On a conjecture of Descartes

by
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To the memory of Eduard Wirsing

1.

It is well known that during the correspondence of Euler and Goldbach the following conjecture – today known as Goldbach's conjecture (or sometimes called even Goldbach conjecture) – was formulated in 1742.

Goldbach Conjecture (Binary Goldbach Conjecture). *Every even number greater than* 2 *can be written as the sum of two primes.*

In his original letter Goldbach formulated two similar, more complicated conjectures which were actually equivalent with the above and it was Euler who used the above formulation in his reply letter. However, he noted in the same letter that Goldbach mentioned him earlier in a conversation the above simpler and more elegant form. So it is fully justified to attribute the conjecture to Goldbach.

It is much less known – I learned it from a manuscript of D. Wolke – that Descartes (1591–1650) mentioned many years before the following assertion (without any proof), what we will call Descartes Conjecture.

Descartes Conjecture. *Every even number can be expressed as the sum of at most three primes.*

This assertion appeared in print first in the collected works of Descartes only in the 1908 edition ([3], Opuscula Posthuma, Excerpta Mathematica, Vol. 10, p. 298), so we can rightly assume that Goldbach and Euler did not hear about this before their correspondence in 1742.

Supported by National Research Development and Innovation Office, NKFIH, K 119528 and KKP 133819.
Keywords and phrases: Descartes conjecture, Goldbach conjecture, Exceptional set in Goldbach's problem.
2020 Mathematics Subject Classification: 11P32, 11P55.

Descartes does not mention odd numbers in his assertion, but the same assertion follows trivially for odd numbers from the assertion for even numbers.

It is also obvious that if an even N satisfies Descartes Conjecture then N or $N-2$ can be expressed as the sum of two primes. The converse is clearly also true.

In the present work we will investigate the number of possible exceptional Descartes numbers below a large bound X, that is ($\mathcal{P}$ denotes the set of primes)

$$D(X) = \#\left\{ n \le X; 2 \mid n, n \ne \sum_{i=1}^{j} p_i,\ p_i \in \mathcal{P},\ \text{for } j \le 3 \right\}. \tag{1.1}$$

It is trivial that $D(X) \le E(X)$, where

$$E(X) = \# \{ n \le X; 2 \mid n,\ n \ne p_1 + p_2,\ p_i \in \mathcal{P} \} \tag{1.2}$$

is the size of the exceptional set for Goldbach's problem.

The strongest published result

$$E(X) \ll X^{0.879} \tag{1.3}$$

is due to Wen Chao Lu [8]. We improved this to

$$E(X) \ll X^{0.72} \tag{1.4}$$

in a work in arXiv ([11]).

Our present goal is to show a sharper estimate for $D(X)$. Earlier methods did not allow to prove a distinctly sharper bound for even exceptional Goldbach numbers n if we knew that $n-2$ is also an exceptional Goldbach number.

The crucial point, which makes a more effective treatment of $D(X)$ possible is an approximate formula for the contribution of the major arcs [10]. This formula shows that a particular $\mathcal{L}$-zero close to the line $\operatorname{Re} s = 1$ can have a bad effect for the number of Goldbach decomposition of an even number n (more precisely, for the contribution of the major arcs to it), but not simultaneously for n and $n-2$. We will prove

Theorem 1. *$D(X) \ll_{\varepsilon} X^{3/5+\varepsilon}$ for any $\varepsilon > 0$.*

2 Notation. The role of the explicit formula

The explicit formula proved in [10] will play a central role in the proof of Theorem 1; in order to formulate it we first need to introduce the notation.

