J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1252

Imre Bárány · Péter Frankl

Cells in the box and a hyperplane

Received April 26, 2020

Abstract. It is well known that a line can intersect at most $2n - 1$ cells of the $n \times n$ chessboard. Here we consider the high-dimensional version: how many cells of the d-dimensional $n \times \cdots \times n$ box can a hyperplane intersect? We also prove the lattice analogue of the following well-known fact: if K, L are convex bodies in $\mathbb{R}^d$ and $K \subset L$, then the surface area of K is smaller than that of L.

Keywords. Lattices, polytopes, lattice points in convex bodies

1. Introduction and main result

It is well-known that a line can intersect the interior of at most $2n - 1$ cells of the $n \times n$ chessboard. What happens in high dimensions? This is the question we address here.

Write $Q_n = Q_n^d = [0, n]^d$, $Q^d = Q_1^d$ so $Q_n^d = nQ^d$. Let $e_1, \dots, e_d$ be the standard basis vectors of $\mathbb{R}^d$ and $\mathbb{Z}^d$. For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ define the unit cube

$$C(z) = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : z_i \le x_i \le z_i + 1,\ i \in [d]\},$$

which we call a *cell* in this paper. Here $[d]$ stands for the set $\{1, \dots, d\}$. For $v \in \mathbb{R}^d$ ($v \neq 0$) let $A(v, t)$ denote the hyperplane $\{x \in \mathbb{R}^d : vx = t\}$ where vx is the scalar product of the two vectors. Define $N^d(n)$ as the maximal number of cells in Q_n^d that a hyperplane $A(v, t)$ can intersect properly, meaning that $A(v, t) \cap \operatorname{int} C(z) \neq \emptyset$.

It is well-known that $N^2(n) = 2n - 1$. Variants of this result have appeared as olympiad problems in several countries. In a seminal paper [5], József Beck used a slightly stronger version of this fact to answer questions of Dirac, Motzkin, and Erdős. In a companion paper [3] we show that $N^3(n) = \frac{9}{4}n^2 + O(n)$. Here we determine the asymptotic behaviour of $N^d(n)$.

Imre Bárány: Rényi Institute of Mathematics, 13-15 Reáltanoda Street, Budapest, 1053 Hungary; and Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK; barany.imre@renyi.hu

Péter Frankl: Rényi Institute of Mathematics, 13-15 Reáltanoda Street, Budapest, 1053 Hungary; peter.frankl@gmail.com

Mathematics Subject Classification (2020): Primary 52B20; Secondary 11H06

We need some definitions. We let $|v|$ resp. $|v|_1$ denote the ℓ_2 and ℓ_1 norm of the vector $v \in \mathbb{R}^d$. Set

$$V_d(v) = \frac{|v|_1}{|v|} \max_{t\in\mathbb{R}} \mathrm{vol}_{d-1}(A(v,t) \cap Q^d),$$
$$V_d = \max\{V_d(v) : v \in \mathbb{R}^d,\ v \neq 0,\ t \in \mathbb{R}\}.$$

It is a consequence of the Brunn–Minkowski theorem (cf. [6] and the proof of Lemma 4.1 below) that for fixed v the quantity $\mathrm{vol}_{d-1}(A(v,t) \cap Q^d)$ is maximal when $A(v,t) \cap Q^d$ is the central section of Q^d, that is, $A(v,t)$ contains the centre of Q^d, which is the point $e/2$ where $e = e_1 + \cdots + e_d$. In this case of course $t = ev/2$. It is known that

$$1 \leq \mathrm{vol}_{d-1}(A(v, ev/2) \cap Q^d) \leq \sqrt{2};$$

the upper bound is a famous result of Keith Ball [2], the lower bound is trivial. This implies that

$$\sqrt{d} \leq V_d \leq \sqrt{2d}.$$

It is known (see [1] or [2]) that the sequence $V_2, V_3, \ldots$ is increasing, $V_2 = 2$, $V_3 = \frac{9}{4}$, $V_4 = \frac{8}{3}$ etc., and its limit is $\sqrt{6d/\pi}$. We conjectured that the vector $v = e$ gives the maximum in the definition of V_d. This has recently been proved by Iskander Aliev [1]. Our main result is

Theorem 1.1. $N^d(n) = V_d n^{d-1}(1 + o(1))$.

In Section 3 we give an outline of the proof.

From now on we assume that $v \in \mathbb{R}^d$ is a unit vector, i.e., $|v| = 1$, and $v \geq 0$; the latter causes no loss generality because of symmetry. Define the (open) strip

$$S(v,t) = \{x \in \mathbb{R}^d : t - ev < vx < t\}.$$

Clearly

$$N^d(n) = \max_{v,t} |S(v,t) \cap Q_n^d \cap \mathbb{Z}^d|.$$

So we have to determine the number of lattice points in the convex set $S(v,t) \cap Q_n^d$. But this convex set is very thin in one direction (of v) and standard methods do not seem to work. In Section 2 we introduce a novel approach to deal with such cases.

Our result extends to any convex body (convex compact set with non-empty interior) $K \subset \mathbb{R}^d$. We define $V(K) = \max\{|v|_1 \,\mathrm{vol}_{d-1}(K \cap A(v,t)) : v \in \mathbb{R}^d,\ |v| = 1,\ t \in \mathbb{R}\}$ and consider the lattice $\frac{1}{n}\mathbb{Z}^d$. Write $N(K,n)$ for the maximal number of cells contained in K that a hyperplane can intersect properly (in the same sense as earlier). A cell in this case is $\frac{1}{n}C(z)$ with $z \in \mathbb{Z}^d$. With this notation $N^d(n) = N(Q^d, n)$. Theorem 1.1 extends to this case as follows.

Theorem 1.2. $N(K,n) = V(K)n^{d-1}(1 + o(1))$.

The proof goes along the same lines as that of Theorem 1.1 and is therefore omitted.

2. Inside cells and boundary cells

For a general convex body K in $\mathbb{R}^d$ a metatheorem says that vol K is approximately equal to $|K \cap \mathbb{Z}^d|$, that is,

$$\text{vol}\, K \approx |K \cap \mathbb{Z}^d|,$$

valid when K is well positioned with respect to $\mathbb{Z}^d$. But this is not necessarily the case with $S(v,t) \cap Q_n^d$. We are going to well-position it or rather choose a suitable basis of $\mathbb{Z}^d$ in which $S(v,t) \cap Q_n^d$ is well positioned. We start out more generally.

Let $K \subset \mathbb{R}^d$ be a convex body. A cell $C(z)$, $z \in \mathbb{Z}^d$, called *inside* if $C(z) \subset K$, *outside* if $C(z) \cap K = \emptyset$, and *boundary* otherwise. The following result will be useful in other applications as well. It is similar to the well-known fact that the surface area of a convex subset of a convex set K is smaller that the surface area of K itself. To our surprise we could not find it anywhere in the literature.

