

Additive solvability and linear independence of the solutions of a system of functional equations

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Abstract. The aim of this paper is twofold. On one hand, the additive solvability of the system of functional equations

$$d_k(xy) = \sum_{i=0}^k \Gamma(i, k-i) d_i(x) d_{k-i}(y) \quad (x, y \in \mathbb{R}, k \in \{0, \dots, n\})$$

is studied, where $\Delta_n := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i, j \text{ and } i + j \leq n\}$ and $\Gamma: \Delta_n \rightarrow \mathbb{R}$ is a symmetric function such that $\Gamma(i, j) = 1$ whenever $i \cdot j = 0$. On the other hand, the linear dependence and independence of the additive solutions $d_0, d_1, \dots, d_n: \mathbb{R} \rightarrow \mathbb{R}$ of the above system of equations is characterized. As a consequence of the main result, for any nonzero real derivation $d: \mathbb{R} \rightarrow \mathbb{R}$, the iterates d^0, d^1, \dots, d^n of d are shown to be linearly independent, and the graph of the mapping $x \mapsto (x, d^1(x), \dots, d^n(x))$ to be dense in \mathbb{R}^{n+1} .

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1. Introduction

Given a real linear space X , a function $a: \mathbb{R} \rightarrow X$ is called *additive* if

$$a(x + y) = a(x) + a(y) \quad (x, y \in \mathbb{R}). \quad (1)$$

It is a nontrivial fact that additive functions may satisfy further functional equations. Among these particular additive functions the so-called derivations

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play an important role. An additive function $d: \mathbb{R} \rightarrow X$ is called a *derivation* (cf. [6], [8]) if it satisfies the (first-order) *Leibniz Rule*:

$$d(xy) = xd(y) + yd(x) \quad (x, y \in \mathbb{R}). \quad (2)$$

Putting $x = y = 1$ into (10), we get $d(1) = 0$, hence, by the \mathbb{Q} -homogeneity of additive functions, it follows that derivations vanish at rational numbers. Therefore, assuming that X is equipped with a Hausdorff vector topology, the only continuous derivation is the identically zero function. It can be shown that derivations with weak regularity properties are necessarily continuous and consequently are identically equal to zero. On the other hand, there exists derivations that are discontinuous and henceforth very irregular (see [6]). More generally, for any algebraic base B of \mathbb{R} , and for any function $d_0: B \rightarrow X$, there exists a unique derivation $d: \mathbb{R} \rightarrow X$ such that $d|_B = d_0$.

Given a real-valued derivation $d: \mathbb{R} \rightarrow \mathbb{R}$, one can prove by induction that the iterates $d^0 := \text{id}$, $d^1 := d$, \dots , $d^n := d \circ d^{n-1}$ of d satisfy the following higher-order Leibniz Rule:

$$d^k(xy) = \sum_{i=0}^k \binom{k}{i} d^i(x) d^{k-i}(y) \quad (x, y \in \mathbb{R}, k \in \{1, \dots, n\}). \quad (3)$$

Motivated by this property, Heyneman–Sweedler [3] introduced the notion of n th-order derivation (in the context of functions mapping rings to modules, however, we will restrict ourselves only to real functions). Given $n \in \mathbb{N}$, a sequence of additive functions $d_0, d_1, \dots, d_n: \mathbb{R} \rightarrow \mathbb{R}$ is termed a *derivation of order n* , if $d_0 = \text{id}$ and, for any $k \in \{1, \dots, n\}$,

$$d_k(xy) = \sum_{i=0}^k \binom{k}{i} d_i(x) d_{k-i}(y) \quad (x, y \in \mathbb{R}) \quad (4)$$

is fulfilled.

Clearly, a pair (id, d) is a first-order derivation if and only if d is a derivation. More generally, if $d: \mathbb{R} \rightarrow \mathbb{R}$ is a derivation, then the sequence (d^0, d^1, \dots, d^n) is a derivation of order n . However, if $\tilde{d}: \mathbb{R} \rightarrow \mathbb{R}$ is a nontrivial derivation and $n \geq 2$, then $(d^0, d^1, \dots, d^{n-1}, d^n + \tilde{d})$ is also an n th-order derivation where the last element is not the n th iterate of the derivation d .

