

A characterization of Hermitian varieties as codewords

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Abstract

It is known that the Hermitian varieties are codewords in the code defined by the points and hyperplanes of the projective spaces $\text{PG}(r, q^2)$. In finite geometry, also quasi-Hermitian varieties are defined. These are sets of points of $\text{PG}(r, q^2)$ of the same size as a non-singular Hermitian variety of $\text{PG}(r, q^2)$, having the same intersection sizes with the hyperplanes of $\text{PG}(r, q^2)$. In the planar case, this reduces to the definition of a unital. A famous result of Blokhuis, Brouwer, and Wilbrink states that every unital in the code of the points and lines of $\text{PG}(2, q^2)$ is a Hermitian curve. We prove a similar result for the quasi-Hermitian varieties in $\text{PG}(3, q^2)$, $q = p^h$, as well as in $\text{PG}(r, q^2)$, $q = p$ prime, or $q = p^2$, p prime, and $r \geq 4$.

Keywords: Hermitian variety; incidence vector; codes of projective spaces; quasi-Hermitian variety.

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1 Introduction

Consider the non-singular Hermitian varieties $\mathcal{H}(r, q^2)$ in $\text{PG}(r, q^2)$. A non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\text{PG}(r, q^2)$ is the set of absolute points of a Hermitian polarity of $\text{PG}(r, q^2)$. Many properties of a non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\text{PG}(r, q^2)$ are known. In particular, its size is $(q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 -$

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1), and its intersection numbers with the hyperplanes of $\text{PG}(r, q^2)$ are equal to $(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})/(q^2 - 1)$, in case the hyperplane is a non-tangent hyperplane to $\mathcal{H}(r, q^2)$, and equal to $1 + q^2(q^{r-1} + (-1)^r)(q^{r-2} - (-1)^r)/(q^2 - 1)$ in case the hyperplane is a tangent hyperplane to $\mathcal{H}(r, q^2)$; see [16].

Quasi-Hermitian varieties \mathcal{V} in $\text{PG}(r, q^2)$ are generalizations of the non-singular Hermitian variety $\mathcal{H}(r, q^2)$ so that \mathcal{V} and $\mathcal{H}(r, q^2)$ have the same size and the same intersection numbers with hyperplanes.

Obviously, a Hermitian variety $\mathcal{H}(r, q^2)$ can be viewed as a trivial quasi-Hermitian variety and we call $\mathcal{H}(r, q^2)$ the *classical quasi-Hermitian variety* of $\text{PG}(r, q^2)$. In the 2-dimensional case, \mathcal{V} is also known as the classical example of a *unital* of the projective plane $\text{PG}(2, q^2)$.

As far as we know, the only known non-classical quasi-Hermitian varieties of $\text{PG}(r, q^2)$ were constructed in [1, 2, 8, 9, 13, 14].

In [6], it is shown that a unital in $\text{PG}(2, q^2)$ is a Hermitian curve if and only if it is in the \mathbb{F}_p -code spanned by the lines of $\text{PG}(2, q^2)$, with $q = p^h$, p prime and $h \in \mathbb{N}$.

In this article, we prove the following result.

Theorem 1.1. *A quasi-Hermitian variety \mathcal{V} of $\text{PG}(r, q^2)$, with $r = 3$ and $q = p^h > 4$, or $r \geq 4$, $q = p > 4$, or $r \geq 4$, $q = p^2$, $p > 3$ prime, is classical if and only if it is in the \mathbb{F}_p -code spanned by the hyperplanes of $\text{PG}(r, q^2)$.*

Furthermore we consider *singular quasi-Hermitian varieties*, that is point sets having the same number of points as a singular Hermitian variety \mathcal{S} and for which each intersection number with respect to hyperplanes is also an intersection number of \mathcal{S} with respect to hyperplanes. We show that Theorem 1.1 also holds in the case in which \mathcal{V} is assumed to be a singular quasi-Hermitian variety of $\text{PG}(r, q^2)$.

2 Preliminaries

A subset \mathcal{K} of $\text{PG}(r, q^2)$ is a k_{n,r,q^2} if n is a fixed integer, with $1 \leq n \leq q^2$, such that:

- (i) $|\mathcal{K}| = k$;
- (ii) $|\ell \cap \mathcal{K}| = 1, n, \text{ or } q^2 + 1$ for each line ℓ ;
- (iii) $|\ell \cap \mathcal{K}| = n$ for some line ℓ .

A point P of \mathcal{K} is *singular* if every line through P is either a unisecant or a line of \mathcal{K} . The set \mathcal{K} is called *singular* or *non-singular* according as it has singular points or not.

Furthermore, a subset \mathcal{K} of $\text{PG}(r, q^2)$ is called *regular* if

- (a) \mathcal{K} is a k_{n,r,q^2} ;
- (b) $3 \leq n \leq q^2 - 1$;
- (c) no planar section of \mathcal{K} is the complement of a set of type $(0, q^2 + 1 - n)$.

Theorem 2.1. [10, Theorem 19.5.13] *Let \mathcal{K} be a $k_{n,3,q^2}$ in $\text{PG}(3, q^2)$, where q is any prime power and $n \neq \frac{1}{2}q^2 + 1$. Suppose furthermore that every point in \mathcal{K} lies on at least one n -secant. Then $n = q + 1$ and \mathcal{K} is a non-singular Hermitian surface.*

Theorem 2.2. [11, Theorem 23.5.19] *If \mathcal{K} is a regular, non-singular k_{n,r,q^2} , with $r \geq 4$ and $q > 2$, then \mathcal{K} is a non-singular Hermitian variety.*

Theorem 2.3. [11, Th. 23.5.1] *If \mathcal{K} is a singular $k_{n,3,q^2}$ in $\text{PG}(3, q^2)$ with $3 \leq n \leq q^2 - 1$, then the following holds: \mathcal{K} is n planes through a line or a cone with vertex a point and base \mathcal{K}' a plane section of type*

- I. a unital;
- II. a subplane $\text{PG}(2, q)$;
- III. a set of type $(0, n - 1)$ plus an external line;
- IV. the complement of a set of type $(0, q^2 + 1 - n)$.

Theorem 2.4. [11, Lemma 23.5.2 and Th. 25.5.3] *If \mathcal{K} is a singular $k_{n,r,q}$ with $r \geq 4$, then the singular points of \mathcal{K} form a subspace Π_d of dimension d and one of the following possibilities holds:*

1. $d = r - 1$ and \mathcal{K} is a hyperplane;
2. $d = r - 2$ and \mathcal{K} consists of $n > 1$ hyperplanes through Π_d ;
3. $d \leq r - 3$ and \mathcal{K} is equal to a cone $\Pi_d \mathcal{K}'$, with π_d as vertex and with \mathcal{K} as base, where \mathcal{K}' is a non singular $k_{n,r-d-1,q}$.

