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SOME NOTES ON MODULES IN WHICH ALL SUBMODULES HAVE A UNIQUE CLOSURE

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Abstract. A module M is called a UC-module if whenever every submodule of M has a unique closure. In this paper, we establish new characterizations of several well-studied classes of rings in terms of UC-modules, and show that UC is not a Morita invariant property. In addition, we study the behaviour of UC-modules under excellent extensions of rings.

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1. INTRODUCTION

Throughout this note all rings are associative with unity and R denotes such a ring. Modules are unital and M_R shall indicate that M is a right R-module. Unless stated otherwise, all R-modules are understood to be right R-modules. Let M be an R-module. A submodule C of M is a *complement* of submodule A in M if C is maximal such that $C \cap A = 0$. A submodule C of M is *closed* in M provided C has no proper essential extension in M. For a submodule C of M, C is a closed submodule if and only if C is a complement submodule [7, 1.10]. The intersection of any two closed submodules of a module may not be closed [13, Example 1.6]. It is well known that, for any submodule A of M, there exists a closed submodule C of M such that A is essential in C, and C is called a *closure* of A (in M). Smith [21] defines a module M to be a *UC-module* if every submodule has a unique closure, or equivalently, the intersection of any two closed submodules of M is also closed. Smith in his study [21] provides 20 different characterizations of UC-modules, and later UC-modules were studied by many authors [2, 6, 8, 11, 15, 16]. We should note that UC-modules are called *dimension modules* in [5]. It is well known that every direct summand of a module M is a closed submodule of M. The converse is not true, generally (for example, the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$). A module M is called an *extending* (or *CS*) *module* if every closed submodule of M is a direct summand of M [9]. Wilson [22] says that a module M has the *summand intersection property* (in short, *SIP-module*) if the intersection of any two direct summands is again a direct summand. UC-modules and SIP-modules coincide when the module is extending [2, Lemma 17].

In this paper, we first prove in Proposition 1 that if M is a UC-module, then for every decomposition $M = A \oplus B$ and every R-homomorphism $f : A \to B$, $Ker(f)$ is a complement submodule of M. This proposition is key to our work in this paper and is used to characterize many well known classes of rings in terms of UC-modules. For example, we show in Theorem 1 that a ring R is semisimple if and only if every R-module is a UC-module; or equivalently, every injective R-module is a UC-module; or equivalently, every UC-module is injective. We prove in Theorem 2 that a ring R is a right V-ring (that is, every simple right R-module is injective) if and only if every finitely cogenerated R-module is a UC-module; or equivalently, every finitely copresented R-module is a UC-module. In Corollary 2, we prove that a ring R is an SSI-ring if and only if R is a right Noetherian ring and every finitely cogenerated (or finitely copresented) right R-module is a UC-module. Later, it is shown in Proposition 2 that if the class of UC-R-modules is closed under finite direct sums, then R is a right V-ring. It is proved in Theorem 3 that a ring R is semisimple if and only if the following are satisfied: (1) R is a right Noetherian ring, and (2) the class of UC-R-modules is closed under arbitrary direct sums. We give a new characterization of SI-rings in Theorem 4 that a ring R is a right SI-ring if and only if $Z(R_R) = 0$ and every singular right R-module is a UC-module. Analogous to an idea of Enochs [10], we introduce the notion of UC-cover, and we prove that R is a semisimple ring if and only if every right R-module has a UC-cover (Theorem 5). At the end of this section, we show in Example 2 that UC is not a Morita invariant property.

Section 4 is devoted to the behaviour of UC-modules under excellent extensions of rings. We prove in Theorem 8 that if M is a right S-module, then M_R is a UC-module if and only if M_S is a UC-module, and prove in Theorem 9 that if M is a right R-module, then $(M \otimes_R S)_S$ is a UC-module if and only M_R is a UC-module. Let S be a right excellent extension of R. Then R is right finitely Σ-UC if and only if S is also (Theorem 10).