The aim of this paper is twofold. On one hand, we study the additive solvability of the following system of functional equations:

$$d_k(xy) = \sum_{i=0}^k \Gamma(i, k-i) d_i(x) d_{k-i}(y) \quad (x, y \in \mathbb{R}, k \in \{0, \dots, n\}), \quad (5)$$

where

$$\Delta_n := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i, j \text{ and } i + j \leq n\}, \quad (6)$$

and $\Gamma: \Delta_n \rightarrow \mathbb{R}$ is a symmetric function such that $\Gamma(i, j) = 1$ whenever $i \cdot j = 0$. On the other hand, we characterize the linear dependence and independence of the additive solutions $d_0, d_1, \dots, d_n: \mathbb{R} \rightarrow \mathbb{R}$ of (5).

2. On the additive solvability of the system of functional equations (5)

We recall first a particular case of the following result of Ebanks [1, Theorem 3] (which generalizes a result of Jessen–Karpf–Thorup [4]):

Lemma 1. *Let X be real linear space and $C, D: \mathbb{R}^2 \rightarrow X$. Then there exists a function $f: \mathbb{R} \rightarrow X$ such that*

$$\begin{aligned} C(x, y) &= f(x + y) - f(x) - f(y) \quad (x, y \in \mathbb{R}), \\ D(x, y) &= f(xy) - xf(y) - yf(x) \quad (x, y \in \mathbb{R}) \end{aligned} \tag{7}$$

if and only if C, D satisfy the following system of equations

$$\begin{aligned} C(x + y, z) + C(x, y) &= C(x, y + z) + C(y, z) & (x, y, z \in \mathbb{R}), \\ D(x, y) &= D(y, x) & (x, y \in \mathbb{R}), \\ D(xy, z) + zD(x, y) &= D(x, yz) + xD(y, z) & (x, y, z \in \mathbb{R}), \\ C(xz, yz) - zC(x, y) &= D(x + y, z) - D(x, z) - D(y, z) & (x, y, z \in \mathbb{R}). \end{aligned} \tag{8}$$

As a trivial consequence of this result, we can characterize those two-variable functions that are identical to the Leibniz difference of an additive function.

Corollary 2. *Let X be a real linear space and $D: \mathbb{R}^2 \rightarrow X$. Then there exists an additive function $f: \mathbb{R} \rightarrow X$ fulfilling functional equation*

$$D(x, y) = f(xy) - xf(y) - yf(x) \quad (x, y \in \mathbb{R}) \tag{9}$$

if and only if D satisfies

$$\begin{aligned} D(x, y) &= D(y, x) & (x, y \in \mathbb{R}), \\ D(xy, z) + zD(x, y) &= D(x, yz) + xD(y, z) & (x, y, z \in \mathbb{R}), \\ D(x + y, z) &= D(x, z) + D(y, z) & (x, y, z \in \mathbb{R}). \end{aligned} \tag{10}$$

Proof. Applying Lemma 1 for the function $C = 0$, (7) is equivalent to the additivity of f and (9), and (8) reduces to (10). □

Our first main result offers a sufficient condition on the recursive additive solvability of the functional equations (5). We deduce this result by using Corollary 2, however, we note that another proof could be elaborated applyin the results of Gselmann [2].

Theorem 3. *Let $n \geq 2$ and $\Gamma: \Delta_n \rightarrow \mathbb{R}$ be a symmetric function such that $\Gamma(i, j) = 1$ whenever $i \cdot j = 0$ and*

$$\Gamma(i + j, k)\Gamma(i, j) = \Gamma(i, j + k)\Gamma(j, k) \quad (0 \leq i, j, k \text{ and } i + j + k \leq n). \tag{11}$$

Let $d_0 = \text{id}$ and let $d_1, \dots, d_{n-1}: \mathbb{R} \rightarrow \mathbb{R}$ be additive functions such that (5) holds for $k \in \{1, \dots, n-1\}$. Then there exists an additive function $d_n: \mathbb{R} \rightarrow \mathbb{R}$ such that (5) is also valid for $k = n$.