Result 2.5 ([15]). *Let \mathcal{M} be a multiset in $\text{PG}(2, q)$, $17 < q$, so that the number of lines intersecting it in not $k \pmod{p}$ points is δ , where $\delta < \frac{3}{16}(q + 1)^2$. Then the number of non $k \pmod{p}$ secants through any point is at most $\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3}$ or at least $q + 1 - (\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^3})$.*

Property 2.6 ([15]). *Let \mathcal{M} be a multiset in $\text{PG}(2, q)$, $q = p^h$, where p is prime. Assume that there are δ lines that intersect \mathcal{M} in not $k \pmod{p}$ points. If through a point there are more than $q/2$ lines intersecting \mathcal{M} in not $k \pmod{p}$ points, then there exists a value r such that the intersection multiplicity of at least $2\frac{\delta}{q+1} + 5$ of these lines with \mathcal{M} is r .*

Result 2.7 ([15]). *Let \mathcal{M} be a multiset in $\text{PG}(2, q)$, $17 < q$, $q = p^h$, where p is prime. Assume that the number of lines intersecting \mathcal{M} in not $k \pmod{p}$ points is δ , where $\delta < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor)$. Assume furthermore that Property 2.6 holds. Then there exists a multiset \mathcal{M}' with the property that it intersects every line in $k \pmod{p}$ points and the number of different points in $(\mathcal{M} \cup \mathcal{M}') \setminus (\mathcal{M} \cap \mathcal{M}')$ is exactly $\lceil \frac{\delta}{q+1} \rceil$.*

Result 2.8 ([15]). *Let B be a proper point set in $\text{PG}(2, q)$, $17 < q$. Suppose that B is a codeword of the lines of $\text{PG}(2, q)$. Assume also that $|B| < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor)$. Then B is a linear combination of at most $\lceil \frac{|B|}{q+1} \rceil$ lines. \square*

3 Proof of Theorem 1.1

Let V be the vector space of dimension $q^{2r} + q^{2(r-1)} + \dots + q^2 + 1$ over the prime field \mathbb{F}_p , where the coordinate positions for the vectors in V correspond to the points of $\text{PG}(r, q^2)$ in some fixed order. If S is a subset of points in $\text{PG}(r, q^2)$ then let v^S denote the vector in V with coordinate 1 in the positions corresponding to the points in S and with coordinate 0 in all other positions; that is v^S is the *characteristic vector* of S . Let \mathcal{C}_p denote the subspace of V spanned by the characteristic vectors of all the hyperplanes in $\text{PG}(r, q^2)$. This code \mathcal{C}_p is called *the linear code of $\text{PG}(r, q^2)$* .

From [12, Theorem 1], we know that the characteristic vector $v^\mathcal{V}$ of a Hermitian variety $\mathcal{V} \in \text{PG}(r, q^2)$ is in \mathcal{C}_p . So from now on, we will assume that \mathcal{V} is a quasi-Hermitian variety in $\text{PG}(r, q^2)$ and $v^\mathcal{V} \in \mathcal{C}_p$. In the remainder of this section we will show that \mathcal{V} is a classical Hermitian variety.

The next lemmas hold for $r \geq 3$ and for any $q = p^h$, p prime, $h \geq 1$.

Lemma 3.1. *Every line of $\text{PG}(r, q^2)$, $q = p^h$, p prime, $h \geq 1$, meets \mathcal{V} in $1 \pmod{p}$ points.*

Proof. We may express

$$v^\mathcal{V} = v^{H_1} + \dots + v^{H_t},$$

where H_1, \dots, H_t are (not necessarily distinct) hyperplanes of $\text{PG}(r, q^2)$. Denote by \cdot the usual dot product. We get $v^\mathcal{V} \cdot v^\mathcal{V} = |\mathcal{V}| \equiv 1 \pmod{p}$. On the other hand

$$v^\mathcal{V} \cdot v^\mathcal{V} = v^\mathcal{V} \cdot (v^{H_1} + \dots + v^{H_t}) \equiv t \pmod{p},$$

since every hyperplane of $\text{PG}(r, q^2)$ meets \mathcal{V} in $1 \pmod{p}$ points. Hence, we have $t \equiv 1 \pmod{p}$. Finally, for a line ℓ of $\text{PG}(r, q^2)$,

$$v^\ell \cdot v^\mathcal{V} = v^\ell \cdot (v^{H_1} + \dots + v^{H_t}) \equiv t \pmod{p},$$

as every line of $\text{PG}(r, q^2)$ meets a hyperplane in 1 or $q^2 + 1$ points. That is, $|\ell \cap \mathcal{V}| \equiv 1 \pmod{p}$ and in particular no lines of $\text{PG}(r, q^2)$ are external to \mathcal{V} . \square

Remark 3.2. The preceding proof also shows that \mathcal{V} is a linear combination of 1 \pmod{p} (not necessarily distinct) hyperplanes, all having coefficient one.

Lemma 3.3. *For every hyperplane H of $\text{PG}(r, q^2)$, $q = p^h$, p prime, $h \geq 1$, the intersection $H \cap \mathcal{V}$ is in the code of points and hyperplanes of H itself.*

Proof. Let Σ denote the set of all hyperplanes of $\text{PG}(r, q^2)$. By assumption,

$$v^{\mathcal{V}} = \sum_{H_i \in \Sigma} \lambda_i v^{H_i}. \quad (1)$$

For every $H \in \Sigma$, let π denote a hyperplane of H ; then $\pi = H_{j_1} \cap \cdots \cap H_{j_{q^2+1}}$, where $H_{j_1}, \dots, H_{j_{q^2+1}}$ are the hyperplanes of $\text{PG}(r, q^2)$ through π . We assume $H = H_{j_{q^2+1}}$. For every hyperplane π of H , we set

$$\lambda_{\pi} = \sum_{k=1, \dots, q^2+1} \lambda_{j_k},$$

where λ_{j_k} is the coefficient in (1) of $v^{H_{j_k}}$ and H_{j_k} is one of the $q^2 + 1$ hyperplanes through π .