For a submodule A of M, the notation $A \leq M$, $A \leq^{ess} M$, $A \leq^{\oplus} M$ and $A \leq^{c} M$ mean that A is a submodule, an essential submodule, a direct summand and a complement submodule of M, respectively. For a module M, we use $Z(M)$ and $E(M)$ to denote the singular submodule and the injective hull, respectively. $\mathbb{CFM}_{\Lambda}(R)$ denotes the column finite $card(\Lambda) \times card(\Lambda)$ matrix ring over R, where $card(\Lambda)$ is the cardinality of Λ. For a module M, $M^{(I)}$ is the direct sum of copies of M indexed by a set I. For definitions and notations which are not given, please see [3].

2. UC-MODULES

First, we begin by proving a useful proposition:

Proposition 1. *If M is a UC-module, then for every decomposition $M = A \oplus B$ and every R-homomorphism $g : A \to B$, $Ker(g) \leq^{c} M$.*

Proof. Assume M is a UC-module. Let $M = A \oplus B$ and $g : A \to B$ be an R-homomorphism. Let $C = \{u + g(u) \mid u \in A\}$. We want to show that $M = C \oplus B$. Let $x \in M$, then $x = u + v$ where $u \in A$ and $v \in B$. Now, $x = u + g(u) - g(u) + v$. But $u + g(u) \in C$ and $-g(u) + v \in B$. So, $M = C + B$. Let us choose $x \in C \cap B$. We can write $x = u + g(u)$ where $u \in A$ and hence $u = x - g(u) \in A \cap B = 0$. Therefore $g(u) = 0$ which gives $x = 0$. So, $M = C \oplus B$. Since M is a UC-module, an intersection of closed submodules is closed, thus $C \cap A$ is a closed submodule of M. It is a straightforward matter to show that $C \cap A = Ker(g)$. Thus, $Ker(g) \leq^c M$. □

Corollary 1. *Let M be an R-module. If $E(M) \oplus E(E(M)/M)$ is a UC-module, then M is injective.*

Proof. It is *mutatis mutandis* the same as the proof (3) ⇒ (1) of [14, Theorem 4.12]. □

3. Characterizations of rings in terms of UC-modules

Let R be a ring. R is semisimple if and only if every R-module is semisimple if and only if every R-module is injective [23, 20.3] if and only if every injective R-module has the SIP [22, Proposition 3]. Recall from [4] that R is said to be an *SSI-ring* if every semisimple R-module is injective; or equivalently, R is a right Noetherian V-ring [4, Proposition 1]. First, we provide some characterizations of semisimple rings:

Theorem 1. *The following conditions are equivalent for a ring R:*

(1) *R is semisimple;*
(2) *Every R-module is a UC-module;*
(3) *Every injective R-module is a UC-module;*
(4) *Every UC-module is injective.*

Proof. (1) ⇒ (2) Since R is semisimple, every R-module is semisimple. Then, every R module is a UC-module (see also [5, Corollary 3]).
(2) ⇒ (3) Clear.
(3) ⇒ (1) We want to show that every injective R-module has the SIP. Since every injective R-module is a UC-module, then every injective R-module has the SIP. So, R is semisimple by [22, Proposition 3].
(1) ⇒ (4) Since R is semisimple then every R-module is injective.
(4) ⇒ (1) Suppose that every UC-module is injective. Then, every semisimple module is injective. Thus, R is an SSI-ring, and hence R is a right Noetherian V-ring by [4, Proposition 1]. Since R is right Noetherian, $E(R) = \bigoplus_{\lambda \in \Lambda} E_\lambda$, where E_λ is an indecomposable injective for each $\lambda \in \Lambda$. Let $0 \neq e \in E_\lambda$. It follows that $eR \leq E_\lambda$ is a uniform submodule of E_λ. Thus, eR is a UC-module, and hence eR is injective by the hypothesis. Hence $eR \leq^{\oplus} E_\lambda$. Since E_λ is indecomposable, $E_\lambda = eR$. Then E_λ is a simple R-module for each $\lambda \in \Lambda$. Therefore, $E(R)$ is a semisimple R-module, and hence R is a semisimple ring. □

Remark 1. The proof of $(4) \Rightarrow (1)$ can also be proved in a different way: Assume (4) holds. Then R is an SSI-ring. Since R is a product of simple rings, it is right nonsingular and thus a UC-module, whence injective. Since simple self-injective rings are von Neumann regular (that is, every principal ideal is a direct summand of R_R), the Noetherian condition implies semisimplicity.