Theorem 2.1. *Assume K, L are convex bodies in $\mathbb{R}^d$ and $K \subset L$. Then*

$$|\text{boundary cells of } K| \le |\text{boundary cells of } L|.$$

We prove this theorem in Section 8.

Now we return to the generic convex body K. Since K contains all inside cells and is contained in the union of inside and boundary cells, we have

$$|\text{inside cells of } K| \le \text{vol}\, K \le |\text{inside or boundary cells of } K|.$$

It is not hard to check that

$$|\text{inside cells of } K| \le |K \cap \mathbb{Z}^d| \le |\text{inside or boundary cells of } K|,$$

implying that

$$\big|\text{vol}\, K - |K \cap \mathbb{Z}^d|\big| \le |\text{boundary cells of } K|. \tag{2.1}$$

Given a basis $F = \{f_1, \dots, f_d\}$ of $\mathbb{Z}^d$ we define the F-*box* with parameters $\alpha, \beta \in \mathbb{R}^d$ as

$$B(\alpha, \beta, F) = \Big\{x = \sum_{i=1}^{d} x_i f_i \in \mathbb{R}^d : \alpha_i \le x_i \le \beta_i,\ i \in [d]\Big\}.$$

This is a parallelotope. We of course assume that $\alpha_i \le \beta_i$ for all i. The minimal box containing K is denoted by $B(K, F)$; it is the F-box $B(\alpha, \beta, F)$ with all α_i maximal and β_i minimal under the condition that $K \subset B(\alpha, \beta, F)$. We will make use of the following theorem of Bárány and Vershik [4] (see also [7]).

Theorem 2.2. *For every convex body K in $\mathbb{R}^d$ there is a basis F such that*

$$\text{vol}\, B(K, F) \ll_d \text{vol}\, K.$$

The notation $\ll_d$ means, as usual, that the quantity on the LHS is smaller than the one on the RHS times a positive constant that only depends on d. When d is clear from the context, we will use $\ll$ instead of $\ll_d$. Of course one can use F-cells (i.e. basic parallelotopes in the basis F) and call them inside, outside, and boundary F-cells with respect to K. Then inequality (2.1) becomes

$$\big|\text{vol}\, K - |K \cap \mathbb{Z}^d|\big| \le |\text{boundary } F\text{-cells of } K|. \tag{2.2}$$

This inequality extends to any lattice Λ and a basis F of Λ in the following form:

$$\left|\frac{1}{\det \Lambda}\text{vol}\, K - |K \cap \Lambda|\right| \le |\text{boundary } F\text{-cells of } K|. \tag{2.3}$$

We need a non-degeneracy condition on K:

$$K \cap \mathbb{Z}^d \text{ contains } d+1 \text{ affinely independent vectors.} \tag{2.4}$$

Under this condition and with minimal box $B(K,F) = B(\alpha,\beta,F)$ we have $\alpha_i \le \lceil \alpha_i \rceil < \lfloor \beta_i \rfloor \le \beta_i$ for all $i \in [d]$. Setting $\gamma_i = \beta_i - \alpha_i$, $\text{vol}\, B(K,F) = \prod_{i=1}^d \gamma_i$. The number of boundary cells of $B(K,F)$ is easy to estimate: it is at most

$$2\sum_{i=1}^d \prod_{j\ne i}(\gamma_j + 2) \ll \sum_{i=1}^d \prod_{j \ne i} \gamma_j = \text{vol}\, B(K,F) \cdot \left(\frac{1}{\gamma_1} + \cdots \frac{1}{\gamma_d}\right).$$

Combining the previous theorems we have

Theorem 2.3. *Let K be a convex body in $\mathbb{R}^d$ satisfying (2.4), and let F be the basis from Theorem 2.2. Then*

$$\big|\text{vol}\, K - |K \cap \mathbb{Z}^d|\big| \ll \text{vol}\, K \cdot \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d}\right).$$

The corresponding version for a general lattice Λ says the following. Assume K is a convex body, Λ a lattice in $\mathbb{R}^d$, and K contains $d+1$ affinely independent points from Λ. Then there is a basis F of Λ such that

$$\left|\frac{1}{\det \Lambda}\text{vol}\, K - |K \cap \mathbb{Z}^d|\right| \ll \frac{1}{\det \Lambda}\text{vol}\, K \cdot \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d}\right). \tag{2.5}$$

Here, just as in Theorem 2.3, the parameters γ_i come from the minimal box $B(K,F)$.

3. Outline of the proof

In this section we give a sketch of the proof of Theorem 1.1. One main ingredient is Theorem 2.3.

The next section establishes some basic properties of $A(v,t)$ and $S(v,t)$. For instance, we show that for fixed v, $\text{vol}(S(v,t) \cap Q^n)$ is maximal when $S(v,t)$ is the central strip (Lemma 4.1). Write $S^*(v,t) = S(v,t) \cap Q_n$ for the strip that maximizes, for fixed v, the number of lattice points in $S(v,t) \cap Q^n$. We also prove the important but not surprising fact (Lemma 4.3) that the convex set $S^*(v,t)$ contains an ellipsoid whose half-axes have lengths of order n apart from one that has length $|v|_1/2$.

The lower bound in Theorem 1.1 is simpler and is based on estimating $|S^*(v,t) \cap \mathbb{Z}^d|$ when $v = z/|z|$ with $z \in \mathbb{Z}^d$ a primitive vector. In this case the points of $S^*(v,t) \cap \mathbb{Z}^d$ lie on $|z|_1$ consecutive lattice hyperplanes $A(z,k)$ where k is an integer, and $|A(z,k) \cap \mathbb{Z}^d|$ is estimated using Theorem 2.3 in the form (2.5).

For the upper bound in Theorem 1.1 we fix a maximizer vector $v = v(n)$ and find a basis $F = F^n = \{f_1, \dots, f_d\}$ of $\mathbb{Z}^d$ using Theorem 2.3. This basis is more suitable than the standard one. The main difficulty is to bound $\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_d}$ on the right hand side of the inequality in Theorem 2.3. Here of course $\gamma_i = \gamma_i(n)$ for all $i \in [d]$. The upper bound is easy when $\gamma_i(n) \to \infty$ for all $i \in [d]$. So we assume that $\gamma_i(n)$ is bounded along a subsequence n' for some $i \in [d]$, for $i = 1$, say.