Now, consider

$$T = \sum_{\pi \in \Sigma'} \lambda_{\pi} v^{\pi}, \quad (2)$$

where Σ' is the set of all hyperplanes in H . We are going to show that

$$T = v^{\mathcal{V} \cap H}.$$

In fact, it is clear that at the positions belonging to the points outside of H we see zeros. At a position belonging to a point in H , we see the original coefficients of $v^{\mathcal{V}}$ plus $(|\Sigma'| - 1)\lambda_{j_{q^2+1}}$. Note that this last term is 0 \pmod{p} , hence $T = v^{\mathcal{V} \cap H}$. \square

Corollary 3.4. *For every subspace S of $\text{PG}(r, q^2)$, $q = p^h$, p prime, $h \geq 1$, the intersection $S \cap \mathcal{V}$ is in the code of points and hyperplanes of S itself.* \square

Remark 3.5. Lemma 3.3 and Corollary 3.4 are valid for \mathcal{V} any set of points in $\text{PG}(r, q^2)$ whose incidence vector belongs to the code of points and hyperplanes of $\text{PG}(r, q^2)$. In particular, it follows that for every plane π the intersection $\pi \cap \mathcal{V}$ is a codeword of the points and lines of π , $\pi \cap \mathcal{V}$ has size 1 \pmod{p} and so it is a linear combination of 1 \pmod{p} not necessarily distinct lines.

Lemma 3.6. *Let ℓ be a line of $\text{PG}(r, q^2)$. Then there exists at least one plane through ℓ meeting \mathcal{V} in δ points, with $\delta \leq q^3 + q^2 + q + 1$.*

Proof. By way of contradiction, assume that all planes through ℓ meet \mathcal{V} in more than $q^3 + q^2 + q + 1$ points. Set $x = |\ell \cap \mathcal{V}|$. We get

$$\frac{(q^{r+1} + (-1)^r)(q^r - (-1)^r)}{q^2 - 1} > m(q^3 + q^2 + q + 1 - x) + x, \quad (3)$$

where $m = q^{2(r-2)} + q^{2(r-3)} + \dots + q^2 + 1$ is the number of planes in $\text{PG}(r, q^2)$ through ℓ . From (3), we obtain $x > q^2 + 1$, a contradiction. \square

Lemma 3.7. *For each line ℓ of $\text{PG}(r, q^2)$, $q > 4$ and $q = p^h$, p prime, $h \geq 1$, either $|\ell \cap \mathcal{V}| \leq q + 1$ or $|\ell \cap \mathcal{V}| \geq q^2 - q + 1$.*

Proof. Let ℓ be a line of $\text{PG}(r, q^2)$ and let π be a plane through ℓ such that $|\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1$; Lemma 3.6 shows that such a plane exists. Set $B = \pi \cap \mathcal{V}$. By Corollary 3.4, B is a codeword of the code of the lines of π , so we can write it as a linear combination of some lines of π , that is $\sum_i \lambda_i v^{e_i}$, where v^{e_i} are the characteristic vectors of the lines e_i in π .

Let B^* be the multiset consisting of the lines e_i , with multiplicity λ_i , in the dual plane of π . The weight of the codeword B is at most $q^3 + q^2 + q + 1$, hence in the dual plane this is the number of lines intersecting B^* in not 0 (mod p) points. Actually, as B is a proper set, we know that each non 0 (mod p) secant of B^* must be a 1 (mod p) secant. Using Result 2.5, the number of non 0 (mod p) secants through any point is at most $q + 1$ or at least $q^2 - q + 1$. In the original plane π , this means that each line intersects B in either at most $q + 1$ or in at least $q^2 - q + 1$ points. \square

Proposition 3.8. *Assume that π is a plane of $\text{PG}(r, q^2)$, $q > 4$, and $q = p^h$, p prime, $h \geq 1$, such that $|\pi \cap \mathcal{V}| \leq q^3 + 2q^2$. Furthermore, suppose also that there exists a line ℓ meeting $\pi \cap \mathcal{V}$ in at least $q^2 - q + 1$ points, when $q^3 + 1 \leq |\pi \cap \mathcal{V}|$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q + 1$ lines, each with weight 1.*

Proof. Let B be the point set $\pi \cap \mathcal{V}$. By Corollary 3.4, B is the corresponding point set of a codeword c of lines of π , that is $c = \sum_i \lambda_i v^{e_i}$, where lines of π are denoted by e_i . Let C^* be the multiset in the dual plane containing the dual of each line e_i with multiplicity λ_i . Clearly the number of lines intersecting C^* in not 0 (mod p) points is $w(c) = |B|$. Note also, that every line that is not a 0 (mod p) secant is a 1 (mod p) secant, as B is a proper point set, hence Property 2.6 trivially holds (with $k = 1$).

Our very first aim is to show that c is a linear combination of at most $q + 3$ different lines. When $|B| < q^3 + 1$, then by Result 2.8 it is a linear combination of at most q different lines.

Next assume that $|B| \geq q^3 + 1$. From the assumption of the proposition, we know that there exists a line ℓ meeting $\pi \cap \mathcal{V}$ in at least $q^2 - q + 1$ points and from Lemma 3.7, we also know that each line intersects B in either at most $q + 1$ or in at least $q^2 - q + 1$ points. Hence, if we add the line ℓ to c with multiplicity -1 , we reduce the weight by at least $q^2 - q + 1 - q$ and at most by $q^2 + 1$. If $w(c - v^\ell) < q^3 + 1$, then from the above we know that $c - v^\ell$ is a linear combination of $\lceil \frac{w(c - v^\ell)}{q^2 + 1} \rceil$ lines. Hence, c is a linear combination of at most $q + 1$ lines. If $w(c - v^\ell) \geq q^3 + 1$, then $w(c) \geq q^3 + q^2 - 2q$ (see above) and so it follows that through any point of B , there passes at least one line intersecting B in at least $q^2 - q + 1$ points. This means that we easily find three lines ℓ_1, ℓ_2 , and ℓ_3 intersecting B in at least $q^2 - q + 1$ points. Since $w(c) \leq q^3 + 2q^2$, we get that $w(c - v^{\ell_1} - v^{\ell_2} - v^{\ell_3}) \leq q^3 + 2q^2 - 3 \cdot (q^2 - 2q - 2) < q^3 + 1$. Hence, similarly as before, we get that c is a linear combination of at most $q + 3$ lines.

