$$\leq \Big(\frac{|\Lambda_2|}{2} + \frac{|\Lambda_1|}{\Gamma(4-\alpha)}\Big)\|u-v\|_C.$$

Then, for each $u,v \in U$, $\|\ominus(u)-\ominus(v)\|_\infty \leq \Big(\frac{|\Lambda_2|}{2} + \frac{|\Lambda_1|}{\Gamma(4-\alpha)}\Big)\|u-v\|_C$.

On the other hand, $\forall\ u,v \in U$ and $\varsigma \in [0,1]$, we have

$$\begin{aligned}|\ominus'(u)(\varsigma)-\ominus'(v)(\varsigma)| &\leq |\Lambda_1||I^{2-\alpha}u'(\varsigma) - I^{2-\alpha}v'(\varsigma)| + |\Lambda_2||I^1 u(\varsigma) - I^1 v(\varsigma)|\\ &\leq \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\|u'-v'\|_\infty + |\Lambda_2|\|u-v\|_\infty\\ &\leq \Big(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\Big)\|u-v\|_C.\end{aligned}$$

Then, for each $u,v \in U$, $\|\ominus'(u)-\ominus'(v)\|_\infty \leq \Big(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\Big)\|u-v\|_C$.

Therefore, we get for each $u,v \in U$,

$$\|\ominus(u)-\ominus(v)\|_C \leq \Big(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\Big)\|u-v\|_C = k\|u-v\|_C.$$

By Banach contraction principle, $\exists\ u^* \in U$ such that $u^* = \ominus u^*$. Also, $\ominus$ is a Picard operator with $\lim_{n\to\infty}\ominus^n u = u^*,\ u \in U$.

The assertion II. of the theorem is an obvious consequence of the Banach contraction mapping principle. Now, we show that the assertion III. of the theorem is correct.

Let (U_0,V_0) be an arbitrary element of $U\times U$ (not necessarily a coupled lower and upper solution). For each $n \in \mathbb{N}$, we let

$$\begin{aligned}U_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) &- \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^\varsigma (\varsigma-\tau)^{2-\alpha}V'_{n-1}(\tau)d\tau\\ &+ \int_0^\varsigma (\varsigma-\tau)[\omega(\tau) - \Lambda_2 U_{n-1}(\tau)]d\tau,\end{aligned}$$

and

$$\begin{aligned}V_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) &- \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^\varsigma (\varsigma-\tau)^{2-\alpha}U'_{n-1}(\tau)d\tau\\ &+ \int_0^\varsigma (\varsigma-\tau)[\omega(\tau) - \Lambda_2 V_{n-1}(\tau)]d\tau\end{aligned}$$

for all $\varsigma \in [0,1]$.

It is easy to see that

$$\|U_n - u^*\|_\infty \leq \max\left\{\frac{|\Lambda_2|}{2}, \frac{|\Lambda_1|}{\Gamma(4-\alpha)}\right\}\Big(\|U_{n-1}-u^*\|_C + \|V_{n-1}-u^*\|_C\Big), \quad (3.3)$$

$$\|V_n - u^*\|_\infty \le \max\left\{\frac{|\Lambda_2|}{2}, \frac{|\Lambda_1|}{\Gamma(4-\alpha)}\right\}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big), \quad (3.4)$$

$$\|U_n' - (u^*)'\|_\infty \le \max\left\{|\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\right\}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big), \quad (3.5)$$

and

$$\|V_n' - (u^*)'\|_\infty \le \max\left\{|\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\right\}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big). \quad (3.6)$$

Here, using (3.3) and (3.5), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|U_n - u^*\|_C &\le \max\left\{|\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\right\}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big)\\ &\le \frac{k_1}{2}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big).\end{aligned} \quad (3.7)$$

Furthermore, using (3.4) and (3.6), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|V_n - u^*\|_C &\le \max\left\{|\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)}\right\}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big)\\ &\le \frac{k_1}{2}\Big(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C\Big).\end{aligned} \quad (3.8)$$

By induction, (3.7) and (3.8), we obtain

$$\|U_n - u^*\|_C \le \frac{k_1^n}{2}\Big(\|U_0 - u^*\|_C + \|V_0 - u^*\|_C\Big), \quad n \in \mathbb{N},$$

and

$$\|V_n - u^*\|_C \le \frac{k_1^n}{2}\Big(\|U_0 - u^*\|_C + \|V_0 - u^*\|_C\Big).$$

Letting $n \to \infty$ in above equations, we see that the sequences $\{U_n\}$ and $\{V_n\}$ converge to u^*.