Let $G = G^{n'}$ be the corresponding dual basis, and $g_1(n') \in \mathbb{Z}^d$ be the corresponding dual basis vector. We show next that $g_1(n')$ is also bounded, implying that $g_1(n'') = g$ is a constant (primitive) vector along a further subsequence n''. This means that the lattice points in $S^*(v,t)$ lie on γ consecutive lattice hyperplanes orthogonal to g. Here γ is the floor of $\gamma_1(n'')$, which we can assume to be a constant since $\gamma_1(n'')$ is bounded. It turns further out that $v(n'')$ tends to $g_0 = g/|g|$ because the angle $\phi_{n''}$ between these two vectors is $\ll \frac{\gamma}{|g|n''}$.

The next step of the argument is 2-dimensional. Let $\Psi = \Psi_n$ denote the orthogonal projection of $\mathbb{R}^d$ to the 2-plane Π spanned by $v(n'')$ and g. The projections of the lattice points in $S^*(v,t)$ lie on γ parallel lines ℓ_h (that are $\frac{1}{|g|}$ apart) see Figure 1. The projected lattice points on the hth line belong to a segment Y_h whose length is $|v(n'')|_1/\sin\phi_{n''}$. We show (Claim 7.1) that any line orthogonal to ℓ_h intersects at most $\gamma+1$ segments Y_h, and, more importantly, any such line intersects at most γ segments Y_h^* where Y_h^* is what you get after deleting a short segment (of length $\sqrt{2d}$) from the left end of Y_h.

The number of lattice points in $S^*(v,t)$ is the sum of the lattice points in $\Psi^{-1}(Y_h)$, which is close to $\frac{1}{|g|}\operatorname{vol}_{d-1}\Psi^{-1}(Y_h)$, which is close to $\frac{1}{|g|}\operatorname{vol}_{d-1}\Psi^{-1}(Y_h^*)$. Estimating the sum of these volumes finishes the proof.

4. Preparations for the proof of Theorem 1.1

In this section we establish some basic properties of the hyperplane $A(v,t)$ and the strip $S(v,t)$ that give the maximal value of $V_d(v)$. We assume again that v is a unit vector, and suppose without loss of generality that $v \ge 0$, that is, $v_i \ge 0$ for all $i \in [d]$. Actually, we can assume that $v_i > 0$ for each i because the requirement $A(v,t) \cap \operatorname{int} C(z) \ne \emptyset$ remains valid even if v_i is modified a little.

For simpler notation we write $A^*(v,t) = A(v,t) \cap Q_n$ and $S^*(v,t) = S(v,t) \cap Q_n$. These intersections of course depend on n, but we suppress this dependence as long as it is not needed. The *central section* is $A^*(v,t_0)$ where $t_0 = n|v|_1/2$; it contains $en/2$, the centre of Q_n. The *central strip* is $S^*(v,t_2)$ where $t_2 = t_0 + |v|_1/2$; it is centrally symmetric with centre $en/2$. We will write $A^*(v)$ resp. $S^*(v)$ for the corresponding central section and strip.

Lemma 4.1. *For a fixed unit vector* $v \in \mathbb{R}^d$, $\mathrm{vol}\, S^*(v,t)$ *is maximal for the central strip and*

$$\max_{t \in \mathbb{R}} \mathrm{vol}\, S^*(v,t) = \mathrm{vol}\, S^*(v,t_2) = V_d(v)n^{d-1} + O(n^{d-2}).$$

Proof. We still assume that $v > 0$ and $|v| = 1$. By the Brunn–Minkowski theorem (see [6]) the function $t \mapsto (\mathrm{vol}_{d-1} A^*(v,t))^{1/(d-1)}$, defined for $t \in [0, n|v|_1]$, is concave. It is also symmetric with respect to $t_0 = n|v|_1/2$, and equals zero at the endpoints of $[0, n|v|_1]$. So its maximum is taken at t_0, implying that $A^*(v) = A^*(v,t_0)$. The integral formula

$$\mathrm{vol}\, S^*(v,t) = \int_{t-|v|_1}^{t} \mathrm{vol}_{d-1} A^*(v,s)\, ds$$

implies that

$$\max_{t \in \mathbb{R}} \mathrm{vol}\, S^*(v,t) \le |v|_1 \max_{t \in \mathbb{R}} \mathrm{vol}_{d-1} A^*(v,t) = |v|_1 \mathrm{vol}_{d-1} A^*(v) = V_d(v)n^{d-1}.$$

The volume of the central strip is

$$\mathrm{vol}\, S^*(v,t_2) = \int_{t_1}^{t_2} \mathrm{vol}_{d-1} A^*(v,t)\, dt = 2\int_{t_1}^{t_0} \mathrm{vol}_{d-1} A^*(v,t)\, dt$$

where $t_1 = t_0 - |v|_1/2$. Concavity implies that on the interval $[t_1, t_0]$,

$$\mathrm{vol}_{d-1} A^*(v,t) \ge \mathrm{vol}_{d-1} A^*(v,t_0)(t/t_0)^{d-1}.$$

We next estimate $D := |v|_1 \mathrm{vol}_{d-1} A^*(v,t_0) - \mathrm{vol}\, S^*(v,t)$ for $t \in [t_1, t_0]$:

$$\begin{aligned} D &= 2\int_{t_1}^{t_0} [\mathrm{vol}_{d-1} A^*(v,t_0) - \mathrm{vol}_{d-1} A^*(v,t)]\, dt \\ &\le 2\int_{t_1}^{t_0} \mathrm{vol}_{d-1} A^*(v) \cdot [1 - (t/t_0)^{d-1}]\, dt \\ &\le |v|_1 \mathrm{vol}_{d-1} A^*(v) \cdot [1 - (t_1/t_0)^{d-1}] \\ &= |v|_1 \mathrm{vol}_{d-1} A^*(v) \cdot [1 - (1 - 1/2n)^{d-1}] < |v|_1 \mathrm{vol}_{d-1} A^*(v) \cdot \frac{d}{2n}. \end{aligned}$$

This shows that $\max_t \mathrm{vol}\, S^*(v,t) \ge V_d(v)\big(1 - \frac{d}{2n}\big)$. ■

Here come the properties of $A(v,t)$ and $S(v,t)$ that we need. Every $A^*(v,t)$ is contained an a $d-1$-dimensional ball of radius $\ll n$ because Q_n is contained in a ball of radius $\sqrt{d}\, n/2$. Fix a unit vector v. The *maximizer* is the slice $A^*(v,t)$ that properly intersects the maximal number of cells in Q_n among all $A^*(v,s)$, $s \in \mathbb{R}$. The corresponding $S^*(v,t)$ is also a *maximizer*.

Lemma 4.2. *There is a maximizer* $A^*(v,t)$ *whose inscribed ball has radius* $\gg n$.