Next we show that each line in the linear combination (that constructs c) has weight 1. Take a line ℓ which is in the linear combination with coefficient $\lambda \neq 0$. Then there are at least $q^2 + 1 - (q + 2)$ positions, such that the corresponding point is in ℓ and the value at that position is λ . As B is a proper set, this yields that $\lambda = 1$. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of c must be $1 \pmod{p}$, $p > 2$; hence it can be at most $q + 1$. \square

Proposition 3.9. *Assume that π is a plane of $\text{PG}(r, q^2)$, $q > 4$, and $q = p^h$, p prime, $h \geq 1$, such that $|\pi \cap \mathcal{V}| \leq q^3 + 2q^2$. Furthermore, suppose that every line meets $\pi \cap \mathcal{V}$ in at most $q + 1$ points. Then $\pi \cap \mathcal{V}$ is a classical unital.*

Proof. Again let $B = \pi \cap \mathcal{V}$ and first assume that $|B| < q^3 + 1$. Proposition 3.8 shows that B is a linear combination of at most $q + 1$ lines, each with weight 1. But this yields that these lines intersect B in at least $q^2 + 1 - q$ points. So this case cannot occur.

Hence, $q^3 + 1 \leq |B| \leq q^3 + 2q^2$. We are going to prove that there exists at least a tangent line to B in π . Let t_i be the number of lines meeting B in i points. Set $x = |B|$. Then double counting arguments give the following equations for the integers t_i .

$$\begin{cases} \sum_{i=1}^{q+1} t_i = q^4 + q^2 + 1 \\ \sum_{i=1}^{q+1} it_i = x(q^2 + 1) \\ \sum_{i=1}^{q+1} i(i-1)t_i = x(x-1). \end{cases} \quad (4)$$

Consider $f(x) = \sum_{i=1}^{q+1} (i-2)(q+1-i)t_i$. From (4), we get

$$f(x) = -x^2 + x[(q^2 + 1)(q + 2) + 1] - 2(q + 1)(q^4 + q^2 + 1).$$

Since $f(q^3/2) > 0$, whereas $f(q^3 + 1) < 0$ and $f(q^3 + 2q^2) < 0$, it follows that if $q^3 + 1 \leq x \leq q^3 + 2q^2$, then $f(x) < 0$ and thus t_1 must be different from zero. Therefore, $x = q^3 + 1$ and $\sum_{i=1}^{q+1} (i-1)(q+1-i)t_i = 0$.

Since $(i-1)(q+1-i) > 0$, for $2 \leq i \leq q$, we obtain $t_2 = t_3 = \dots = t_q = 0$, that is, B is a set of $q^3 + 1$ points such that each line is a 1-secant or a $(q+1)$ -secant of B . Namely, B is a unital and precisely a classical unital since B is a codeword of π [6]. \square

The above two propositions and Lemma 3.7 imply the following corollary.

Corollary 3.10. *Assume that π is a plane of $\text{PG}(r, q^2)$, $q > 4$ and $q = p^h$, p prime, $h \geq 1$, such that $|\pi \cap \mathcal{V}| \leq q^3 + 2q^2$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q+1$ lines, each with weight 1, or it is a classical unital. \square*

Corollary 3.11. *Suppose that π is a plane of $\text{PG}(r, q^2)$, $q > 4$ and $q = p^h$, p prime, $h \geq 1$, containing exactly $q^3 + 1$ points of \mathcal{V} . Then $\pi \cap \mathcal{V}$ is a classical unital.*

Proof. Let B be the point set $\pi \cap \mathcal{V}$. We know that B is the support of a codeword of lines of π . By Proposition 3.8, if there is a line intersecting B in at least $q^2 - q + 1$ points, then B is a linear combination of at most $q+1$ lines, each with multiplicity 1. First of all note that a codeword that is a linear combination of $q+1$ lines has weight at least $(q^2 + 1)(q+1) - 2\binom{q+1}{2}$, that is exactly $q^3 + 1$. To achieve this, we need that the intersection points of any two lines from a linear combination are all different and the sum of the coefficients of any two lines is zero; which is clearly not the case (as all the coefficients are 1). From Remark 3.5, in this case B would be a linear combination of at most $q+1-p$ lines and so its weight would be less than $q^3 + 1$, a contradiction. Hence, there is no line intersecting B in at least $q^2 - q + 1$ points, so Proposition 3.9 finishes the proof. \square

3.1 Case $r = 3$

In $\text{PG}(3, q^2)$, each plane intersects \mathcal{V} in either $q^3 + 1$ or $q^3 + q^2 + 1$ points since these are the intersection numbers of a quasi-Hermitian variety with a plane of $\text{PG}(3, q^2)$.

Lemma 3.12. *Let π be a plane in $\text{PG}(3, q^2)$ such that $|\pi \cap \mathcal{V}| = q^3 + q^2 + 1$, then every line in π meets $\pi \cap \mathcal{V}$ in either 1, $q+1$ or $q^2 + 1$ points.*

Proof. Set $C = \pi \cap \mathcal{V}$ and let m be a line in π such that $|m \cap C| = s$, with $s \neq 1$ and $s \neq q+1$. Thus, from Corollary 3.11, every plane through m has to meet \mathcal{V} in $q^3 + q^2 + 1$ points and thus

$$|\mathcal{V}| = (q^2 + 1)(q^3 + q^2 + 1 - s) + s,$$

which gives $s = q^2 + 1$. □

Proof of Theorem 1.1 (case $r = 3$): From Corollary 3.11 and Lemma 3.12, it follows that every line in $\text{PG}(3, q^2)$ meets \mathcal{V} in either 1, $q + 1$, or $q^2 + 1$ points. Now, suppose on the contrary that there exists a singular point P on \mathcal{V} ; this means that all lines through P are either tangents or $(q^2 + 1)$ -secants to \mathcal{V} . Take a plane π which does not contain P . Then $|\mathcal{V}| = q^2|\pi \cap \mathcal{V}| + 1$ and since the two possible sizes of the planar sections are $q^3 + 1$ or $q^3 + q^2 + 1$, we get a contradiction. Thus, every point in \mathcal{V} lies on at least one $(q + 1)$ -secant and, from Theorem 2.1, we obtain that \mathcal{V} is a Hermitian surface. □

3.2 Case $r \geq 4$ and $q = p$

We first prove the following result.

Lemma 3.13. *If π is a plane of $\text{PG}(r, p^2)$, which is not contained in \mathcal{V} , then either*

$$|\pi \cap \mathcal{V}| = p^2 + 1 \text{ or } |\pi \cap \mathcal{V}| \geq p^3 + 1.$$

Proof. Let π be a plane of $\text{PG}(r, p^2)$ and set $B = \pi \cap \mathcal{V}$. By Remark 3.5, B is a linear combination of 1 mod $(\text{mod } p)$ not necessarily distinct lines.