Let ε and ε_0 be small positive numbers, X be a number large enough ($X > X_0(\varepsilon, \varepsilon_0)$), and let us define

$$\text{(2.1)}\quad X_1 := X^{1-\varepsilon_0}, \;\; e(u) := e^{2\pi i u}, \;\; S(\alpha) := \sum_{X_1 < p \le X} \log p \, e(p\alpha), \;\; \mathcal{L} = \log X,$$

where p, p', p_i will always denote primes. $|\mathcal{M}|$ will denote the cardinality of the finite set $\mathcal{M}$. We will define the major ($\mathfrak{M}$) and minor ($\mathfrak{m}$) arcs through the parameters P and Q satisfying (c and C will denote generic absolute constants)

$$\text{(2.2)}\qquad (\log X)^C \le P \le X^{4/9-\varepsilon}, \quad Q = \frac{X}{P},$$

$$\text{(2.3)}\qquad \mathfrak{M} = \bigcup_{q \le P} \bigcup_{\substack{a \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right], \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}.$$

We will examine the number of Goldbach decompositions of even numbers $m \in [X/2, X]$ in the form

$$\text{(2.4)}\qquad R(m) = \sum_{\substack{p+p'=m \\ p,p' \ge X_1}} \log p \cdot \log p' = R_1(m) + R_2(m),$$

where

$$\text{(2.5)}\qquad R_1(m) = \int\limits_{\mathfrak{M}} S^2(\alpha) e(-m\alpha) d\alpha, \qquad R_2(m) = \int\limits_{\mathfrak{m}} S^2(\alpha) e(-m\alpha) d\alpha.$$

The now standard treatment of the minor arcs (Parseval's theorem and the estimate of Vinogradov, reproved in a simpler way by Vaughan) gives

$$\text{(2.6)}\qquad |R_2(m)| \le \frac{X}{\sqrt{\log X}} \quad \text{for } \; P \le X^{2/5}$$

apart from at most $C\frac{X}{P}\log^{10} X$ exceptional values m (see Section 5 of [10], for example).

In order to formulate the explicit formula for the major arcs in Goldbach's problem we will define the set $\mathcal{E} = \mathcal{E}(H, P, T, X)$ of generalized exceptional singularities of the functions L'/L for all primitive L-functions mod r, $r \leq P$, as follows ($\chi_0 = \chi_0(\bmod 1)$ is considered as a primitive character mod 1)

$$(2.7)\qquad \begin{aligned} &(\varrho_0, \chi_0) \in \mathcal{E} \quad \text{if} \ \ \varrho_0 = 1,\\ &(\varrho_i, \chi_i) \in \mathcal{E} \quad \text{if} \ \ \exists \chi_i, \ \operatorname{cond} \chi_i = r_i \leq P, \ L(\varrho_i, \chi_i) = 0,\\ &\qquad \beta_i \geq 1 - \frac{H}{\log X}, \quad |\gamma_i| \leq T, \end{aligned}$$

where zeros of L-functions are denoted by $\varrho = \beta + i\gamma = 1 - \delta + i\gamma$ and $\operatorname{cond} \chi$ denotes the conductor of χ. Zeros are counted with multiplicity. Let further

$$(2.8)\qquad \begin{aligned} A(\varrho) &= 1 \quad \ \ \text{if} \ \ \varrho = 1,\\ A(\varrho) &= -1 \quad \text{if} \ \ \varrho \neq 1. \end{aligned}$$

The expected main term of $R_1(m)$ is the well-known singular series of Hardy and Littlewood, arising from the effect of the pole of $\zeta(s)$ at $s = 1$:

$$(2.9)\qquad \mathfrak{S}(m) := \mathfrak{S}(\chi_0, \chi_0, m) := \prod_{p \mid m} \left(1 + \frac{1}{p-1}\right) \prod_{p \nmid m} \left(1 - \frac{1}{(p-1)^2}\right).$$

However, if we have zeros of moderate height near to the line $\operatorname{Re} s = 1$ then we necessarily have a number of secondary terms with coefficients $\mathfrak{S}(\chi_i, \chi_j, m)$ corresponding to the primitive characters belonging to generalized exceptional zeros. We will call these characters generalized exceptional characters, the corresponding singular series $\mathfrak{S}(\chi_i, \chi_j, m)$ generalized exceptional singular series. They can be expressed in a very complicated explicit form, proven in the Main Lemma of [10]. However, the important properties of it can be incorporated into the following theorem, where we use the notation and conditions of the present section.