Proof. Recall that $e = e_1 + \cdots + e_d$ where $e_1, \ldots, e_d$ form the standard basis of $\mathbb{R}^d$. We can assume by symmetry that the hyperplane $A(v,t)$ satisfies $v > 0$ and $t \le ve/2$; for each $i \in [d]$, $A(v,t)$ contains the (unique) point $a_i e_i$, and $a_i > 0$, of course. We choose $A(v,t)$ so that $\min\{a_i : i \in [d]\}$ is maximal. We claim that this maximum is at least $n-1$. Assume that, on the contrary, $a_1 = \min\{a_i : i \in [d]\} < n-1$. If $A(v,t)$ intersects the cell $C(z) \subset Q_n$, then the hyperplane $A(v,t) + e_1$ intersects the cell $C(z) + e_1$ which lies in Q_n, so it intersects at least as many cells as $A(v,t)$. It is easy to check that for each $i \in [d]$, $A(v,t) + e_1$ contains the (unique) point $a_i' e_i$ with $a_i' > a_i$, a contradiction.

Then the $d-1$-dimensional ball inscribed in $A^*(v,t)$ has radius at least n/d, as one can see easily. ■

We now fix this maximizer $A^*(v,t)$ together with $S^*(v,t)$.

Lemma 4.3. *The maximizer $S^*(v,t)$ contains an ellipsoid with all half-axes length $\gg n$ apart from one whose length is $|v|_1/2$, which is between $1/2$ and $\sqrt{d}/2$.*

Proof. The middle section $A^*(v, t - ev/2)$ of $S^*(v,t)$ contains a $d-1$-dimensional ball of radius $\gg n$. This follows from Lemma 4.2 for n large. The width of the strip in direction v is $|v|_1$. ■

5. Lattice points in $A^*(z,h)$

Given a primitive vector $z \in \mathbb{Z}^d$ we are going to estimate the number of lattice points in $A^*(z,h)$ where $h \in \mathbb{Z}$. We will need a more general setting so assume K is a convex subset of $A^*(z,h)$ and we will estimate $|K \cap \mathbb{Z}^d|$. As $A^*(z,h)$ is $d-1$-dimensional, condition (2.4) requires having d affinely independent points in $K \cap \mathbb{Z}^d$.

Lemma 5.1. *If K does not satisfy the non-degeneracy condition* (2.4)*, then*

$$|K \cap \mathbb{Z}^d| \ll n^{d-2}.$$

Proof. Under the above conditions the lattice points in K lie on a hyperplane in $A(z,t)$, that is, a $d-2$-dimensional affine (lattice) subspace. One can project K orthogonally to a facet of Q^n so that distinct lattice points project to distinct (lattice) points. An induction argument on dimension finishes the proof. ■

Lemma 5.2. *If K satisfies the non-degeneracy condition* (2.4)*, then*

$$|K \cap \mathbb{Z}^d| \ll \frac{1}{|z|} \operatorname{vol}_{d-1} K.$$

Proof. We can apply the general lattice version of Theorem 2.3, i.e., (2.5). The lattice now is $\Lambda = A(z,h) \cap \mathbb{Z}^d$, it is $d-1$-dimensional and its determinant equals $|z|$, the ℓ_2 norm of z. So there is a basis $F = \{f_1, \ldots, f_{d-1}\}$ of Λ such that $\operatorname{vol}_{d-1} B(K,F) \ll \operatorname{vol}_{d-1} K$. Here $B(K,F)$ is the minimal box in Λ containing K, and so it is of the form

$\{x = \sum_{i=1}^{d-1} x_i f_i : \alpha_i \le x_i \le \beta_i,\ i \in [d-1]\}$ with suitable α_i, β_i. Because of the non-degeneracy assumption, $\gamma_i := \beta_i - \alpha_i \ge 1$. Theorem 2.3 shows now that

$$\left|\frac{1}{|z|}\operatorname{vol}_{d-1} K - |K \cap \mathbb{Z}^d|\right| \ll \frac{1}{|z|}\operatorname{vol}_{d-1} K \cdot \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}}\right).$$

As $\gamma_i \ge 1$ for all i; this implies the statement. ∎

We assume now that $K \subset A^*(z,h)$ contains a $d-1$-dimensional ball of radius $c_1 n$ where $c_1 > 0$ is a constant depending only on d. Of course K lies in a $d-1$-dimensional ball of radius $\sqrt{d}\,n/2$ because Q_n lies in the d-dimensional ball of the same radius and centre $en/2$.

Lemma 5.3. *Assume further that K contains d affinely independent points from $\mathbb{Z}^d$. Then*

$$|K \cap \mathbb{Z}^d| = \frac{1}{|z|}\operatorname{vol}_{d-1} K \cdot \left(1 + |z| O\left(\frac{1}{n}\right)\right),$$

where the implicit constant depends only on d.

Proof. We assume $z \ge 0$ because of symmetry. Again there is a basis $F = \{f_1, \ldots, f_{d-1}\}$ of Λ such that $\operatorname{vol}_{d-1} B(K,F) \ll \operatorname{vol}_{d-1} K \ll n^{d-1}$ where $B(K,F)$ is the minimal box in Λ containing K which is of the form $\{x = \sum_{i=1}^{d-1} x_i f_i : \alpha_i \le x_i \le \beta_i,\ i \in [d-1]\}$ with suitable α_i, β_i. Set $\gamma_i = \beta_i - \alpha_i$ again and note that $\operatorname{vol}_{d-1} B(K,F) = |z| \prod_{i=1}^{d-1} \gamma_i$.

Claim 5.1. *$n \ll \gamma_i |f_i| \ll n$ for every $i \in [d-1]$.*

Proof. Let E be the largest volume ($d-1$-dimensional) ellipsoid contained in $B(K,F)$ and define E^* as the blown-up copy of E from its centre by the factor $d-1$. Then $B(K,F)$ is contained in E^* by the well-known Loewner–John theorem. The volume of E^* is $\ll n^{d-1}$ and E^* contains the ball of radius $c_1 n$. This implies that each axis of E^* has length $\gg_d n$, which implies in turn that each axis has length $\ll_d n$. Then the diameter of E^* is $\ll n$, and then so is the diameter of $B(K,F)$ as well. Thus every edge of the parallelotope $B(K,F)$ has length $\ll n$. These edges are of the form $\gamma_i f_i$, so $\gamma_i |f_i| \ll n$ follows.