An R-module M is finitely cogenerated if and only if $Soc(M)$ is finitely generated and essential in M [23, 21.3]. An R-module X is called *finitely copresented* if (i) X is finitely cogenerated and (ii) in every exact sequence $0 \to X \to Y \to Z \to 0$ in Mod-R with Y finitely cogenerated, Z is also finitely cogenerated [23, p.248]. Recall from [23] that a ring R is called a right *V-ring* (*co-semisimple*) if every simple right R-module is injective. A ring R is a right V-ring if and only if every finitely cogenerated R-module is semisimple if and only if every finitely cogenerated R-module is injective [23, 23.1] if and only if every finitely copresented R-module is semisimple if and only if every finitely copresented R-module is injective [23, 31.7].

Theorem 2. *The following conditions are equivalent for a ring R:*

(1) *R is a right V-ring;*
(2) *Every finitely cogenerated R-module is a UC-module;*
(3) *Every finitely copresented R-module is a UC-module.*

Proof. $(1) \Rightarrow (2)$ This is immediate, since it has already been noted that if R is a right V-ring, then every finitely cogenerated R-module is semisimple and thus UC.
$(2) \Rightarrow (3)$ It is clear since every finitely copresented module is finitely cogenerated.
$(3) \Rightarrow (1)$ Let M be a finitely copresented R-module. We will show that M is injective. By [23, 30.1], $E(M)$ and $E(M)/M$ are finitely cogenerated. Since $E(M)/M$ is finitely cogenerated, $E(E(M)/M)$ is finitely cogenerated. Since any finitely cogenerated injective module is finitely copresented, by condition (3) and [23, 21.4], $E(M) \oplus E(E(M)/M)$ is a UC-module. By Corollary 1, M is injective. By [23, 31.7], R is a right V-ring. □

Corollary 2. *A ring R is an SSI-ring if and only if R is a right Noetherian ring and every finitely cogenerated (or finitely copresented) right R-module is a UC-module.*

Proof. Note that R is an SSI-ring if and only if R is a right Noetherian, right V-ring [4, Proposition 1]. The equivalence holds true by Theorem 2. □

Proposition 2. *If the class of UC-R-modules is closed under finite direct sums, then R is a right V-ring.*

Proof. Let M be a finitely cogenerated R-module. Then M is a finite direct sum of uniform R-modules. Since uniform modules are UC-modules, M is a UC-module by the hypothesis. Hence, by Theorem 2, R is a right V-ring. □

Theorem 3. *A ring R is semisimple if and only if the following are satisfied:*

(1) *R is a right Noetherian ring,*

(2) *The class of UC-R-modules is closed under arbitrary direct sums.*

Proof. ($\Rightarrow$:) (1) Clear.
(2) If R is semisimple, then every R-module is a UC-module by Theorem 1(2).
($\Leftarrow$:) Let M be an injective R-module. Since R is a right Noetherian ring, M is a direct sum of uniform modules. Since uniform modules are UC-modules, M is a UC-module. Hence, by Theorem 1(3), R is semisimple. □

Any module isomorphic to the factor M/N of an essential extension $N \leq M$ is called a *singular module*. Now, we recall some facts about singular modules:
Fact 1: If $Z(R_R) = 0$, the class of all singular right R-modules is closed under essential extensions [13, Proposition 1.23(c)].
Fact 2: The class of all singular right R-modules is closed under factor modules, and direct sums [13, Proposition 1.22(b)].
Fact 3: Let M be an R-module. M/N is singular whenever $N \leq^{ess} M$. Thus, $E(M)/M$ is always singular. The converse of this assertion is not true in general, please see [13, p. 32].
A ring R is called a *right SI-ring* if every singular right R-module is injective [12]. In the next theorem we give a new characterization of SI-rings:

Theorem 4. *The following are equivalent for a ring R:*

(1) *R is a right SI-ring;*
(2) $Z(R_R) = 0$ *and every singular right R-module is a UC-module.*

Proof. (1) $\Rightarrow$ (2) If R is a right SI-ring, then every singular right R-module is semisimple by [20, Lemma 3.1]. On the other hand, it is clear that R is right nonsingular. Hence, (2) holds.
(2) $\Rightarrow$ (1) Let M be a singular right R-module. We want to show that M is injective. By Fact 1, $E(M)$ is singular. Moreover, by Facts 1 and 3, $E(E(M)/M)$ is a singular R-module. It follows that $E(M) \oplus E(E(M)/M)$ is a singular module from Fact 2. Then, by the hypothesis, $E(M) \oplus E(E(M)/M)$ is a UC-module. Hence, by Corollary 1, M is injective. So, R is a right SI-ring. □

It is well-known that R is right Noetherian if and only if every direct sum of injective modules is injective.

Lemma 1. *If the direct sum of any family of injective envelopes of simple right R-modules is a UC-module, then R is a right Noetherian ring.*

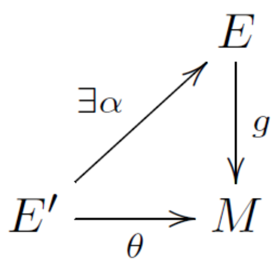
Proof. Let $\{S_i\}_{i \in I}$ be a family of simple modules. Set $M = \oplus_{i \in I} E(S_i)$. We will show that M is injective. By the hypothesis, $E(M) \oplus E(E(M)/M)$ is a UC-module. By Corollary 1, M is injective. So, R is right Noetherian. □

The next example shows that converse of Lemma 1 is not true generally.

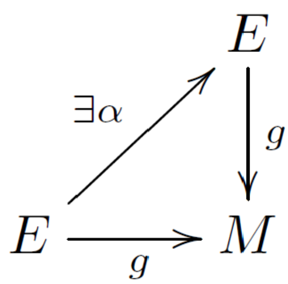
Example 1. Consider $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. It is well-known that the ring of integer $\mathbb{Z}$ is a right Noetherian ring, and M is an injective $\mathbb{Z}$-module. It is proved in [14, Example 2.4(1)] that M does not have the *SIP*. Then, M is not a UC-module by [2, Lemma 17].

Enochs [10] introduced the injective cover notion which is the dual to the injective envelope, and showed that a ring R is a right Noetherian ring if and only if every right R-module has an injective cover. Now, we introduce the UC-cover notion.

Definition 1. An R-homomorphism $g : E \to M$ is called a *UC-cover* of a right R-module M if E is a UC-module such that any diagram



with E' a UC-module can be completed; and the diagram



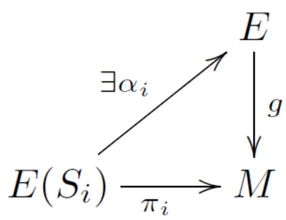
can be completed only by an automorphism α.

Now, we prove in Theorem 5 that a ring R is semisimple if and only if every right R-module has a UC-cover.

Theorem 5. *The following are equivalent for a ring R:*

(1) *R is semisimple;*
(2) *Every right R-module has a UC-cover.*

Proof. $(1) \Rightarrow (2)$ Obvious.
$(2) \Rightarrow (1)$ First, we want to prove R is right Noetherian. Let $\{S_i\}_{i\in I}$ be a family of simple right R-module and let $M = \bigoplus_{i\in I} E(S_i)$. Call $g : E \to M$ a UC-cover of M. Consider the following diagram:



where π_i is the canonical injection for $i \in I$. Note that all modules $E(S_i)$ are uniform and injective modules. It follows that all modules $E(S_i)$ are UC-modules. By the definition of UC-cover, there exists a homomorphism $\alpha_i : E(S_i) \to E$ such that $g\alpha_i = \pi_i$ for $i \in I$. Define $\alpha : M \to E$ by $\alpha(\Sigma_{i=1}^n x_i) = \Sigma_{i=1}^n \alpha_i(x_i)$ for $x_i \in (E(S_i))$ and $i \in I$. It can easily be checked that α is well-defined and we have

$$g\alpha(\Sigma_{i=1}^n x_i) = \Sigma_{i=1}^n g\alpha_i(x_i) = \Sigma_{i=1}^n \pi_i(x_i) = \Sigma_{i=1}^n x_i.$$

Thus, $g\alpha = 1_M$, and $\alpha : M \to E$ is a split monomorphism. Then $M \cong D \leq^{\oplus} E$. Since a direct summand of a UC-module is again a UC-module, M is a UC-module. By Lemma 1, R is a right Noetherian ring. With similar argument, we can prove that an arbitrary direct sum $M = \bigoplus_{i \in I} M_i$ of right UC-modules is again a UC-module. Hence, by Theorem 3, R is semisimple. □

Theorem 6. *Let $R = \prod_{\alpha \in \Lambda} R_\alpha$ be a product of rings. Then R is right UC if and only if each R_α is right UC.*

Proof. Let π_α be the αth projection map and i_α the αth inclusion map canonically.
($\Rightarrow$:) Let $0 \neq C_\alpha$ be a closed right ideal of R_α for each R_α. We show that $C = i_\alpha(C_\alpha)$ is a closed right ideal of R. If C is not a closed right ideal of R, there is a right ideal B of R such that C is properly contained in B and $C \leq^{ess} B$. If $\beta \neq \alpha$, then $C \leq^{ess} B$ implies that $\pi_\beta(B) = 0$. Thus $B = i_\alpha \pi_\alpha(B)$. Since C is properly contained in B and $C \leq^{ess} B$, C_α is properly contained in $\pi_\alpha(B)$ and $C_\alpha \leq^{ess} \pi_\alpha(B)$. This is impossible because C_α is a closed right ideal of R_α. Thus C is a nonzero closed right ideal of R. The rest is straightforward.
($\Leftarrow$:) Let $0 \neq C$ be a closed right ideal of R. Set $C_\alpha = \pi_\alpha(C)$, $\alpha \in \Lambda$. It can easily be checked that $C = \prod_{\alpha \in \Lambda} C_\alpha$. Since C is a closed right ideal of R, clearly, C_α is a closed right ideal of R_α for each $\alpha \in \Lambda$. Since $C \neq 0$, there exists $\alpha \in \Lambda$ such that C_α is a nonzero closed right ideal of R_α. The rest is straightforward. □

In this section, we give some results about UC-rings. Recall that a ring R is said to be a *right UC-ring* if the module R_R is a UC-module. Let R be a ring, e an idempotent in R such that $R = ReR$, and S the subring eRe. It is clear that if M is a right R-module, then Me is a right S-module.

Theorem 7. *Using the above notation, the module $(Me)_S$ is a UC-module if and only if the module M_R is a UC-module.*

Proof. Immediate by [1, Lemma 5(i)]. □

Corollary 3. *Using the above notation, the ring R is a UC-ring if and only if the module $(Re)_{eRe}$ is a UC-module.*

Proof. This follows immediately from Theorem 7. □

In this note, R^n denotes the set of all $n \times 1$ column matrices over R. Let R be a ring, n a positive integer, $M_n(R)$ the ring of $n \times n$ matrices over R, and e_{11} the matrix in $M_n(R)$ with $(1,1)$ entry 1 and all other entries 0. It is well known that e_{11} is idempotent, $R \cong e_{11}M_n(R)e_{11}$ and $M_n(R) = M_n(R)e_{11}M_n(R)$. Thus, Corollary 3 gives the next two results without further proof.