Theorem A (Explicit formula). *Let $0 < \varepsilon < \varepsilon_0$, $2\varepsilon < \vartheta < \frac{4}{9} - \varepsilon$ be any numbers, $2 \mid m \in \left[\frac{X}{2}, X\right]$. Then there exists $P \in (X^{\vartheta - \varepsilon}, X^{\vartheta})$ such that for $X > X_0(\varepsilon)$*

$$(2.10)\qquad \begin{aligned} R_1(m) = &\sum_{\varrho_i \in \mathcal{E}} \sum_{\varrho_j \in \mathcal{E}} A(\varrho_i) A(\varrho_j) \mathfrak{S}(\chi_i, \chi_j, m) \frac{\Gamma(\varrho_i)\Gamma(\varrho_j)}{\Gamma(\varrho_i + \varrho_j)} m^{\varrho_i + \varrho_j - 1}\\ &+ O_\varepsilon\left(X e^{-cH} + \frac{X}{\sqrt{T}} + X^{1-\varepsilon}\right), \end{aligned}$$

where the generalized singular series satisfy

$$|\mathfrak{S}(\chi_i, \chi_j, m)| \le \mathfrak{S}(\chi_0, \chi_0, m) = \mathfrak{S}(m); \tag{2.11}$$

further for any η small enough

$$|\mathfrak{S}(\chi_i, \chi_j, m)| \le \eta, \tag{2.12}$$

unless the following three conditions all hold,

$$r_i | C(\eta) m, \;\; r_j | C(\eta) m, \quad \operatorname{cond} \chi_i \chi_j < \eta^{-3} \tag{2.13}$$

where $C(\eta)$ is a suitable constant depending only on η.

Its proof follows from Theorem 1 [10] and Main Lemma 1 of [10].

Remark 1. A very important feature of the explicit formula is that the number K of generalized exceptional zeros appearing in (2.10) is by the log-free zero density theorem of Jutila [4].

$$N^*(\alpha, T, Q) \ll_\varepsilon (Q^2 T)^{(2+\varepsilon)(1-\alpha)} \quad \text{for} \;\; \varepsilon > 0, \;\; \alpha \ge 4/5 \tag{2.14}$$

from which

$$K \le C e^{2H}, \tag{2.15}$$

so it is bounded by an absolute constant (depending on ε), if we choose H as a sufficiently large absolute constant depending on ε, which we suppose later on in the proof of Theorem 1. Similarly, we will choose T as a sufficiently large constant depending on ε.

Remark 2. A very important information of the explicit formula is the relation (2.13) which shows that a generalized exceptional character χ_i causes a problem only for the quasi-multiples m of its conductor r_i.

Although the quoted explicit formula is in general a good starting point for the proof of

$$R_1(m) > \varepsilon \mathfrak{S}(m) m \tag{2.16}$$

if ϑ is small enough, the argument breaks down in case of the existence of a Siegel-zero $1 - \delta$ corresponding to $L(s, \chi_1)$, in which case we might have $\mathfrak{S}(\chi_1, \chi_1, m) = -\mathfrak{S}(m)$ and we cannot show the crucial relation (2.22) if δ is small enough. In this case the Deuring–Heilbronn phenomenon can help. This case was worked out as Theorem 2 in [10] which we quote now as

Theorem B. *Let $\varepsilon' > 0$ be arbitrary. If $X > X(\varepsilon')$, ineffective constant and there exists a Siegel zero β_1 of $L(s, \chi_1)$ with*

$$\beta_1 > 1 - h/\log X, \quad \operatorname{cond} \chi_1 \le X^{\frac{4}{9}-\varepsilon'}, \tag{2.17}$$

where h is a sufficiently small constant depending on ε', then

$$E(X) < X^{\frac{3}{5}+\varepsilon'}. \tag{2.18}$$

Remark 3. Let us fix a sufficiently small $\varepsilon > 0$. Then in the proof of Theorem 1 we are entitled to suppose that all $L(s, \chi)$ functions $\bmod r \le P$ satisfy

$$L(s, \chi) \ne 0 \quad \text{for} \quad s \in [1 - c_0/\log X,\ 1] \tag{2.19}$$

if we choose $\vartheta \le 0.44$. In other words, we can suppose that there are no exceptional zeros $1-\delta$ satisfying $\delta < c_0/\log X$ with a small but fixed $c_0 > 0$.