A similar argument can also obtain the following relations:

$$\|U_{n+1} - U_n\|_C \le \frac{k_1^n}{2}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big),$$

and

$$\|V_{n+1} - V_n\|_C \le \frac{k_1^n}{2}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big).$$

Then for any $m > n \ge 1$,

$$\begin{aligned}\|U_m - U_n\|_C &\le \sum_{j=0}^{m-n-1} \|U_{n+j+1} - U_{n+j}\|_C\\ &\le \sum_{j=0}^{m-n-1} \frac{k_1^{n+j}}{2}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big)\end{aligned}$$

$$\leq \frac{1}{2}\frac{k_1^n - k_1^m}{1-k_1}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big),$$

and

$$\begin{aligned}\|V_m - V_n\|_C &\leq \sum_{j=0}^{m-n-1} \|U_{n+j+1} - U_{n+j}\|_C \\ &\leq \sum_{j=0}^{m-n-1} \frac{k_1^{n+j}}{2}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big) \\ &\leq \frac{1}{2}\frac{k_1^n - k_1^m}{1-k_1}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big).\end{aligned}$$

Letting $m \to \infty$, we conclude the error estimate (3.1) as following.

$$\|U_n - u^*\|_C \leq \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big),$$

and

$$\|V_n - u^*\|_C \leq \frac{1}{2}\frac{k_1^n}{1-k_1}\Big(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C\Big).$$

Finally, (3.2) follows immediately from (3.1) □

Remark 1. Consider the following equations

$$\begin{cases} u''(\varsigma) + \Lambda_1 \mathcal{D}^\alpha u(\varsigma) = h(\varsigma), \quad 0 < \varsigma < 1, 1 < \alpha < 2, \\ u(0) = \mu,\ u'(0) = \nu, \end{cases} \tag{3.9}$$

and

$$\begin{cases} v''(\varsigma) + \Lambda_1 \mathcal{D}^\alpha v(\varsigma) = h(\varsigma), \quad 0 < \varsigma < 1, 1 < \alpha < 2, \\ v(1) = 0,\ v'(0) = 0. \end{cases} \tag{3.10}$$

One may observe that if u is a solution of (3.9), then $v(\varsigma) = u(\varsigma) - u(1) + (1-\varsigma)u'(0)$ is a solution of (3.10). Further, if v is a solution of (3.10), then $u(\varsigma) = v(\varsigma) - v(0) + \mu + \nu t$ is a solution of (3.9). Note that, $v''(\varsigma) = u''(\varsigma)$ and $\mathcal{D}^\alpha v(\varsigma) = \mathcal{D}^\alpha u(\varsigma)$.

4. A Concluding remark and some examples

An improved existence-uniqueness theorem is established for FBTE under more relaxed assumptions. We prove our main result of this paper by using entirely different approach. The feature of Theorem 3 in this paper compared to the consequences of Theorem 3.4 and Theorem 3.6 of [15] can be viewed under:

- In Theorem 3.4 and Theorem 3.6 of [15], the authors needed the concept of coupled solutions for FBTE, while Theorem 3 does not require this notion.
- Theorem 3.4 and Theorem 3.6 of [15] are based on the product $\Lambda_1 \cdot \Lambda_2$. The proof of Theorem 3, however, is freelance of the $\Lambda_1 \cdot \Lambda_2$.

- $\ominus$ is a Picard operator, that is,

$$\lim_{n\to\infty} \ominus^n u = u^*, \quad u \in U.$$

Therefore, for each $u_0 \in U$ (not necessarily a lower solution), $\lim_{n\to\infty} u_n = u^*$, where

$$u_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^\varsigma (\varsigma-\tau)^{2-\alpha} u'_{n-1}(\tau)d\tau$$
$$+ \int_0^\varsigma (\varsigma-\tau)[\omega(\tau) - \Lambda_2 u_{n-1}(\tau)]d\tau,$$

for each $n \in \mathbb{N}$.

- That $\max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$ is an essential condition to prove the existence and uniqueness in Theorem 3.4 of [15]. In Theorem 3, the existence and uniqueness is established under the condition $|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$ which is less restrictive.
- Remark 1 demonstrates that part (i) of Theorem 3.4 and Theorem 3.6 in [15] are a weak version of Theorem 4.1 of [24] where we put $T = 1, \mu = 0, \varphi_1 = |\Lambda_2|$ and $\varphi_2 = \varphi_3 = 0$.