If $|B| < p^3 + 1$, then by Result 2.5, B is a linear combination of at most p distinct lines. This and the previous observation yield that when $|B| < p^3 + 1$, then it is the scalar multiple of one line; hence $|B| = p^2 + 1$. □

Proposition 3.14. *Let π be a plane of $\text{PG}(r, p^2)$, such that $|\pi \cap \mathcal{V}| \leq p^3 + p^2 + p + 1$. Then $B = \pi \cap \mathcal{V}$ is either a classical unital or a linear combination of $p + 1$ concurrent lines or just one line, each with weight 1.*

Proof. From Corollary 3.10, we have that B is either a linear combination of at most $p + 1$ lines or a classical unital. In the first case, since B intersects every line in 1 mod p points and B is a proper point set, the only possibilities are that B is a linear combination of $p + 1$ concurrent lines or just one line, each with weight 1. □

Proof of Theorem 1.1 (case $r \geq 4$, $q = p$): Consider a line ℓ of $\text{PG}(r, p^2)$ which is not contained in \mathcal{V} . By Lemma 3.6, there is a plane π through ℓ such that $|\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1$. From Proposition 3.14, we have that ℓ is either a unisecant or a $(p + 1)$ -secant of \mathcal{V} and we also have that \mathcal{V} has no plane section of size $(p + 1)(p^2 + 1)$. Finally, it is easy to see like in the previous case $r = 3$, that \mathcal{V} has no singular points, thus \mathcal{V} turns out to be a Hermitian variety of $\text{PG}(r, p^2)$ (Theorem 2.2).

3.3 Case $r \geq 4$ and $q = p^2$

Assume now that \mathcal{V} is a quasi-Hermitian variety of $\text{PG}(r, p^4)$, with $r \geq 4$.

Lemma 3.7 states that every line contains at most $p^2 + 1$ points of \mathcal{V} or at least $p^4 - p^2 + 1$ points of \mathcal{V} .

Lemma 3.15. *If ℓ is a line of $\text{PG}(r, p^4)$, such that $|\ell \cap \mathcal{V}| \geq p^4 - p^2 + 1$, then $|\ell \cap \mathcal{V}| \geq p^4 - p + 1$.*

Proof. Set $|\ell \cap \mathcal{V}| = p^4 - x + 1$, where $x \leq p^2$. It suffices to prove that $x < p + 2$. Let π be a plane through ℓ and $B = \pi \cap \mathcal{V}$. Choose π such that $|B| = |\pi \cap \mathcal{V}| \leq p^6 + p^4 + p^2 + 1$ (Lemma 3.6). Then, by Proposition 3.8, B is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Let c be the codeword corresponding to B . We observe that ℓ must be one of the lines of c , otherwise $|B \cap \ell| \leq p^2 + 1$, which is impossible. Thus if P is a point in $\ell \setminus B$, then through P there pass at least $p - 1$ other lines of c . If $x \geq p + 2$, then the number of lines necessary to define the codeword c would be at least $(p + 2)(p - 1) + 1$, a contradiction. \square

Lemma 3.16. *For each plane π of $\text{PG}(r, p^4)$, either $|\pi \cap \mathcal{V}| \leq p^6 + 2p^4 - p^2 - p + 1$ or $|\pi \cap \mathcal{V}| \geq p^8 - p^5 + p^4 - p + 1$.*

Proof. Let $B = \pi \cap \mathcal{V}$, $x = |B|$, and let t_i be the number of lines in π meeting B in i points. Then, in this case, Equations (4) read

$$\begin{cases} \sum_{i=1}^{p^4+1} t_i = p^8 + p^4 + 1 \\ \sum_{i=1}^{p^4+1} it_i = x(p^4 + 1) \\ \sum_{i=1}^{p^4+1} i(i-1)t_i = x(x-1). \end{cases} \quad (5)$$

Set $f(x) = \sum_{i=1}^{p^4+1} (p^2 + 1 - i)(i - (p^4 - p + 1))t_i$. From (5) we obtain

$$f(x) = -x^2 + [(p^4 + 1)(p^4 + p^2 - p + 1) + 1]x - (p^8 + p^4 + 1)(p^2 + 1)(p^4 - p + 1).$$

Because of Lemma 3.15, we get $f(x) \leq 0$, while $f(p^6 + 2p^4 - p^2 + 1) > 0$, $f(p^8 - p^5 + p^4 - p) > 0$. This finishes the proof of the lemma. \square

Lemma 3.17. *If π is a plane of $\text{PG}(r, p^4)$, such that $|\pi \cap \mathcal{V}| \geq p^8 - p^5 + p^4 - p + 1$, then either π is entirely contained in \mathcal{V} or $\pi \cap \mathcal{V}$ consists of $p^8 - p^5 + p^4 + 1$ points and it only contains i -secants, with $i \in \{1, p^4 - p + 1, p^4 + 1\}$.*

Proof. Set $S = \pi \setminus \mathcal{V}$. Suppose that there exists some point $P \in S$. We have the following two possibilities: either each line of the pencil with center at P is a $(p^4 - p + 1)$ -secant or only one line through P is an i -secant, with $1 \leq i \leq p^2 + 1$,

whereas the other p^4 lines through P are $(p^4 - p + 1)$ -secants. In the former case, when there are no i -secants, $1 \leq i \leq p^2 + 1$, each line ℓ in π either is disjoint from S or it meets S in p points since ℓ is a $(p^4 - p + 1)$ -secant. This implies that S is a maximal arc and this is impossible for $p \neq 2$ [4, 5].

In the latter case, we observe that the size of $\pi \cap \mathcal{V}$ must be $p^8 - p^5 + p^4 + i$, where $1 \leq i \leq p^2 + 1$. Next, we denote by t_s the number of s -secants in π , where $s \in \{i, p^4 - p + 1, p^4 + 1\}$. We have that

$$\begin{cases} \sum_s t_s = p^8 + p^4 + 1 \\ \sum_s s t_s = (p^4 + 1)(p^8 - p^5 + p^4 + i) \\ \sum_s s(s-1)t_s = (p^8 - p^5 + p^4 + i)(p^8 - p^5 + p^4 + i - 1). \end{cases} \quad (6)$$

From (6) we get

$$t_i = \frac{p(p^4 - p - i + 1)(p^5 - i + 1)}{p(p^4 - p - i + 1)(p^4 - i + 1)} = \frac{p^5 - i + 1}{p^4 - i + 1} \quad (7)$$

and we can see that the only possibility for t_i to be an integer is $ip - p - i + 1 = 0$, that is $i = 1$. For $i = 1$, we get $|B| = p^8 - p^5 + p^4 + 1$. \square

Lemma 3.18. *If π is a plane of $\text{PG}(r, p^4)$, not contained in \mathcal{V} and which does not contain any $(p^4 - p + 1)$ -secant, then $\pi \cap \mathcal{V}$ is either a classical unital or the union of i concurrent lines, with $1 \leq i \leq p^2 + 1$.*

Proof. Because of Lemmas 3.15, 3.16 and 3.17, the plane π meets \mathcal{V} in at most $p^6 + 2p^4 - p^2 - p + 1$ points. Furthermore, each line of π which is not contained in \mathcal{V} is an i -secant, with $1 \leq i \leq p^2 + 1$ (Lemma 3.15 and the sentence preceding Lemma 3.15). Set $B = \pi \cap \mathcal{V}$. If in π there are no $(p^4 + 1)$ -secants to B , then $|B| \leq p^6 + p^2 + 1$ and by Proposition 3.9 it follows that B is a classical unital.