Corollary 4. *The ring $M_n(R)$ is a UC-ring if and only if the free R-module R^n is a UC-module.*

Corollary 5. *Let R be a ring, and let Λ be an infinite set. Then $\mathbb{CFM}_\Lambda(R)$ is right UC if and only if $R_R^{(\Lambda)}$ is UC.*

In the following example we see that UC is not a Morita invariant property.

Example 2. Consider the ring $\mathbb{Z}_4$. Although $\mathbb{Z}_4$ is UC, the ring $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ of 2×2 matrices over $\mathbb{Z}_4$ is not UC. If R were UC, then by Corollary 4, the right $\mathbb{Z}_4$-module $\mathbb{Z}_4^2$ would be UC. One can now argue that this is not so, since if A and B are the submodules of $\mathbb{Z}_4^2$ generated by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively, then A and B are both direct summands of $\mathbb{Z}_4^2$ (and thus closed submodules), yet their intersection, which is the submodule generated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, is not closed.

4. UC-MODULES AND EXCELLENT EXTENSIONS

Recall from [17] that let R be a subring of a ring S such that they have the same identity. The ring S is called a right excellent extension of R if the following two conditions are satisfied:

(1) ${}_SR$ and R_S are free modules with a basis $\{1 = a_1, a_2, \ldots, a_n\}$ such that $a_iR = Ra_i$ for $i = 1, \ldots, n$.

(2) For any submodule A_S of a module M_S, if A_R is a direct summand of M_R, then A_S is a direct summand of M_S.

Lemma 2 ([18, Proposition 1.6]). *Let A_S be a submodule of an S-module M. Then A_S is closed in M_S if and only if A_R is closed in M_R.*

Lemma 3 ([19, Lemma 2.4]). *Let A_R be a submodule of M_R. Then A_R is a closed submodule of M_R if and only if $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$.*

Theorem 8. *Let M be a right S-module. Then M_R is a UC-module if and only if M_S is a UC-module.*

Proof. Let M_R be a UC-module and A_S and B_S are closed submodules of M_S. By Lemma 2, A_R and B_R are closed submodules of M_R. Since M_R is UC, $A_R \cap B_R = C_R$ is a closed submodule of M_R. By Lemma 2, $A_S \cap B_S = C_S$ is a closed submodule of M_S. So, M_S is a UC-module. The converse can be proved similarly. □

Theorem 9. *Let M be a right R-module. Then $(M \otimes_R S)_S$ is a UC-module if and only M_R is a UC-module.*

Proof. Immediate by the definition of UC-modules and Lemma 3. □

An R-module M is called $finitely\ \Sigma\text{-}UC$ if every finite direct sum of copies of M is UC. The ring R is called $right\ finitely\ \Sigma\text{-}UC$ if R_R is $finitely\ \Sigma\text{-}UC$.

Theorem 10. *Let S be a right excellent extension of R. Then R is right finitely Σ-UC if and only if S is also.*

Proof. Suppose R_R is finitely Σ-UC. Then for any $k > 0$, $(S^k)_R \cong (R^{nk})_R$ is UC. Thus, $(S^k)_S$ is UC by Theorem 8.

For the converse, suppose S_S is finitely Σ-UC. Then for any $k > 0$, $(R^k \otimes_R S)_S \cong (S^k)_S$ is UC. By Theorem 9, $(R^k)_R$ is UC. □

It can easily be checked that the theorem still holds (with the same proof) if "finitely Σ-UC" is replaced by "countably Σ-UC" or "Σ-UC".

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References

[1] E. Akalan, G. F. Birkenmeier, and A. Tercan, "Goldie extending rings." *Comm Algebra.*, vol. 40, pp. 423–428, 2012, doi: 10.1080/00927872.2010.529096.

[2] M. Alkan and A. Harmancı, "On summand sum and summand intersection property of modules," *Turk J Math*, vol. 26, no. 2, pp. 131–147, 2002.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules.* New York: Springer Science+Business Media, LLC, 2012. doi: 10.1007/978-1-4612-4418-9.