The well-known relation (cf. [5], p. 46) ($\operatorname{Re} w, \operatorname{Re} z > 0$)

$$\frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)} = B(w, z) = \int_0^1 x^{w-1}(1-x)^{z-1}dx \tag{2.20}$$

tells us that
(2.21)
$$|B(\varrho_i, \varrho_j)| \le |B(\operatorname{Re} \varrho_i, \operatorname{Re} \varrho_j)| = B(1,1) + O(1/\log X) = 1 + O(1/\log X).$$

Hence, taking into account the relations (2.11)–(2.13) we see that the estimation (2.16) will follow, if we can show

$$\sideset{}{^*}\sum_{\substack{\varrho_i, \varrho_j \in \mathcal{E} \\ (\varrho_i, \varrho_j) \ne (1,1)}} X^{-\delta_i - \delta_j} \le 1 - \frac{5}{2}\varepsilon, \tag{2.22}$$

where the $*$ means that the additional condition (2.13) is satisfied for the pairs (ϱ_i, ϱ_j) of zeros in the summation with η chosen as in (4.3) of Section 4.

The expression (2.22) can be estimated directly by density theorems and the Deuring–Heilbronn phenomenon, as done in the earlier estimates of Chen-Liu [1], Hongze Li [6], [7], and Lu [8]. It also resembles the well-studied problem of the Linnik-constant, with the seemingly major disadvantage that

(†) the zeros do not belong to a fixed modulus $q \le P$

but to a set of different moduli $r_i \le P$.

During the proof we will show that this disadvantage can be overwhelmed thanks to the information (2.13) supplied by the explicit formula.

3 Contribution of the minor arcs

We will use the same treatment for the minor arcs as all earlier works beginning with the pioneering one of I. M. Vinogradov [13] in which he proved the ternary Goldbach conjecture, the so-called three primes theorem for every sufficiently large odd numbers, i.e.

$$2N + 1 = p_1 + p_2 + p_3, \quad p_i \in \mathcal{P}, \quad \text{for} \quad N > N_0. \tag{3.1}$$

This is based for his estimation of trigonometric sums for primes, simplified later by Vaughan (see [2], Chapter 25)

$$S(\alpha) \ll \left(\frac{X}{\sqrt{q}} + X^{4/5} + (Xq)^{1/2} \right) \log^4 X \quad \text{if} \quad \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}, \quad (a, q) = 1. \tag{3.2}$$

This implies by Parseval's identity

$$\sum_{m \leq x} R_2^2(m) = \int_{\mathfrak{m}} |S^4(\alpha)| d\alpha \leq \big(\max_{\mathfrak{m}} |S(\alpha)| \big)^2 \int_0^1 |S(\alpha)|^2 d\alpha \ll \max \left(\frac{X^2}{P}, X^{8/5} \right) X \mathcal{L}^9. \tag{3.3}$$

Vinogradov chose $P = \mathcal{L}^A$ (with any large A). This choice makes an asymptotic evaluation of $\mathcal{R}_1(n)$ possible for all $n \leq X$. On the other hand, in this case we get a relatively weak upper estimate for the contribution of the minor arcs due to the moderate size of P (cf. (3.3)). It was the idea of Vaughan [12] and Montgomery–Vaughan [9] to choose P larger. However, then we lose the possibility of asymptotic evaluation of $R_1(n)$ due to the possible existence of a Siegel-zero. The situation is somewhat easier by a result of Landau and Page (see [2], Chapter 14) according to which for a given large X we might have only at most one Siegel-zero with a character with conductor $\leq X$. Then the idea of [12] and [9] was to evaluate the effect of the possible single Siegel-zero for $R_1(n)$. In [9] they are able to choose $P = X^c$ in such a way with a small fixed absolute constant $c > 0$.