On the other hand, the parallelotope $B(K,F)$ contains the ball of radius $c_1 n$ so its edges have length at least $c_1 n$, showing that $n \ll \gamma_i |f|_i$. ∎

We remark that in view of the claim,

$$\begin{aligned} n^{d-1} &\gg \prod \gamma_i \prod |f_i| = \frac{1}{|z|}\operatorname{vol}_{d-1} B(K,F) \cdot \prod |f_i| \\ &\gg \frac{1}{|z|} n^{d-1} \prod |f_i|, \end{aligned}$$

implying $\prod |f_i| \ll |z|$ and so $|f_i| \ll |z|$ as $|f_i| \ge 1$ for all i.

As $\mathbb{Z}^d \cap K$ contains d affinely independent vectors, Theorem 2.3, or rather its lattice version (2.5), applies. Using $|f_i| \ll |z|$ we see that

$$\left|\frac{1}{|z|}\mathrm{vol}_{d-1} K - |K \cap \mathbb{Z}^d|\right| \ll \frac{1}{|z|}\mathrm{vol}_{d-1} K \cdot \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}}\right)$$
$$\ll \frac{1}{|z|}\mathrm{vol}_{d-1} K \cdot \left(\frac{|f_1|}{n} + \cdots + \frac{|f_{d-1}|}{n}\right) \ll \frac{1}{n}\mathrm{vol}_{d-1} K.$$

So we have indeed

$$|\mathbb{Z}^d \cap K| = \frac{1}{|z|}\mathrm{vol}_{d-1} K \cdot \left(1 + |z| O\left(\frac{1}{n}\right)\right). \quad \blacksquare$$

Set $z_0 = z/|z|$ and define

$$M_d(z,n) = \max_t |\mathbb{Z}^d \cap S^*(z_0,t)|.$$

The lattice points in a maximizer $S^*(z_0,t)$ (in the sense used in Lemma 4.2) are all contained in $|z|_1$ consecutive lattice hyperplanes of the form $A(z,h)$. Consequently,

$$M_d(z,n) = \max_{k \in \mathbb{Z}} \sum_{h=1}^{|z|_1} |\mathbb{Z}^d \cap A^*(z,k-h)|. \tag{5.1}$$

Theorem 5.1. *For any primitive vector $z \in \mathbb{Z}^d$ there is $n_0(z) \in \mathbb{Z}$ such that for all $n > n_0(z)$,*

$$M_d(z,n) = n^{d-1} V_d(z_0) + O(n^{d-2}),$$

where the implied constant depends only on d.

Proof. We will use Lemma 5.3 with $K = A^*(z,k-h)$. By Lemma 4.2 the maximizer $A^*(z,k)$ contains a ball of radius $\gg n$. It also contains d affinely independent lattice points if n is large enough (depending on z). The same applies to all $A^*(z,k-h)$ with $h \in [|z|_1]$ because for large n the slice $A^*(z,k-h)$ is very close to $A^*(z,k)$. We can use Lemma 5.3 in (5.1) to get

$$\sum_{h=1}^{|z|_1} |\mathbb{Z}^d \cap A^*(z,k-h)| = \sum_{h=1}^{|z|_1} \frac{1}{|z|}\mathrm{vol}_{d-1} A^*(z,k-h) \cdot \left(1 + |z| O\left(\frac{1}{n}\right)\right).$$

As we have seen, $\mathrm{vol}_{d-1} A^*(z,k-h)$ is at most the $d-1$-dimensional volume of the central slice $A^*(z) = A^*(z,t_0)$. So the sum of $\mathrm{vol}_{d-1} A^*(z,k-h)$ for $|z|_1$ consecutive slices is at most $|z|_1 \mathrm{vol}_{d-1} A^*(z)$. This sum is maximal when the slices are as close to the central slice as possible. This follows from the concavity of the function $t \mapsto (\mathrm{vol}_{d-1} A^*(z,t))^{1/(d-1)}$. The sum of these central slices is estimated as in the proof of Lemma 4.1. We omit the details. $\blacksquare$

Corollary 5.1. $N^d(n) \geq V_d n^{d-1}(1 + o(1))$.

Proof. Denote by $A^0(v)$ the central section $A(v,t) \cap Q^d$. Since the function $v \mapsto |v|_1 \operatorname{vol}_{d-1} A^0(v)$ (for unit vectors in $\mathbb{R}^d$) is continuous, for any $\varepsilon > 0$ we can choose a primitive vector $z \in \mathbb{Z}^d$ such that $V_d(z_0) \geq V_d - \varepsilon/2$ where $z_0 = z/|z|$. Then for all large enough n,

$$\begin{aligned} M_d(z,n) &\geq n^{d-1} V_d(z_0) + O(n^{d-2}) \geq n^{d-1}(V_d - \varepsilon/2) + O(n^{d-2}) \\ &\geq n^{d-1}(V_d - \varepsilon). \end{aligned}$$

■

6. Proof of the upper bound in Theorem 1.1

Let $S_n = S^*(v,t)$ be the maximizer for $N_d(n)$; of course $v = v(n)$ and $t = t(n)$ but we suppress this dependence as long as possible. We are to show that for every $\varepsilon > 0$,

$$|S_n \cap \mathbb{Z}^d| \leq (V_d + \varepsilon) n^{d-1} \tag{6.1}$$

for all large enough n. Fix $\varepsilon > 0$.

We claim first that S_n satisfies the non-degeneracy condition (2.4). Otherwise $S_n \cap \mathbb{Z}^d$ is contained in a hyperplane of normal w with $we_i \neq 0$ for some $i \in [d]$, $i = d$ say. Projecting the points of $S_n \cap \mathbb{Z}^d$ to the hyperplane $x_d = 0$ we get lattice points on a facet of Q_n, and distinct points project to distinct points. No facet contains more than $(n+1)^{d-1}$ lattice points, so $|S_n \cap \mathbb{Z}^d| \leq (n+1)^{d-1}$, which is smaller than $M_d(n) \geq \sqrt{d}\, n^{d-1} + O(n^{d-2})$. The last inequality follows from Corollary 5.1 and from $V_d \geq \sqrt{d}$.

Now Theorem 2.3 gives

$$\left| \operatorname{vol} S_n - |S_n \cap \mathbb{Z}^d| \right| \ll \operatorname{vol} S_n \cdot \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_{d-1}} \right). \tag{6.2}$$

Here of course $\alpha_i = \alpha_i(n)$, $\beta_i = \beta_i(n)$ and $\gamma_i = \gamma_i(n) = \beta_i(n) - \alpha_i(n)$. A simple case is when there is a sequence n' of positive integers such that $\lim \gamma_i(n') = \infty$ for every $i \in [d]$. For simplicity of writing we use n instead of n'. Then (6.2) implies that

$$|S_n \cap \mathbb{Z}^d| = \operatorname{vol} S_n \cdot (1 + o(1)) \leq V_d n^{d-1}(1 + o(1)),$$

so (6.1) holds true indeed.