If there is a $(p^4 + 1)$ -secant to B in π , then arguing as in the proof of Proposition 3.8, we get that B is still a linear combination of m lines, with $m \leq p^2 + 1$. Each of these m lines is a $(p^4 + 1)$ -secant to \mathcal{V} . In fact if one of these lines, say v , was an s -secant, with $1 \leq s \leq p^2 + 1$, then through each point $P \in v \setminus B$, there would pass at least p lines of the codeword corresponding to B and hence B would be a linear combination of at least $(p^4 + 1 - s)(p - 1) + 1 > p^2 + 1$ lines, which is impossible.

We are going to prove that these m lines, say ℓ_1, \dots, ℓ_m , are concurrent. Assume on the contrary that they are not. We can assume that through a point $P \in \ell_n$, there pass at least $p + 1$ lines of our codeword but there is a line ℓ_j which does not pass through P . Thus through at least $p + 1$ points on ℓ_j , there are at least $p + 1$ lines of our codeword and thus we find at least $(p + 1)p + 1 > m$ lines of B , a contradiction. \square

Lemma 3.19. *A plane π of $\text{PG}(r, p^4)$ meeting \mathcal{V} in at most $p^6 + 2p^4 - p^2 - p + 1$ points and containing a $(p^4 - p + 1)$ -secant to \mathcal{V} has at most $(p^2 + 1)(p^4 - p + 1)$ points.*

Proof. Let ℓ be a line of π which is a $(p^4 - p + 1)$ -secant to \mathcal{V} . In this case, $\pi \cap \mathcal{V}$ is a linear combination of at most $p^2 + 1$ lines, each with weight 1 (Proposition 3.8). In particular ℓ is a line of the codeword and hence through each of the missing points of ℓ there are at least p lines of the codeword corresponding to B . On these p lines we can see at most $p^4 - p + 1$ points of \mathcal{V} .

So let $\ell_1, \ell_2, \dots, \ell_p$ be p lines of the codeword through a point of ℓ . Each of these lines contains at most $p^4 - p + 1$ points of \mathcal{V} . Thus these p lines contain together at most $p(p^4 - p + 1)$ points of \mathcal{V} . Now take any other line of the codeword, say e . If e goes through the common point of the lines ℓ_i , then there is already one point missing from e , so adding e to our set, we can add at most $p^4 - p + 1$ points. If e does not go through the common point, then it intersects ℓ_i in p different points. These points either do not belong to the set $\pi \cap \mathcal{V}$ or they belong to the set $\pi \cap \mathcal{V}$, but we have already counted them when we counted the points of ℓ_i , so again e can add at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Thus adding the lines of the codewords one by one to ℓ_i and counting the number of points, each time we add only at most $p^4 - p + 1$ points to the set $\pi \cap \mathcal{V}$. Hence, the plane π contains at most $(p^2 + 1)(p^4 - p + 1)$ points of \mathcal{V} . \square

Lemma 3.20. *Let π be a plane of $\text{PG}(r, p^4)$, containing an i -secant, $1 < i < p^2 + 1$, to \mathcal{V} . Then $\pi \cap \mathcal{V}$ is either the union of i concurrent lines or it is a linear combination of $p^2 + 1$ lines (each with weight 1) so that they form a subgeometry of order p , minus p concurrent lines.*

Proof. By Lemmas 3.16, 3.17 and 3.18, π meets \mathcal{V} in at most $p^6 + 2p^4 - p^2 - p + 1$ points and must contain a $(p^4 - p + 1)$ -secant to \mathcal{V} or $\pi \cap \mathcal{V}$ is the union of i concurrent lines. Hence from now on, we assume that π contains a $(p^4 - p + 1)$ -secant. By Result 2.8, such a plane is a linear combination of at most $p^2 + 1$ lines. As before each line from the linear combination has weight 1. Note that the above two statements imply that a line of the linear combination will be either a $(p^4 + 1)$ -secant or a $(p^4 - p + 1)$ -secant.

As we have at most $p^2 + 1$ lines, the line of a $(p^4 - p + 1)$ -secant must be one of the lines from the linear combination. This also means that through each of the p missing points of this line, there must pass at least $p - 1$ other lines from the linear combination. Hence, we already get $(p - 1)p + 1$ lines.

In the case in which the linear combination contains exactly $p^2 - p + 1$ lines, then from each of these lines there are exactly p points missing and through each missing point there are exactly p lines from the linear combination. Hence, the

missing points and these lines form a projective plane of order $p-1$, a contradiction as $p > 3$.

Therefore, as the number of the lines of the linear combination must be $1 \pmod{p}$ and at most $p^2 + 1$, we can assume that the linear combination contains $p^2 + 1$ lines. We are going to prove that through each point of the plane there pass either 0 , 1 , p or $p + 1$ lines from the linear combination. From earlier arguments, we know that the number of lines through one point P is 0 or $1 \pmod{p}$. Assume to the contrary that through P there pass at least $p + 2$ of such lines. These $p^2 + 1$ lines forming the linear combination are not concurrent, so there is a line ℓ not through P . Through each of the intersection points of ℓ and a line through P , there pass at least $p - 1$ more other lines of the linear combination, so in total we get at least $(p - 1)(p + 2) + 1$ lines forming the linear combination, a contradiction.

Since there are $p^2 + 1$ lines forming the linear combination and through each point of the plane there pass either 0 , 1 , p or $p + 1$ of these lines, we obtain that on a $(p^4 - p + 1)$ -secant there is exactly one point, say P , through which there pass exactly $p + 1$ lines from the linear combination and p points, not in the quasi Hermitian variety, through each of which there pass exactly p lines.