Proof. Using $\Gamma(0, n) = \Gamma(n, 0) = 1$, the functional equation for $d_n: \mathbb{R} \rightarrow \mathbb{R}$ can be rewritten as

$$\begin{aligned} d_n(xy) - xd_n(y) - yd_n(x) &= D_n(x, y) \\ &:= \sum_{i=1}^{n-1} \Gamma(i, n-i) d_i(x) d_{n-i}(y) \quad (x, y \in \mathbb{R}). \end{aligned} \quad (12)$$

Thus, in view of Corollary 2, in order that there exist an additive function d_n such that (12) hold, it is necessary and sufficient that $D = D_n$ satisfy the conditions in (10). The symmetry of Γ implies the symmetry, the additivity of d_1, \dots, d_{n-1} results the biadditivity of D_n . Thus, it suffices to prove that $D = D_n$ also satisfies the second identity in (10). This is equivalent to showing that, for all fixed $y \in \mathbb{R}$, the mapping $(x, z) \mapsto D_n(xy, z) + zD_n(x, y)$ is symmetric. Using equations (5) for $k \in \{1, \dots, n-1\}$, we obtain

$$\begin{aligned} &D_n(xy, z) + zD_n(x, y) \\ &= \sum_{k=1}^{n-1} \Gamma(k, n-k) d_k(xy) d_{n-k}(z) + z \sum_{i=1}^{n-1} \Gamma(i, n-i) d_i(x) d_{n-i}(y) \\ &= \sum_{k=1}^{n-1} \Gamma(k, n-k) \left(\sum_{i=0}^k \Gamma(i, k-i) d_i(x) d_{k-i}(y) \right) d_{n-k}(z) \\ &\quad + z \sum_{i=1}^{n-1} \Gamma(i, n-i) d_i(x) d_{n-i}(y) \\ &= \sum_{k=0}^n \sum_{i=0}^k \Gamma(k, n-k) \Gamma(i, k-i) d_i(x) d_{k-i}(y) d_{n-k}(z) \\ &\quad - xy d_n(z) - xz d_n(y) - yz d_n(x) \\ &= \sum_{\alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = n} \Gamma(\alpha + \beta, \gamma) \Gamma(\alpha, \beta) d_\alpha(x) d_\beta(y) d_\gamma(z) \\ &\quad - xy d_n(z) - xz d_n(y) - yz d_n(x). \end{aligned}$$

The sum of the last three terms in the above expression is symmetric in (x, z) . The symmetry of the first summand is the consequence of the symmetry of $(\alpha, \gamma) \mapsto \Gamma(\alpha + \beta, \gamma) \Gamma(\alpha, \beta)$ which follows from property (11). \square

In what follows, we describe the nowhere zero solutions of (11).

Theorem 4. *Let $n \geq 2$ and $\Gamma: \Delta_n \rightarrow \mathbb{R} \setminus \{0\}$ be a symmetric function so that $\Gamma(i, j) = 1$ whenever $i \cdot j = 0$. Then Γ satisfies the functional equation (11) if and only if there exists a function $\gamma: \{0, 1, \dots, n\} \rightarrow \mathbb{R} \setminus \{0\}$ such that*

$$\Gamma(i, j) = \frac{\gamma(i+j)}{\gamma(i)\gamma(j)} \quad ((i, j) \in \Delta_n). \quad (13)$$

Proof. Define the function $\gamma: \{0, 1, \dots, n\} \rightarrow \mathbb{R} \setminus \{0\}$ through

$$\gamma(k) = \prod_{\ell=1}^{k-1} \Gamma(\ell, 1) \quad (k \in \{0, 1, \dots, n\}).$$

The empty product being equal to 1, we have that $\gamma(0) = \gamma(1) = 1$.

To complete the proof, we have to show that, for any $(i, j) \in \Delta_n$,

$$\Gamma(i, j) = \frac{\gamma(i+j)}{\gamma(i)\gamma(j)}.$$

This equivalent to proving that

$$\Gamma(i, j) \prod_{\ell=1}^{i-1} \Gamma(\ell, 1) = \prod_{\ell=j}^{i+j-1} \Gamma(\ell, 1) \quad ((i, j) \in \Delta_n). \quad (14)$$