Here, we provide an example that can not be discussed using Theorem 3.4 and Theorem 3.6 of [15].

Consider the following particular FBTE

$$\begin{cases} u''(\varsigma) + \frac{3}{5}\mathcal{D}^{\frac{4}{3}}u(\varsigma) - \frac{1}{4}u(\varsigma) = \frac{1}{10}\varsigma^2 - \frac{1}{4}\varsigma - \frac{8\sqrt{\varsigma}}{\sqrt{\pi}}, \quad 0 < \varsigma < 1, \\ u(0) = 0,\ u'(0) = 1. \end{cases} \tag{4.1}$$

Therefore, we get $k = |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} = 0.9146393007 < 1$. Thus, by Theorem 3, equation (4.1) has a unique solution in $\mathcal{C}$.

We observe that the product $\Lambda_1 \cdot \Lambda_2 < 0$ and

$$k_1 = \max\left\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\right\} = 1.354055000 > 1.$$

Thus, Theorem 3.4 of [15] does not work herein.

Consider the following particular FBTE

$$\begin{cases} u''(\varsigma) - \frac{2}{5}\mathcal{D}^{\frac{3}{2}}u(\varsigma) - \frac{1}{2}u(\varsigma) = \omega(\varsigma), \quad 0 < \varsigma < 1, \\ u(0) = 0,\ u'(0) = \frac{9}{16}, \end{cases} \tag{4.2}$$

where $\omega(\varsigma) = \frac{-1}{2}\varsigma^3 - \frac{3}{4}\varsigma^2 + \frac{183}{32}\varsigma - 3 - \frac{4\sqrt{\varsigma}(-3+4\varsigma)}{5\sqrt{\pi}}$. Here $\Lambda_1 = \frac{-2}{5}, \Lambda_2 = \frac{-1}{2}, \alpha = \frac{3}{2}, \mu = 0$ and $\nu = \frac{9}{16}$. Therefore, we can obtain that $k = |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} = 0.95135 < 1$ and the exact solution is $u^*(\varsigma) = \varsigma^3 - \frac{3}{2}\varsigma^2 + \frac{9}{16}\varsigma$. Thus, by Theorem 3, equation (4.2) has a unique solution $u^* \in \mathcal{C}$. Moreover, the iterative sequence $\{u_n\}$ with an initial function u_0 (not necessarily a lower solution) defined by

$$u_n(\varsigma) = \mu + \nu\Big(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)}\varsigma^{3-\alpha}\Big) - \frac{\Lambda_1}{\Gamma(3-\alpha)}\int_0^{\varsigma}(\varsigma-\tau)^{2-\alpha}u'_{n-1}(\tau)d\tau$$
$$+\int_0^{\varsigma}(\varsigma-\tau)[\omega(\tau) - \Lambda_2 u_{n-1}(\tau)]d\tau,$$

converges to u^* in $\mathcal{C}$. The graphs of $u_n, n = 0,1,2$ with the initial point $u_0(\varsigma) = \sin(\varsigma)$ are shown in Figure 1 whereas graphs with the initial point $u_0(\varsigma) = \cos(\varsigma)$ are shown in Figure 2. Here, it is important to say that Figure 3. shows that equation (2.5) is not true for initial function $u_0(\varsigma) = \sin(\varsigma)$. Then the mapping $u_0(\varsigma) = \sin(\varsigma)$ is not a lower solution of the problem (4.2). Hence, we can not conclude this approximations by the consequences reported in [15].