If all the $p^2 + 1$ lines forming the linear combination, were $(p^4 - p + 1)$ -secants then the number of points through which there pass exactly p lines would be $(p^2 + 1)p/p$. On the other hand, through P there pass $p + 1$ $(p^4 - p + 1)$ -secants, hence we already get $(p + 1)p$ such points, a contradiction. Thus, there exists a line m of the linear combination that is a $(p^4 + 1)$ -secant. From the above arguments, on this line there are exactly p points through each of which there pass exactly $p + 1$ lines, whereas through the rest of the points of the line m there pass no other lines of the linear combination.

Assume that there is a line $m' \neq m$ of the linear combination that is also a $(p^4 + 1)$ -secant. Then there is a point Q on m' but not on m through which there pass $p + 1$ lines. This would mean that there are at least $p + 1$ points on m , through which there pass more than 2 lines of the linear combination, a contradiction.

Hence, there is exactly one line m of the linear combination that is a $(p^4 + 1)$ -secant and all the other lines of the linear combination are $(p^4 - p + 1)$ -secants. It is easy to check that the points through which there are more than 2 lines plus the $(p^4 - p + 1)$ -secants form a dual affine plane. Hence our lemma follows. \square

Lemma 3.21. *There are no i -secants to \mathcal{V} , with $1 < i < p^2 + 1$.*

Proof. By Lemma 3.20, if a plane π contains an i -secant, $1 < i < p^2 + 1$, then $\pi \cap \mathcal{V}$ is a linear combination of either i concurrent lines or lines of an embedded subplane of order p minus p concurrent lines. In the latter case, if $i > 1$ then an i -secant is at least a $(p^2 - p + 1)$ -secant. Hence, if there is an i -secant with $1 < i < p^2 - p + 1$, say ℓ , we get that for each plane α through ℓ , $\alpha \cap \mathcal{V}$ is a linear

combination of i concurrent lines. Therefore

$$|\mathcal{V}| = m(ip^4 + 1 - i) + i, \quad (8)$$

where $m = p^{4(r-2)} + p^{4(r-3)} + \dots + p^4 + 1$ is the number of planes in $\text{PG}(r, p^4)$ through ℓ .

Setting $r = 2\sigma + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to r is even or odd, we can write

$$|\mathcal{V}| = 1 + p^4 + \dots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \dots + p^{4(r-1)})p^2$$

Hence, (8) becomes

$$\begin{aligned} & 1 + p^4 + \dots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon+1)} + \dots + p^{4(r-1)})p^2 \\ & - (p^{4(r-2)} + p^{4(r-3)} + \dots + p^4 + 1) = ip^{4(r-1)} \end{aligned} \quad (9)$$

Since $\sigma \geq 2$, we see that $p^{4(r-1)}$ does not divide the left hand side of (9), a contradiction.

Thus, there can only be 1-, $(p^2 - p + 1)$ -, $(p^2 + 1)$ -, $(p^4 - p + 1)$ - or $(p^4 + 1)$ -secants to \mathcal{V} . Now, suppose that ℓ is a $(p^2 - p + 1)$ -secant to \mathcal{V} . Again by Lemma 3.20, each plane through ℓ either has $x = (p^2 - p + 1)p^4 + 1$ or $y = p^2(p^4 - p) + p^4 + 1$ points of \mathcal{V} . Next, denote by t_j the number of j -secant planes through ℓ to \mathcal{V} . We get

$$\begin{cases} t_x + t_y = m \\ t_x(x - p^2 + p - 1) + t_y(y - p^2 + p - 1) + p^2 - p + 1 = |\mathcal{V}| \end{cases} \quad (10)$$

Recover the value of t_y from the first equation and substitute it in the second. We obtain

$$(m - t_y)(p^6 - p^5 + p^4 - p^2 + p) + t_y(p^6 + p^4 - p^3 - p^2 + p) + p^2 - p + 1 = |\mathcal{V}|$$

that is,

$$p^3(p^2 - 1)t_y = |\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1.$$

It is easy to check that $|\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1$ is not divisible by $p + 1$ and hence, t_y turns out not to be an integer, which is impossible. \square

Lemma 3.22. *No plane meeting \mathcal{V} in at most $p^6 + 2p^4 - p^2 - p + 1$ points contains a $(p^4 - p + 1)$ -secant.*

Proof. Let π be a plane of $\text{PG}(r, p^4)$ such that $|\pi \cap \mathcal{V}| \leq p^6 + 2p^4 - p^2 - p + 1$. It can contain only 1-, $(p^2 + 1)$ -, $(p^4 - p + 1)$ -, $(p^4 + 1)$ -secants (Lemma 3.15 and Lemma 3.21). If $\pi \cap \mathcal{V}$ contains a $(p^4 - p + 1)$ -secant, we know from Proposition 3.8 that it

is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Suppose that e is a $(p^4 - p + 1)$ -secant to $\pi \cap \mathcal{V}$. Let P and Q be two missing points of e . We know that there must be at least $p - 1$ other lines of the codeword through P and Q . Let f and g be two such lines through Q . We can find a line, say m , of the plane through P , that intersects f and g in a point of \mathcal{V} and that is not a line of the codeword. Then $|m \cap \mathcal{V}| \geq 1 + p$ since $|m \cap \mathcal{V}| \equiv 1 \pmod{p}$. Thus m contains at least two points of \mathcal{V} , but in P it meets at least p lines of the codeword. Hence, $p + 1 \leq |m \cap \mathcal{V}| \leq p^2 - p + 1$, and this contradicts Lemma 3.21. \square

Lemma 3.23. *There are no $(p^4 - p + 1)$ -secants to \mathcal{V} .*

Proof. If there was a $(p^4 - p + 1)$ -secant to \mathcal{V} , say ℓ , then, by Lemma 3.22, all the planes through ℓ would contain at least $p^8 - p^5 + p^4 + 1$ points of \mathcal{V} , and thus

$$|\mathcal{V}| \geq (p^{4(r-2)} + p^{4(r-3)} + \dots + p^4 + 1)(p^8 - p^5 + p) + p^4 - p + 1, \quad (11)$$

a contradiction. \square

Proof of Theorem 1.1 (case $r \geq 4$ and $q = p^2$): Consider a line ℓ which is not contained in \mathcal{V} . From the preceding lemmas we have that ℓ is either a 1-secant or a $(p^2 + 1)$ -secant of \mathcal{V} . Furthermore, \mathcal{V} has no plane section of size $(p^2 + 1)(p^4 + 1)$. Finally, as in the case $r = 3$, it is easy to see that \mathcal{V} has no singular points, thus, by Theorem 2.2, \mathcal{V} turns out to be a Hermitian variety of $\text{PG}(r, p^4)$.