This identity trivially holds for $i = 0, i = 1$ and for any $j \in \{0, \dots, n-i\}$. Let $j \in \{0, \dots, n-2\}$ be fixed. We prove (14) by induction on $i \in \{1, \dots, n-j\}$. Assume that (14) holds for $i \in \{1, \dots, n-j-1\}$. Then,

$$\begin{aligned} \Gamma(i+1, j) \prod_{\ell=1}^i \Gamma(\ell, 1) &= \frac{\Gamma(i+1, j)\Gamma(i, 1)}{\Gamma(i, j)} \left(\Gamma(i, j) \prod_{\ell=1}^{i-1} \Gamma(\ell, 1) \right) \\ &= \frac{\Gamma(i+1, j)\Gamma(i, 1)}{\Gamma(i, j)} \prod_{\ell=j}^{i+j-1} \Gamma(\ell, 1) \\ &= \frac{\Gamma(i+1, j)\Gamma(i, 1)}{\Gamma(i, j)\Gamma(i+j, 1)} \prod_{\ell=j}^{i+j} \Gamma(\ell, 1). \end{aligned} \quad (15)$$

Using (11), it follows that $\Gamma(i+1, j)\Gamma(i, 1) = \Gamma(i, j)\Gamma(i+j, 1)$, hence (15) yields (14) for $i+1$ instead of i .

Conversely, suppose that there exists a function $\gamma: \{0, 1, \dots, n\} \rightarrow \mathbb{R} \setminus \{0\}$ such that

$$\Gamma(i, j) = \frac{\gamma(i+j)}{\gamma(i)\gamma(j)} \quad ((i, j) \in \Delta_n).$$

Then, for any $i, j, k \geq 0$ with $i+j+k \leq n$, we have

$$\begin{aligned} \Gamma(i+j, k)\Gamma(i, j) &= \frac{\gamma(i+j+k)}{\gamma(i+j)\gamma(k)} \cdot \frac{\gamma(i+j)}{\gamma(i)\gamma(j)} \\ &= \frac{\gamma(i+j+k)}{\gamma(i)\gamma(j+k)} \cdot \frac{\gamma(j+k)}{\gamma(j)\gamma(k)} = \Gamma(i, j+k)\Gamma(j, k), \end{aligned}$$

which completes the proof. □

When Γ is of the form (13), then Theorem 3 reduces to the following statement.

Corollary 5. *Let $n \geq 2$ and $\gamma: \{0, 1, \dots, n\} \rightarrow \mathbb{R} \setminus \{0\}$ with $\gamma(0) = 1$. Let $d_0 = \text{id}$ and let $d_1, \dots, d_{n-1}: \mathbb{R} \rightarrow \mathbb{R}$ be additive functions such that*

$$d_k(xy) = \sum_{i=0}^k \frac{\gamma(k)}{\gamma(i)\gamma(k-i)} d_i(x)d_{k-i}(y) \quad (x, y \in \mathbb{R}) \quad (16)$$

holds for $k \in \{1, \dots, n-1\}$. Then there exists an additive function $d_n: \mathbb{R} \rightarrow \mathbb{R}$ such that (16) is also valid for $k = n$.

We note that if in the above corollary $\gamma(k) = k!$, then (16) is equivalent to (4), that is $\text{id}, d_1, \dots, d_n$ is a derivation of order n .

3. A characterization of the linear dependence of additive functions

Theorem 6. *Let X be a Hausdorff locally convex linear space and let $a: \mathbb{R} \rightarrow X$ be an additive function. Then the following statements are equivalent:*

- (i) *There exists a nonzero continuous linear functional $\varphi \in X^*$ such that $\varphi \circ a = 0$;*
- (ii) *There exists an upper semicontinuous function $\Phi: X \rightarrow \mathbb{R}$ such that $\Phi \not\geq 0$ and $\Phi \circ a \geq 0$;*
- (iii) *The range of a is not dense in X , i.e., $\overline{a(\mathbb{R})} \neq X$.*

Proof. The implication (i) \Rightarrow (ii) is obvious, because Φ can be chosen as φ .

To prove (ii) \Rightarrow (iii), assume that there exists an upper semicontinuous function $\Phi: X \rightarrow \mathbb{R}$ such that $\Phi \not\geq 0$ and $\Phi \circ a \geq 0$. Then $U := \{x \in X \mid \Phi(x) < 0\}$ is a nonempty and open set. The inequality $\Phi \circ a \geq 0$ implies that $U \cap a(\mathbb{R}) = \emptyset$, which proves that the range of a cannot be dense in X .