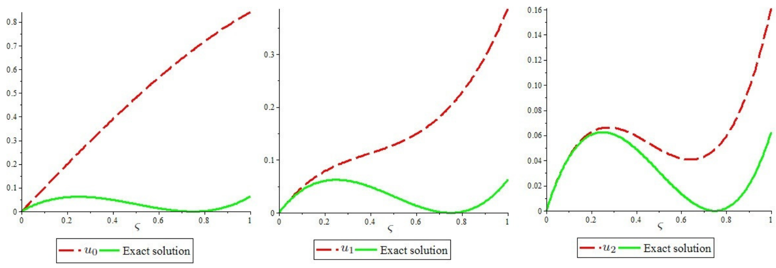

FIGURE 1. The initial function is $u_0(\varsigma) = \sin(\varsigma)$

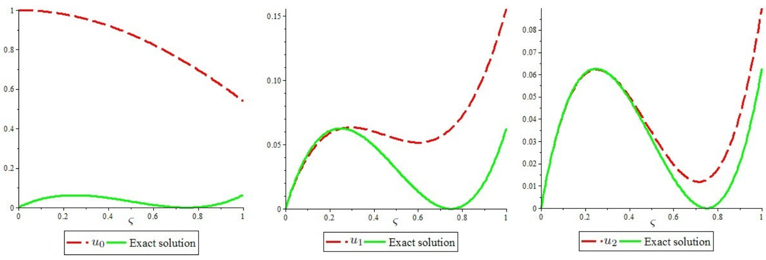

FIGURE 2. The initial function is $u_0(\varsigma) = \cos(\varsigma)$

DECLARATIONS

Availability of data and material
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

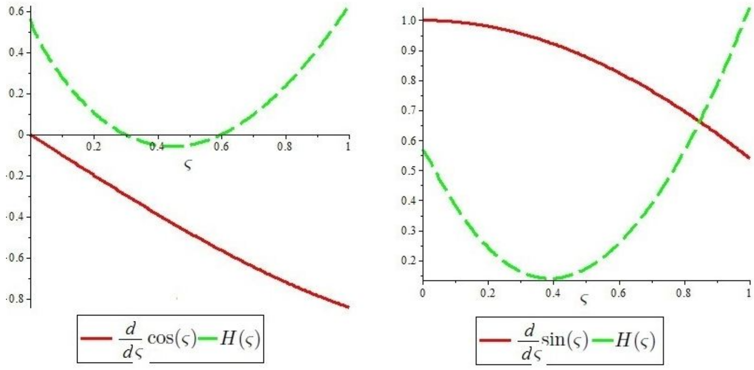

FIGURE 3. $u_0(\varsigma) = \sin(\varsigma)$ is not a lower solution of the problem (4.2)

Competing Interests
The authors declare no competing interests in this work.

Funding
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments
M. Fečkan is thankful to the Slovak Research and Development Agency under the contract No. APVV-18-0308, and the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20. J. Alzabut is thankful to Prince Sultan University and OSTIM Technical University for supporting this work.

REFERENCES

[1] A. Berhail, N. Bouache, M. M. Matar, J. Alzabut, "On nonlocal integral and derivative boundary value problem of nonlinear Hadamard Langevin equation with three different fractional orders," *Bol. Soc. Mat. Mex.*, vol. 26, pp. 303–318, 2019, doi: 10.1007/s40590-019-00257-z.

[2] M. Ahmad, A. Zada, and J. Alzabut, "Stability analysis of a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian," *Adv. Difference Equ.*, vol. 2019, p. 22, 2019, id/No 436, doi: 10.1186/s13662-019-2367-y.

[3] H. Baghani, "Existence and uniqueness of solutions to fractional Langevin equations involving two fractional orders," *J. Fixed Point Theory Appl.*, vol. 20, no. 2, p. 7, 2018, id/No 63, doi: 10.1007/s11784-018-0540-7.

[4] H. Baghani, "An analytical improvement of a study of nonlinear Langevin equation involving two fractional orders in different intervals," *J. Fixed Point Theory Appl.*, vol. 21, no. 4, p. 11, 2019, id/No 95, doi: 10.1007/s11784-019-0734-7.

[5] H. Baghani and J. J. Nieto, “On fractional Langevin equation involving two fractional orders in different intervals,” *Nonlinear Anal., Model. Control*, vol. 24, no. 6, pp. 884–897, 2019, doi: 10.15388/NA.2019.6.3.

[6] R. L. Bagley and P. J. Torvik, “A theoretical basis for the application of fractional calculus to viscoelasticity,” *J. Rheol.*, vol. 27, pp. 201–210, 1983, doi: 10.1122/1.549724.

[7] R. Bagley, “On the equivalence of the Riemann-Liouville and the Caputo fractional order derivatives in modeling of linear viscoelastic materials,” *Fract. Calc. Appl. Anal.*, vol. 10, no. 2, pp. 123–126, 2007.

[8] R. L. Bagley and P. J. Torvik, “Fractional calculus in the transient analysis of viscoelastically damped structures,” *AIAA J.*, vol. 23, pp. 918–925, 1985, doi: 10.2514/3.9007.