[4] K. A. Byrd, "Rings whose quasi-injective modules are injective," *Proceedings of the American Mathematical Society*, vol. 33, no. 2, pp. 235–240, 1972, doi: 10.1090/S0002-9939-1972-0310009-7.

[5] V. Camillo and J. Zelmanowitz, "Dimension modules," *Pacific J. Math.*, vol. 91, no. 2, pp. 249–261, 1980, doi: 10.2140/pjm.1980.91.249.
[6] S. Çeken and M. Alkan, "UC modules with respect to a torsion theory," *Turk J Math*, vol. 36, no. 3, pp. 376–385, 2012, doi: 10.3906/mat-1009-33.
[7] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules: supplements and projectivity in module theory*. Springer Science & Business Media, 2008.
[8] S. Doğruöz, A. Harmanci, and P. F. Smith, "Modules with unique closure relative to a torsion theory II," *Turk J Math*, vol. 33, no. 2, pp. 111–116, 2009, doi: 10.3906/mat-0712-16.
[9] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending modules*. Pitman RN Mathematics 313, Longman, Harlow, 2019. doi: 10.1201/9780203756331.
[10] E. E. Enochs, "Injective and flat covers, envelopes and resolvents," *Israel Journal of Mathematics*, vol. 39, no. 3, pp. 189–209, 1981, doi: 10.1007/BF02760849.
[11] L. Ganesan and N. Vanaja, "Modules for which every submodule has a unique coclosure," *Comm Algebra.*, vol. 30, no. 5, pp. 2355–2377, 2002, doi: 10.1081/AGB-120003473.
[12] K. R. Goodearl, *Singular torsion and the splitting properties*. American Mathematical Soc., 1972, no. 1-124.
[13] K. R. Goodearl, *Ring theory: Nonsingular rings and modules*. CRC Press, 1976, vol. 33.
[14] A. Hamdouni, A. C. Ozcan, and A. Harmanci, "Characterization of modules and rings by the summand intersection property and the summand sum property," *JP J. Algebra Number Theory Appl*, vol. 5, no. 3, pp. 469–490, 2005.
[15] F. Karabacak, M. T. Koşan, T. C. Quynh, D. D. Tai, and Ö. Taşdemir, "On NCS modules and rings," *Comm Algebra.*, vol. 48, no. 12, pp. 5236–5246, 2020, doi: 10.1080/00927872.2020.1784910.
[16] C. Lomp and E. Puczyłowski, "A note on dimension modules," *Comm Algebra.*, vol. 43, no. 6, pp. 2267–2271, 2015, doi: 10.1080/00927872.2014.890725.
[17] M. M. Parmenter and P. N. Stewart, "Excellent extensions," *Comm Algebra.*, vol. 16, no. 4, pp. 703–713, 1988, doi: 10.1080/00927878808823597.
[18] M. M. Parmenter and Y. Zhou, "Relative injectivity of modules and excellent extensions," *Quaestiones Mathematicae*, vol. 22, no. 1, pp. 101–107, 1999, doi: 10.1080/16073606.1999.9632062.
[19] M. M. Parmenter and Y. Zhou, "Finitely $\sum$-cs property of excellent extensions of rings," in *Algebra Colloquium*, vol. 10, no. 1, doi: 10.1007/s100110300003, 2003, pp. 17–21.
[20] S. T. Rizvi and M. F. Yousif, "On continuous and singular modules," in *Non-commutative ring theory*. Springer, 1990, pp. 116–124.
[21] P. F. Smith, "Modules for which every submodule has a unique closure," *Proceedings of the Biennial Ohio-Denison Conference*, pp. 302–313, 1992.
[22] G. V. Wilson, "Modules with the summand intersection property," *Comm Algebra.*, vol. 14, no. 1, pp. 21–38, 1986, doi: 10.1080/00927878608823297.
[23] R. Wisbauer, *Foundations of Module and Ring Theory*. Gordon and Breach Science Publishers, Reading, 1991. doi: 10.1201/9780203755532.

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