Assume next that there is a subsequence n' of the previous subsequence such that $\gamma_i(n')$ is bounded for some $i \in [d]$, $i = 1$ say. We write again n instead of n'. Let $G^n = \{g_1^n, \dots, g_d^n\}$ be the dual basis of $F = F^n$. Set

$$\alpha(n) = \min\{g_1^n x : x \in S_n\} \quad \text{and} \quad \beta(n) = \max\{g_1^n x : x \in S_n\}.$$

Of course $\beta(n) - \alpha(n) = \gamma_1(n)$ and $\gamma_1(n)$ is bounded. So along another subsequence (to be denoted invariably by n) $\lim(\beta(n) - \alpha(n)) = \gamma$ for some $\gamma \geq 0$.

We claim now that the corresponding dual basis vector g_1^n is also bounded. This is simple again: otherwise the width of S_n in direction g_1^n is $\gamma/|g_1^n|$, which tends to zero as $n \to \infty$. But S_n contains a ball of radius $\gg 1$ (by Lemma 4.3), a contradiction. This implies that along a further subsequence, g_1^n is equal to a fixed primitive vector, g, say.

Define the strip

$$T_n = \{x \in \mathbb{R}^d : \alpha(n) \le gx \le \beta(n)\}.$$

Then $S_n \cap \mathbb{Z}^d \subset T_n$ because of the definition of $\alpha(n)$ and $\beta(n)$. Set $g_0 = g/|g|$. Let ϕ_n be the angle between g and $v(n)$, so $\cos\phi_n = v(n)g_0$. Define $\Psi : \mathbb{R}^d \to \Pi_n$ as the orthogonal projection to the 2-dimensional plane spanned by $v(n)$ and g. Note that here we can assume $g \ne v(n)$ since a minute change of $v(n)$ does not influence what cells the hyperplane $A(v(n), t)$ intersects.

Claim 6.1. *Along the present subsequence, $\phi_n \ll \frac{\gamma}{|g|n}$ and so $v(n) \to g_0$.*

Proof. We drop the subscript n whenever possible. $\Psi(Q_n)$ is a centrally symmetric convex polygon. The Ψ-image of the lattice hyperplane $A(g, \lceil\alpha(n)\rceil + h)$ is the line ℓ_h on Π_n, represented by a horizontal line in Figure 1, $h = 0, 1, \dots, \gamma$. Here we take the upper integer part of $\alpha(n)$ because we need lattice hyperplanes. We should also take $h = 0, 1, \dots, \lfloor\gamma\rfloor$ because γ may not be an integer. But for simplicity we keep writing γ now and in what follows.

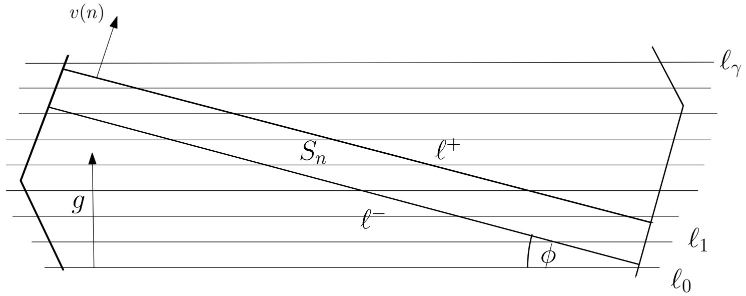

Fig. 1. $v(n)$ tends to g_0.

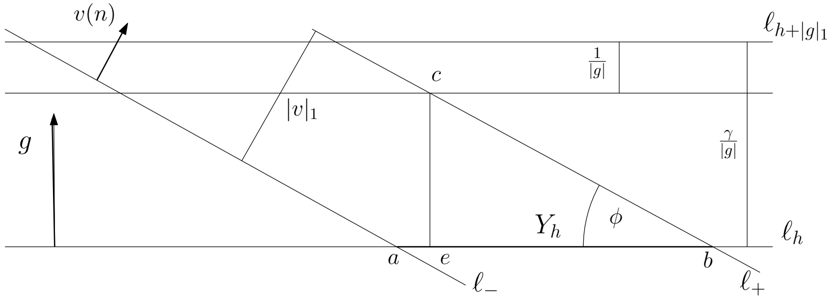

Fig. 2. Projection onto Π.

The Ψ-image of the two hyperplanes bounding $S_n = S(v(n), t(n))$ are the lines ℓ^+ and ℓ^- in Figure 2. Their distance is $|v|_1$. The length of the segments $\ell^+ \cap \Psi(Q^n)$ and $\ell^+ \cap \Psi(Q^n)$ is $\gg n$ because S_n contains the ellipsoid from Lemma 4.3 and $S_n \subset T_n$. So with $\phi = \phi_n$,

$$\sin\phi \ll \frac{\gamma}{n|g|}. \qquad \blacksquare$$

Define now for $h = 0, 1, \dots, \gamma$ the $d-1$-dimensional convex polytope

$$P_h^n = S_n \cap A(g, \lceil \alpha(n) \rceil + h).$$

Every lattice point in S_n belongs to some P_h^n. The proof of the upper bound on $M_d(n)$ is based on estimating $\sum_{h=0}^{\gamma} |P_h^n \cap \mathbb{Z}^d|$.

Define a map $\Phi = \Phi_n : \mathbb{R}^d \to \mathbb{R}^d$ by $\Phi(x) = x/n$. Then $\Phi(P_h^n)$ is a convex compact set in Q^d for all $h \in \{0, 1, \dots, \gamma\}$. We use the Blaschke selection theorem (see for instance [6]): along a subsequence (denoted by n again) $\Phi(P_h^n)$ tends to a convex polytope P_h for $h \in \{0, 1, \dots, \gamma\}$. Note also that each P_h lies in $A(g, t) \cap Q^d$ for some fixed t.

Let I denote the set of $h \in \{0, \dots, \gamma\}$ with $\text{vol}_{d-1} \Psi(P_h) > C_0$ where $C_0 > 0$ will be specified later. Write J_1 resp. J_0 for those $h \notin I$ for which P_h^n does (resp. does not) contain d affinely independent vectors from $\mathbb{Z}^d$. We are going to estimate $|P_h^n \cap \mathbb{Z}^d|$ separately for h in I, in J_0 and in J_1.

When $h \in J_0$, Lemma 5.1 applies and gives $|P_h^n \cap \mathbb{Z}^d| \ll n^{d-2}$. The total contribution of such P_h^ns to $|S_n \cap \mathbb{Z}^d|$ is at most $\ll |J_0| n^{d-2}$.