[9] A. B. Basset, “On the motion of a sphere in a viscous liquid.” *Philos. Trans. R. Soc. Lond., A*, vol. 179, pp. 43–64, 1888, doi: 10.1098/rsta.1888.0003.

[10] A. B. Basset, “On the descent of a sphere in a viscous liquid.” *Quart. J.*, vol. 41, pp. 369–381, 1910.

[11] W. T. Coffey, Y. P. Kalmykov, and J. T. Waldron, *The Langevin equation. With applications to stochastic problems in physics, chemistry and electrical engineering.*, 2nd ed., ser. World Sci. Ser. Contemp. Chem. Phys. River Edge, NJ: World Scientific, 2004, vol. 14.

[12] K. Diethelm and N. J. Ford, “Numerical solution of the Bagley-Torvik equation.” *BIT*, vol. 42, no. 3, pp. 490–507, 2002.

[13] K. Diethelm and N. J. Ford, “Analysis of fractional differential equations,” *J. Math. Anal. Appl.*, vol. 265, no. 2, pp. 229–248, 2002.

[14] M. El-Gamel and M. A. El-Hady, “Numerical solution of the Bagley-Torvik equation by Legendre-collocation method,” *S⃗eMA J.*, vol. 74, no. 4, pp. 371–383, 2017, doi: 10.1007/s40324-016-0089-6.

[15] H. Fazli and J. J. Nieto, “An investigation of fractional Bagley-Torvik equation,” *Open Math.*, vol. 17, pp. 499–512, 2019, doi: 10.1515/math-2019-0040.

[16] H. Baghani, “On a coupled system of fractional differential equations: Existence and uniqueness revisited,” Submitted.

[17] H. Baghani, J. Alzabut, J. J. Nieto, “Further Results on the Existence of Solutions for Generalized Fractional Basset - Boussinesq - Oseen Equation,” *Iran. J. Sci. Technol. Trans. A-Sci*, vol. 44, pp. 1461–1467, 2020, doi: 10.1007/s40995-020-00942-z.

[18] H. Zhou, J. Alzabut, L. Yang, “On fractional Langevin differential equations with anti-periodic boundary conditions,” *The European Physical Journal Special Topics*, vol. 226, pp. 3577–3590, 2017, doi: 10.1140/epjst/e2018-00082-0.

[19] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, ser. North-Holland Math. Stud. Amsterdam: Elsevier, 2006, vol. 204.

[20] R. C. Koeller, “Applications of fractional calculus to the theory of viscoelasticity,” *J. Appl. Mech.*, vol. 51, pp. 299–307, 1984, doi: 10.1115/1.3167616.

[21] W. Labecca, O. Guimarães, and J. R. C. Piqueira, “Analytical solution of general Bagley-Torvik equation,” *Math. Probl. Eng.*, vol. 2015, p. 4, 2015, id/No 591715, doi: 10.1155/2015/591715.

[22] D. Pang, W. Jiang, J. Du, and A. U. K. Niazi, “Analytical solution of the generalized Bagley-Torvik equation,” *Adv. Difference Equ.*, vol. 2019, p. 13, 2019, id/No 207, doi: 10.1186/s13662-019-2082-8.

[23] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, ser. Math. Sci. Eng. San Diego, CA: Academic Press, 1999, vol. 198.

[24] S. Staněk, “Two-point boundary value problems for the generalized Bagley-Torvik fractional differential equation,” *Cent. Eur. J. Math.*, vol. 11, no. 3, pp. 574–593, 2013, doi: 10.2478/s11533-012-0141-4.

[25] P. J. Torvik and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials," *J. Appl. Mech.*, vol. 51, pp. 294–298, 1984, doi: 10.1115/1.3167615.

Authors' addresses

Hamid Baghani
(corresponding) Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.
E-mail address: h.baghani@gmail.com

Michal Fečkan
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia;, and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia.
E-mail address: Michal.Feckan@gmail.com

Javad Farokhi-Ostad
Department of Basic Sciences, Birjand University of Technology, Birjand, Iran.
E-mail address: j.farrokhi@birjandut.ac.ir

Jehad Alzabut
Department of Mathematics and Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia., and Department of Industrial Engineering, OSTIM Technical University,Ankara 06374, Türkiye
E-mail address: jalzabut@psu.edu.sa and jehad.alzabut@ostimteknik.edu.tr