In our present work we are able to work with a $P = X^\vartheta$ for any fixed constant $\vartheta < 4/9$, e.g. with $\vartheta = 0.4$ or 0.44.

Consequently (choosing $P \geq X^{2/5}$) we have

$$\big| R_2(m) \big| \leq X^{1-\varepsilon} \quad \text{with} \quad \mathcal{O}_\varepsilon\big(X^{3/5+3\varepsilon} \big) \quad \text{exceptions.} \tag{3.4}$$

4 Proof of Theorem 1

We will choose $P_0 = X^{\vartheta+2\varepsilon}$, so our P will satisfy

$$P \in \left[X^{\vartheta+\varepsilon}, X^{\vartheta+2\varepsilon}\right]. \tag{4.1}$$

Thus the exceptional set arising from the minor arcs (2.6) will be $o(X^{1-\vartheta})$ (cf. (3.3)–(3.4)).

We will distinguish two cases.

Case 1. All zeros of all $\mathcal{L}$-functions with a conductor $\leq P$ satisfy $\delta \geq 5\varepsilon/\mathcal{L}$ i.e. $\beta = \text{Re } \varrho \leq 1 - 5\varepsilon/\mathcal{L}$.

Case 2. There exists a (real) Siegel zero with a conductor $\leq P$ satisfying $\delta < 5\varepsilon/\mathcal{L}$ i.e. $\beta > 1 - 5\varepsilon/\mathcal{L}$.

In Case 1 we consider the set $\mathcal{R}$ of the K generalized exceptional zeros appearing in (2.10) whose number K is bounded by an absolute constant depending on ε,

$$0 \leq K \leq K(\varepsilon) - 1 \tag{4.2}$$

according to (2.15) since we will choose H as a big constant depending on ε. (If $K = 0$ we are ready.)

Let us choose now

$$\eta = \frac{\varepsilon}{K^2(\varepsilon)}, \tag{4.3}$$

and write

$$C(\eta) = C_1(\varepsilon). \tag{4.4}$$

In this case the total contribution of terms not satisfying (2.13) will be really less than εX in (2.10), so (2.22) will really imply (2.16). Let us divide now the even numbers m in $[X/2, X]$ into at most $2^{|\mathcal{R}|}$ different classes $\mathcal{M}(\mathcal{R}')$ according to the subset $\mathcal{R}' \subset \mathcal{R}$ of generalized exceptional zeros which belong to primitive characters with moduli dividing $C_1(\varepsilon)m$

$$\mathcal{M}(\mathcal{R}') = \left\{m \in [X/2, X],\ 2 \mid m,\ r_i \mid C_1(\varepsilon)m \Leftrightarrow r_i \in \mathcal{R}'\right\}. \tag{4.5}$$

(The subset might be empty for some $\mathcal{R}' \subset \mathcal{R}$; for example, if $\underset{r_i \in \mathcal{R}'}{\text{l.c.m.}}[r_i] > XC_1(\varepsilon)$.)

We have clearly

$$q(\mathcal{R}') := \text{l.c.m.}[r_i;\ r_i \in \mathcal{R}'] \;\Big|\; C_1(\varepsilon)m \quad \text{for } m \in \mathcal{M}(R'). \tag{4.6}$$

Let us consider now a pair of classes $\mathcal{R}_1', \mathcal{R}_2'$ and the quantities

$$\mathcal{M}(\mathcal{R}_1'), \quad \mathcal{M}(R_2'), \quad q(\mathcal{R}_1'), \quad q\mathcal{R}_2'). \tag{4.7}$$