For $h \in J_1$, Lemma 5.2 shows that $|P_h^n \cap \mathbb{Z}^d| \le C_d \frac{1}{|g|} \text{vol}_{d-1} P_h^n$. Here $C_d > 0$ is the constant implicit in the $\ll$ notation. The total contribution of such P_h^ns to $|S_n \cap \mathbb{Z}^d|$ is at most $\ll |J_1| \frac{C_d C_0}{|g|} n^{d-1} \le |J_1| C_d C_0 n^{d-1}$.

For $h \in I$, let E_h^n be the ellipsoid of largest volume inscribed in P_h^n with half-axes of length $a_1, \dots, a_{d-1}$. The Loewner–John theorem implies that

$$\text{vol}_{d-1} E_h^n \ge (d-1)^{-(d-1)} \text{vol}_{d-1} P_h^n \ge C_0 \left(\frac{n}{d-1} \right)^{d-1}.$$

Also $\text{vol}_{d-1} E_h^n = \kappa_{d-1} \prod_{i=1}^{d-1} a_i$ where κ_{d-1} is the volume of the $d-1$-dimensional unit ball. As $a_i \le \sqrt{d}\, n$ for all i, the minimal a_i satisfies $a_i \gg C_0 n$. So P_h^n contains a ball of radius $\gg C_0 n$. It is also clear that for large enough n, P_h^n contains d affinely independent points from $\mathbb{Z}^d$. So we can apply Lemma 5.3: for $h \in I$,

$$|P_h^n \cap \mathbb{Z}^d| \le \frac{1}{|g|} \text{vol}_{d-1} P_h^n \cdot \left(1 + |g| O\left(\frac{1}{n} \right) \right),$$

showing that the total contribution of those P_h^ns to $|S_n \cap \mathbb{Z}^d|$ is at most

$$\frac{1}{|g|} \sum_{i \in I} \text{vol}_{d-1} P_h^n \cdot \left(1 + |g| O\left(\frac{1}{n} \right) \right).$$

Lemma 6.1. *With the previous notation,*

$$\frac{1}{|g|} \sum_{h=0}^{\gamma} \text{vol}_{d-1} P_h^n \le V_d(g) n^{d-1} (1 + o(1)).$$

We postpone the proof to the next section. We show now how to complete the proof of Theorem 1.1 using this lemma.

The number of lattice points in S_n is $V_d(g)n^{d-1}(1+o(1)) \le V_d n^{d-1} + \frac{1}{2}\varepsilon n^{d-1}$ if n is large enough plus an error term of the form

$$|J_0|n^{d-2} + |J_1|C_d C_0 n^{d-1}$$

times a constant depending only on d. Here $|J_1|, |J_0| \le \gamma$, and g and γ are fixed. So if we choose $C_0 > 0$ small enough the error term becomes smaller than $\frac{1}{2}\varepsilon n^{d-1}$. ∎

7. Proof of Lemma 6.1

We first note that $(v(n)-g_0)^2 = 2-2\cos\phi = 2\sin^2\phi/(1+\cos\phi)$. Set $Y_h = \ell_h \cap \Psi(S_n)$; it is a segment of length $|v|_1/\sin\phi$. Let $Y_h^* \subset Y_h$ be the segment that one gets after deleting the segment of length $\sqrt{2d}$ from the left end of Y_h.

Claim 7.1. *Each vertical line intersects at most $|g|_1 + 1$ segments Y_h and at most $|g|_1$ segments Y_h^*, $h = 0, 1, \dots, \gamma$.*

Proof. This is elementary plane geometry using the fact that $v(n)$ and g_0 are very close to each other. We assume $v(n) > 0$; then $g \ge 0$ as well and $|v(n)|_1 = v(n)e$, $|g|_1 = ge$. Assume ℓ^- intersects ℓ_h in a point a, and ℓ^+ intersects ℓ_h resp. $\ell_{h+|g|_1}$ in points b and c, and let e denote the orthogonal projection of c to ℓ_h. We consider a, b, e as real numbers on the x-axis. The length of Y_h is $b - a = ve/\sin\phi$, and $b - e = |g_1|/(|g|\tan\phi) = g_0 e/\tan\phi$ and

$$\begin{aligned} e - a &= \frac{ve}{\sin\phi} - \frac{g_0 e}{\tan\phi} = \frac{1}{\sin\phi}(ve - g_0 e\cos\phi) \\ &= \frac{1}{\sin\phi}[(v - g_0)e + g_0 e(1-\cos\phi)] \\ &\le \frac{1}{\sin\phi}\left(\frac{\sqrt{2}\sin\phi}{\sqrt{1+\cos\phi}}\sqrt{d} + \frac{\sin^2\phi}{1+\cos\phi}\right) < \sqrt{2d}, \end{aligned}$$

as one can check easily. This implies that Y_h^* is contained in the interval $[e, b]$. Moreover, a vertical line intersecting the segment $[a, e]$ intersect $Y_h, Y_{h+1}, \dots, Y_{h+|g|_1}$ but no other Y_i. And a vertical line intersecting $(e, b]$ intersects $Y_h, \dots, Y_{h+|g|_1-1}$ but no other Y_i. ∎

The claim implies what we need. Note that $P_h^n = \Psi^{-1}(Y_h) \cap Q^n$, and define $P_h^{n*} = \Psi^{-1}(Y_h^*) \cap Q^n$. Then $P_h^{n*} \subset P_h^n$ and evidently

$$\mathrm{vol}_{d-1} P_h^n - \mathrm{vol}_{d-1} P_h^{n*} = O(n^{d-2}).$$

Recalling that $\Phi(x) = x/n$ we have

$$\sum_{h=0}^{\gamma} \mathrm{vol}_{d-1} P_h^{n*} = n^{d-1} \sum_{h=0}^{\gamma} \mathrm{vol}_{d-1} \Phi(P_h^{n*}).$$

The sets $\Phi(P_h^{n*})$ tend to a set $P^h \subset A(g,t) \cap Q^d$ for the same t as before, so $\mathrm{vol}_{d-1}\, \Phi(P_h^{n*}) = n^{d-1}\, \mathrm{vol}_{d-1}\, P^h \cdot (1+o(1))$. The sets P^h for $h = 0, 1, \dots, \gamma$ cover $A(g,t) \cap Q^d$ at most $|g|_1$ times. So their total $d-1$-volume is at most $|g|_1\, \mathrm{vol}_{d-1}\, A(g,t) \cap Q^d$. Thus

$$\begin{aligned}\sum_{i=0}^{\gamma} \mathrm{vol}_{d-1}\, P_h^n &\le n^{d-1} \sum_{i=0}^{\gamma} \mathrm{vol}_{d-1}\, \Phi(P_h^{n*}) + O(n^{d-2}) \\ &\le n^{d-1} \sum_{i=0}^{\gamma} \mathrm{vol}_{d-1}\, P^h \cdot (1+o(1)) \\ &\le n^{d-1} |g|_1\, \mathrm{vol}_{d-1}(A(g,t) \cap Q^d)(1+o(1)).\end{aligned}$$