Finally, suppose that $\overline{a(\mathbb{R})} \neq X$. By the additivity of a , the set $a(\mathbb{R})$ is closed under addition and multiplication by rational numbers. Therefore, the closure of $a(\mathbb{R})$ is a proper closed linear subspace of X . Then, by the Hahn–Banach theorem, there exists a nonzero continuous linear functional $\varphi \in X^*$ which vanishes on $a(\mathbb{R})$, i.e., $\varphi \circ a = 0$ is satisfied. \square

By taking $X = \mathbb{R}^n$, the above theorem immediately simplifies to the following consequence which characterizes the linear dependence of finitely many additive functions.

Corollary 7. *Let $n \in \mathbb{N}$ and $a_1, \dots, a_n: \mathbb{R} \rightarrow \mathbb{R}$ be additive functions. Then the following statements are equivalent:*

- (i) *The additive functions a_1, \dots, a_n are linearly dependent, i.e., there exist $c_1, \dots, c_n \in \mathbb{R}$ such that $c_1^2 + \dots + c_n^2 > 0$ and $c_1 a_1 + \dots + c_n a_n = 0$;*
- (ii) *There exists an upper semicontinuous function $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\Phi \not\geq 0$ and*

$$\Phi(a_1(x), \dots, a_n(x)) \geq 0 \quad (x \in \mathbb{R});$$

- (iii) *The set $\{(a_1(x), \dots, a_n(x)) \mid x \in \mathbb{R}\}$ is not dense in \mathbb{R}^n .*

In the particular case of this corollary, namely when Φ is an indefinite quadratic form, the equivalence of statements (i) and (ii) is the main result of the paper [5] by Kocsis. A former result in this direction is due to Maksa and Rätz [7]: If two additive functions $a, b: \mathbb{R} \rightarrow \mathbb{R}$ satisfy $a(x)b(x) \geq 0$ then a and b are linearly dependent.

4. Linear independence of iterates of nonzero derivations

In this section we apply Corollary 7 to the particular case when the additive functions are iterates of a real derivation. However, firstly we prove the following for higher order derivations.

Theorem 8. *Let $n \in \mathbb{N}$, let $\Gamma: \Delta_n \rightarrow \mathbb{R}$ be a symmetric function such that $\Gamma(i, j) = 1$ whenever $i \cdot j = 0$, (11) is satisfied and, for all $k \in \{2, \dots, n\}$ there exists $i \in \{1, \dots, k - 1\}$ such that $\Gamma(i, k - i) \neq 0$. Assume that $d_0 = \text{id}$ and $d_1, \dots, d_n: \mathbb{R} \rightarrow X$ are additive functions satisfying (5) for all $k \in \{1, \dots, n\}$. Then the following statements are equivalent:*

(i) *There exist $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that $c_0^2 + c_1^2 + \dots + c_n^2 > 0$ and*

$$c_0x + c_1d_1(x) + \dots + c_nd_n(x) = 0 \quad (x \in \mathbb{R}); \tag{17}$$

(ii) *There exists an upper semicontinuous function $\Phi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that $\Phi \not\geq 0$ and*

$$\Phi(x, d_1(x), \dots, d_n(x)) \geq 0 \quad (x \in \mathbb{R});$$

(iii) *The set $\{(x, d_1(x), \dots, d_n(x)) \mid x \in \mathbb{R}\}$ is not dense in \mathbb{R}^{n+1} ;*

(iv) *$d_1 = 0$.*

Proof. Applying Corollary 7 to the additive functions $a_i(x) = d_i(x)$ ($i \in \{0, 1, \dots, n\}$), it follows that (i), (ii) and (iii) are equivalent. The implication (iv) \Rightarrow (i) is obvious since if $d_1 = 0$, then (i) holds with $c_1 = 1$ and $c_0 = c_2 = \dots = c_n = 0$.