From (4.5)–(4.7), applied for m and $m-2$ we obtain with the notation

$$\text{g.c.d.}\big(q(\mathcal{R}_1'), q(\mathcal{R}_2')\big) = d, \quad q(\mathcal{R}_1') = q_1 d, \ q(\mathcal{R}_2') = q_2 d \tag{4.8}$$

the relation

$$d \mid 2C_1(\varepsilon). \tag{4.9}$$

Hence

$$C_1(\varepsilon)m \equiv 0 \pmod{q_1}, \quad C_1(\varepsilon)m \equiv 2C_1(\varepsilon) \pmod{q_2} \tag{4.10}$$

which implies that there is an $a_\varepsilon(m)$ with

$$C_1(\varepsilon)m \equiv a_\varepsilon(m) \qquad \pmod{q_1 q_2}. \tag{4.11}$$

The number of $m \le x$ with (4.11) is by (4.8)–(4.11)

$$\ll_\varepsilon \frac{X}{q(\mathcal{R}_1)q(\mathcal{R}_2')} + 1. \tag{4.12}$$

This means that from the point of proving Theorem 1 we can restrict our attention to the case when

$$\min\big(q(\mathcal{R}_1'), q(\mathcal{R}_2')\big) \le X^{1/5}. \tag{4.13}$$

Summarizing the content of Sections 2–4 let us suppose that

$$P \in \big[X^{2/5}, X^{2/5+\varepsilon}\big], \quad m \le X$$

and

$$R(m) = R(m-2) = 0. \tag{4.14}$$

Taking into account that the mean square of the contribution $R_2(m)$ of the minor arcs is small in case of $P \ge X^{2/5}$, i.e., by (3.3)–(3.4) it is sufficient to show that

$$\max\big(R_1(m), R_1(m-2)\big) > X\mathcal{L}^{-1/2}. \tag{4.15}$$

The arguments of (4.7)–(4.13) show that apart from a possible exceptional set of size $\mathcal{O}(X^{3/5+\varepsilon})$ we can suppose that (4.13) holds, so WLOG we can assume that

$$q_0 := q(\mathcal{R}_1') \le X^{1/5}, \quad A := \log X / \log q_0 \ge 5. \tag{4.16}$$

However, in Case 1, i.e. if there are no Siegel-zeros then the main result of [11] asserts that if (4.16) holds, then by restricting the set $\mathcal{E}$ to $\mathcal{E}'$ for $\mathcal{L}$-zeros with cond $\chi \mid q_0$ and cond $\chi_i \chi_j < C_0(\varepsilon)$ we have to show

$$S := \sum_{\varrho_i, \varrho_j \in \mathcal{E}', (\varrho_i, \varrho_j) \ne (1,1)} q_0^{-A(\delta_i + \delta_j)} < 1 - 5\varepsilon. \tag{4.17}$$

This is proved in [11] even for $A = \frac{25}{7}$. The case when the LHS of (4.17) is maximal is treated in (9.36)–(9.37) of [11] (cf. the notation (2.38) of [11]) and yields ($\lambda_1 = \delta_1 \log q_0$) for $A = 5$, λ_1 small

$$S \le e^{-2A\lambda_1} + 5\lambda_1 < 1 - A\lambda_1/2 = 1 - \delta_1 \log X/2 \le 1 - 5\varepsilon/2. \tag{4.18}$$

The arguments of Section 2, actually a summary of the results of [10], show that (4.17) really proves that if m is not in an exceptional set of size $O(X^{3/5+\varepsilon})$ estimated in (4.12) then

$$R_1(m) > (1 - \varepsilon) m \mathfrak{S}(m) \tag{4.19}$$

Repeating the arguments for the intervals $\left[X \cdot 2^{-\nu-1}, X \cdot 2^{-\nu}\right]$ we obtain

$$D(X) \ll_\varepsilon X^{3/5+\varepsilon} \tag{4.20}$$

Finally in Case 2 Theorem B proves (4.20) even in the sharper form $E(X) \ll_\varepsilon X^{3/5+\varepsilon}$ which clearly implies the same inequality for $E(X)$ replaced by $D(X)$, i.e. (4.20).

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