So indeed

$$\begin{aligned}\frac{1}{|g|} \sum_{i=0}^{\gamma} \mathrm{vol}_{d-1}\, P_h^n &\le \frac{|g|_1}{|g|}\, \mathrm{vol}_{d-1}(A(g,t) \cap Q^d)(1+o(1)) \\ &= V_d(g_0)(1+o(1))\end{aligned}$$

because $\frac{|g|_1}{|g|}\, \mathrm{vol}_{d-1}(A(g,t) \cap Q^d) \le V_d(g_0)$ by the definition of $V_d(g_0)$. ■

8. Proof of Theorem 2.1

We construct a homotopy $t \mapsto K_t$ where $t \in [0,1]$, K_t is a convex body in $\mathbb{R}^d$ satisfying $K_0 = K,\ K_1 = L$ and the monotonicity condition $K_t \subset K_s$ for $t < s$. By monotonicity, boundary cells of K_t may become inside cells for K_s, and the point in the argument is that whenever a boundary cell is lost, another one emerges.

The simplest homotopy is $K_t = (1-t)K + tL$, and this works under the following non-degeneracy condition:

(∗) whenever $w \in \partial K_t \cap \mathbb{Z}^d$, then $w \notin K_s$ for $s < t$ and $w \in \mathrm{int}\, K_s$ for all $s > t$, and K_t has an outer normal u at $w \in \partial K_t$ with no coordinate zero.

Under this condition the proof is easy. As t increases, a cell $C(z)$, say, is boundary for K_t with $t < t_0$ just slightly smaller than t_0 but $C(z) \subset K_{t_0}$ and so it becomes inside for $t > t_0$. Then there is a vertex w of $C(z)$ such that $w \notin K_t$ for $t < t_0$, but of course $w \in K_{t_0}$ and even $w \in \partial K_{t_0}$. Let H be a supporting hyperplane to K_{t_0} at w whose outer normal has no zero coordinate. Then $w \in K_{t_0}$ and $C(z)$ and K_{t_0} are on the same side of H. There is a unique cell $C(z')$ (unique because of condition (∗)) on the other side of H with $w \in C(z')$. This unique cell was outside for K_t with $t < t_0$ and becomes boundary for K_t for $t \in [t_0, t_0 + \delta)$ for a suitable small $\delta > 0$. So when the boundary cell $C(z)$ is lost at t_0, another boundary cell appears. Note that $C(z) \cap H = C(z') \cap H = \{w\}$.

We still have to check that the same cell $C(z')$ cannot appear twice. So assume the contrary, that is, there is another cell $C(z^*)$ that is boundary for K_t, for t slightly smaller than t_0 but $C(z^*) \subset K_{t_0}$ and $C(z^*)$ has a vertex w^* with $w^* \notin K_t$ for $t < t_0$ but $w^* \in \partial K_{t_0}$. We cannot have $w = w^*$ here since that would imply $C(z) = C(z^*)$. Then w and w^* are distinct vertices of $C(z')$ and the segment $[w, w^*]$ is on the boundary of both $C(z')$ and K_{t_0}. Then $[w, w^*] \cap H = \{w\}$ for the previous hyperplane H supporting K_{t_0} at w with no zero coordinate, so $w^* \in K_{t_0}$ cannot hold.

To guarantee the non-degeneracy condition we proceed first by assuming that $K \subset \operatorname{int} L$ and that both K and L have smooth boundaries such that for every unit vector u there is a single point on ∂K resp. on ∂L where the outer normal to K and L is u. If this were not the case, we can replace K, L by suitable (and very close to K and L) convex bodies satisfying these conditions and having the same inside and boundary cells. With the new K and L the homotopy $K_t = (1-t)K + tL$ has the property that for every unit vector u there is a single point on ∂K_t where the outer normal to K_t is u. To see that this is indeed the case, let x_K and x_L be the unique points on the boundary of K and L with outer normal u. Then the maximum of $\{ux : x \in K_t\}$ is reached at the unique point $(1-t)x_K + tx_L \in K_t$, and the outer normal to K_t there is u.

This condition also guarantees that K_t has no line segment on its boundary. Assume that, on the contrary, ∂K_t contains a line segment and let u be the outer normal to the tangent hyperplane to K_t containing this segment. Then there is no unique point with outer normal u as every point on the segment has outer normal u.

Let us see finally that K_t satisfies condition $(*)$. Assume the cell $C(z)$ is boundary for K_t for $(t_0 - \delta, t_0)$ and is inside for K_{t_0}. Then there is a vertex w of $C(z)$ on ∂K_{t_0} with outer normal $u = (u_1, \dots, u_d)$ at w to K_{t_0}. Assume some coordinate of u is equal to zero, say $u_1 = 0$. Either $w + e_1$ or $w - e_1$ is in $C(z)$, say $w + e_1$. Then the segment $[w, w + e_1]$ lies both in K_{t_0} and in $C(z)$, and actually in the boundary of both because the hyperplane $\{x : ux = uw\}$ is tangent to both K_{t_0} and $C(z)$. ∎

Funding. Support from Hungarian National Research Grants No 131529, 131696, and 133919 is acknowledged.

References

[1] Aliev, I.: On the volume of hyperplane sections of a d-cube. Acta Math. Hungar. **163**, 547–551 (2021) Zbl 07399069 MR 4227797

[2] Ball, K.: Cube slicing in $\mathbf{R}^n$. Proc. Amer. Math. Soc. **97**, 465–473 (1986) Zbl 0601.52005 MR 840631

[3] Bárány, I., Frankl, P.: How (not) to cut your cheese. Amer. Math. Monthly **128**, 543–552 (2021) Zbl 1475.52020 MR 4265480

[4] Bárány, I., Vershik, A. M.: On the number of convex lattice polytopes. Geom. Funct. Anal. **2**, 381–393 (1992) Zbl 0772.52010 MR 1191566

[5] Beck, J.: On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. Combinatorica **3**, 281–297 (1983) Zbl 0533.52004 MR 729781

[6] Schneider, R.: Convex Bodies: the Brunn–Minkowski Theory. Encyclopedia Math. Appl. 151, Cambridge Univ. Press, Cambridge (2014) Zbl 1287.52001 MR 3155183

[7] Tao, T., Vu, V. H.: Additive Combinatorics. Cambridge Stud. Adv. Math. 105, Cambridge Univ. Press, Cambridge (2010) Zbl 1179.11002 MR 2573797