Thus, it remains to show that (i) implies (iv). Assume that (i) holds. Then there exist a smallest $1 \leq m \leq n$ and $c_0, \dots, c_m \in \mathbb{R}$ such that $c_0^2 + c_1^2 + \dots + c_m^2 > 0$ and

$$c_0x + c_1d_1(x) + \dots + c_md_m(x) = 0 \quad (x \in \mathbb{R}). \tag{18}$$

This means that the equality

$$\gamma_0x + \gamma_1d_1(x) + \dots + \gamma_{m-1}d_{m-1}(x) = 0 \quad (x \in \mathbb{R})$$

can only hold for $\gamma_0 = \dots = \gamma_{m-1} = 0$.

Observe, that $d_1(1) = \dots = d_n(1) = 0$. Indeed, $d_1(1) = 0$ is a consequence of (5) when $k = 1$ because this equation means that d_1 is a derivation. The rest easily follows by induction on k from (5).

Putting $x = 1$ into (18), it follows that $c_0 = 0$. If $m = 1$, then c_1 cannot be zero, hence we obtain that $d_1 = 0$. Thus, we may assume that the minimal

m for which (18) is satisfied is non-smaller than 2. Replacing x by xy in (18) and applying (5), for all $x, y \in \mathbb{R}$, we get

$$\begin{aligned}
0 &= \sum_{k=1}^m c_k d_k(xy) = \sum_{k=1}^m c_k \left(\sum_{i=0}^k \Gamma(i, k-i) d_i(x) d_{k-i}(y) \right) \\
&= \sum_{k=2}^m c_k \left(\sum_{i=1}^{k-1} \Gamma(i, k-i) d_i(x) d_{k-i}(y) \right) + x \left(\sum_{k=1}^m c_k d_k(y) \right) + y \left(\sum_{k=1}^m c_k d_k(x) \right) \\
&= \sum_{k=2}^m \sum_{i=1}^{k-1} c_k \Gamma(i, k-i) d_i(x) d_{k-i}(y) = \sum_{i=1}^{m-1} \sum_{k=i+1}^m c_k \Gamma(i, k-i) d_i(x) d_{k-i}(y) \\
&= \sum_{i=1}^{m-1} \left(\sum_{j=1}^{m-i} c_{i+j} \Gamma(i, j) d_j(y) \right) d_i(x).
\end{aligned}$$

By the minimality of m , it follows from the above equality that, for all $y \in \mathbb{R}$,

$$\sum_{j=1}^{m-i} c_{i+j} \Gamma(i, j) d_j(y) = 0 \quad (i \in \{1, \dots, m-1\}).$$

Again, by the minimality of m , this implies that $c_{i+j} \Gamma(i, j) = 0$ for $(i, j) \in \Delta_m$ with $i, j \geq 1$. By the assumption of the theorem, for all $k \in \{2, \dots, n\}$ there exists $i \in \{1, \dots, k-1\}$ such that $\Gamma(i, k-i) \neq 0$. Thus, $c_2 = \dots = c_m = 0$. Therefore, by (18), c_1 cannot be equal to zero. Then (18) simplifies to $d_1 = 0$, which was to be proved. \square

Let $n \in \mathbb{N}$ be arbitrary and $d: \mathbb{R} \rightarrow \mathbb{R}$ be a derivation. Then the $(n+1)$ -tuple $(\text{id}, d, d^2, \dots, d^n)$ is a derivation of order n . Thus from the previous theorem we immediately get the following.

Corollary 9. *Let $n \in \mathbb{N}$ and let $d: \mathbb{R} \rightarrow \mathbb{R}$ be a derivation. Then the following statements are equivalent:*

- (i) *There exist $c_0, c_1, \dots, c_n \in \mathbb{R}$ such that $c_0^2 + c_1^2 + \dots + c_n^2 > 0$ and*

$$c_0 x + c_1 d(x) + \dots + c_n d^n(x) = 0 \quad (x \in \mathbb{R}); \quad (19)$$
- (ii) *There exists an upper semicontinuous function $\Phi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ such that $\Phi \not\equiv 0$ and*

$$\Phi(x, d(x), \dots, d^n(x)) \geq 0 \quad (x \in \mathbb{R});$$
- (iii) *The set $\{(x, d(x), \dots, d^n(x)) \mid x \in \mathbb{R}\}$ is not dense in \mathbb{R}^{n+1} ;*
- (iv) *$d = 0$.*

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