4 Singular quasi-Hermitian varieties

In this section, we consider sets having the same behavior with respect to hyperplanes as singular Hermitian varieties.

Definition 4.1. *A d -singular quasi-Hermitian variety is a subset of points of $\text{PG}(r, q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension d .*

We prove the following result.

Theorem 4.2. *Let \mathcal{S} be a d -singular quasi-Hermitian variety in $\text{PG}(r, q^2)$. Suppose that either*

- $r = 3, d = 0, q = p^h > 4, h \geq 1$, or
- $r \geq 4, d \leq r - 3, q = p > 4$, or
- $r \geq 4, d \leq r - 3, q = p^2, p > 3$.

Then \mathcal{S} is a singular Hermitian variety with a singular space of dimension d if and only if its incidence vector is in the \mathbb{F}_p -code spanned by the hyperplanes of $\text{PG}(r, q^2)$.

Proof. Let \mathcal{S} be a singular Hermitian variety of $\text{PG}(r, q^2)$. The characteristic vector $v^{\mathcal{S}}$ of \mathcal{S} is in \mathcal{C}_p since [12, Theorem 1] also holds for singular Hermitian varieties. Now assume that \mathcal{S} is a d -singular quasi-Hermitian variety. As in the non-singular case, by Lemma 3.1, each line of $\text{PG}(r, q^2)$ intersects \mathcal{S} in $1 \pmod{p}$ points.

4.1 Case $r = 3$

Suppose that $r = 3$ and therefore $d = 0$. Let π be a plane of $\text{PG}(3, q^2)$. In this case π meets \mathcal{S} in either $q^2 + 1$, or $q^3 + 1$ or $q^3 + q^2 + 1$ points. Therefore Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10, are still valid in the singular case for $r = 3$.

Thus, if $|\pi \cap \mathcal{S}| = q^2 + 1$, then Proposition 3.8 implies that $\pi \cap \mathcal{S}$ is a line of π , whereas if $|\pi \cap \mathcal{S}| = q^3 + 1$, then Corollary 3.10 gives that $\pi \cap \mathcal{S}$ is a classical unital of π . Now suppose that $|\pi \cap \mathcal{S}| = q^3 + q^2 + 1$. Let ℓ be a line of π such that $|\ell \cap \mathcal{S}| = s$ with $s \neq 1, q + 1, q^2 + 1$.

Each plane through ℓ must meet \mathcal{S} in $q^3 + q^2 + 1$ points and this gives

$$(q^3 + q^2 + 1 - s)(q^2 + 1) + s = q^5 + q^2 + 1$$

that is, $s = q^2 + q + 1$ which is impossible.

Thus each line of $\text{PG}(3, q^2)$ intersects \mathcal{S} in either 1 or, $q + 1$ or, $q^2 + 1$ points and hence \mathcal{S} is a $k_{q+1, 3, q^2}$. Also, \mathcal{S} cannot be non-singular by assumption, hence Theorem 2.3 applies and \mathcal{S} turns out to be a cone $\Pi_0 \mathcal{S}'$ with \mathcal{S}' of type I, II, III or IV as the possible intersection sizes with planes are $q^2 + 1, q^3 + 1, q^3 + q^2 + 1$.

Possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $\mathcal{S} = \Pi_0 \mathcal{H}$, where \mathcal{H} is a non-singular Hermitian curve.

4.2 Case $r \geq 4$

Let ℓ be a line of $\text{PG}(r, q^2)$ containing $x < q^2 + 1$ points of \mathcal{S} . We are going to prove that there exists at least one plane through ℓ containing less than $q^3 + q^2 + q + 1$ points of \mathcal{S} . If we suppose that all the planes through ℓ contain at least $q^3 + q^2 + q + 1$ points of \mathcal{S} , then

$$q^{2(d+1)} \frac{(q^{r-d} + (-1)^{r-d-1})(q^{r-d-1} - (-1)^{r-d-1})}{q^2 - 1} + q^{2d} + q^{2(d-1)} + \cdots + q^2 + 1 \geq m(q^3 + q^2 + q + 1 - x) + x,$$

where $m = q^{2(r-2)} + q^{2(r-3)} + \dots + q^2 + 1$ is the number of planes through ℓ in $\text{PG}(r, q^2)$. We obtain $x > q^2 + 1$, a contradiction.

Therefore, there exists at least one plane through ℓ having less than $q^3 + q^2 + q + 1$ points of \mathcal{S} and hence Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10, are still valid in this singular case for any $q > 4$.

Next, we are going to prove that \mathcal{S} is a k_{q+1,r,q^2} , with $q = p^h > 4$, $h = 1, 2$.
case $q = p$: Let ℓ be a line of $\text{PG}(r, p^2)$. As we have seen there is a plane π through ℓ such that $|\pi \cap \mathcal{V}| \leq p^3 + p^2 + p + 1$. Proposition 3.14 is still valid in this case and thus we have that ℓ is either a unisecant or a $(p + 1)$ -secant of \mathcal{S} . Furthermore, we also have that \mathcal{S} has no plane section of size $(p + 1)(p^2 + 1)$ and hence \mathcal{S} is a regular k_{p+1,r,p^2} .

case $q = p^2$: We first observe that (8) and (11) hold true in the case in which \mathcal{V} is assumed to be a singular quasi-Hermitian variety. This implies that all lemmas stated in the subparagraph 3.3 are valid in our case. Thus, we obtain that \mathcal{S} is a k_{p^2+1,r,p^4} and it is straightforward to check that \mathcal{S} is also regular.

Finally, in both cases $q = p$ or $q = p^2$, we have that \mathcal{S} is a singular k_{q+1,r,q^2} because if \mathcal{S} were a non-singular k_{q+1,r,q^2} then, from Theorem 2.2, \mathcal{S} would be a non-singular Hermitian variety and this is not possible by our assumptions.

Therefore, by Theorem 2.4, the only possibility is that \mathcal{S} is a cone $\Pi_d \mathcal{S}'$, with \mathcal{S}' a non-singular $k_{q+1,r-d-1,q^2}$. By Lemma 3.3, \mathcal{S}' belongs to the code of points and hyperplanes of $\text{PG}(r-d-1, q^2)$. Since $r-d-1 \geq 2$, then, by [6] and Theorem 1.1, \mathcal{S}' is a non-singular Hermitian variety and, therefore, \mathcal{S} is a singular Hermitian variety with a vertex of dimension d . \square

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