3-NETS REALIZING A DIASSOCIATIVE LOOP IN A PROJECTIVE PLANE

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Abstract. A 3-*net* of order n is a finite incidence structure consisting of points and three pairwise disjoint classes of lines, each of size n, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. The current interest around 3-nets (embedded) in a projective plane $\mathrm{PG}(2,\mathbb{K})$, defined over a field $\mathbb{K}$ of characteristic p, arose from algebraic geometry; see [5, 12, 14, 17, 18]. It is not difficult to find 3-nets in $\mathrm{PG}(2,\mathbb{K})$ as far as $0 < p \le n$. However, only a few infinite families of 3-nets in $PG(2,\mathbb{K})$ are known to exist whenever $p = 0$, or $p > n$. Under this condition, the known families are characterized as the only 3-nets in $\mathrm{PG}(2,\mathbb{K})$ which can be coordinatized by a group; see [10]. In this paper we deal with 3-nets in $PG(2,\mathbb{K})$ which can be coordinatized by a diassociative loop G but not by a group. We prove two structural theorems on G. As a corollary, if G is commutative then every non-trivial element of G has the same order, and G has exponent 2 or 3. We also discuss the existence problem for such 3-nets.

Keywords 3-net - projective plane - diassociative loop - Latin square - transversal design
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1. Introduction

The concept of a 3-net comes from classical differential geometry via the combinatorial abstraction of the concept of a 3-web. Formally, a 3-*net* of order n is a finite incidence structure consisting of points and three pairwise disjoint classes of lines, each of size n, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the three classes. It is well known that every 3-net can be coordinatized by a loop. The set Q endowed with a binary operation "$\cdot$" is a *quasigroup,* if for any $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in Q. A quasigroup with a multiplicative unit element is called a *loop.* For a general reference on nets, loops and quasigroups see for instance [1, 4].

In this paper we deal with 3-nets (embedded) in $\mathrm{PG}(2,\mathbb{K})$, the projective plane over a field $\mathbb{K}$ of characteristic $p \ge 0$. Such a 3-net, with line classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and coordinatizing loop $G = (G, \cdot)$, is equivalently defined by a triple of bijective maps from G to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, say

$$\alpha : G \to \mathcal{A},\ \beta : G \to \mathcal{B},\ \gamma : G \to \mathcal{C}$$

such that $a \cdot b = c$ if and only if $\alpha(a), \beta(b), \gamma(c)$ are three concurrent lines in $PG(2,\mathbb{K})$, for any $a, b, c \in G$. If this is the case, the 3-net in $PG(2,\mathbb{K})$ is said to *realize* the loop G.

For the purpose of investigating 3-nets in $\mathrm{PG}(2,\mathbb{K})$, the groundfield $\mathbb{K}$ may be assumed to be algebraically closed. In order to present the key examples of embedded 3-nets, it is convenient to work with the dual concept. Formally, a *dual 3-net* of order n in $PG(2,\mathbb{K})$ consists of a triple $(\Lambda_1,\Lambda_2,\Lambda_3)$ with $\Lambda_1,\Lambda_2,\Lambda_3$ pairwise disjoint point-sets of size n, called *components*, such that every line meeting two distinct components meets each component in precisely one point. We notice that finite dual 3-nets are also called *transversal designs.*

The following concepts and results have a detailed exposition in [10]. We say that an embedded dual 3-net is *algebraic,* if its point set $\Lambda_1\cup\Lambda_2\cup\Lambda_3$ is contained in a cubic curve $\mathcal{F}$. If $\mathcal{F}$ is reducible then we speak of *pencil type, triangular type* or *conic-line type* dual 3-net. Except for the pencil type, all algebraic (dual) 3-nets are coordinatized by either a cyclic group or by a direct product of two cyclic groups. Finite dihedral groups can be realized by dual 3-nets of *tetrahedron type;* in this case the point set is contained in six lines joining four independent points. Finally, we mention that the quaternion group $\mathbf{Q}_8$ has an exceptional realization, cf. [16].

In recent years, finite 3-nets realizing a group have been investigated also in connection with complex line arrangements and resonance theory; see [2, 3, 5, 10, 11, 12, 14, 17, 18]. The following almost complete classification of such 3-nets is proven in [10].

Theorem 1.1. *In the projective plane $PG(2,\mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p\geq 0$, let $(\Lambda_1,\Lambda_2,\Lambda_3)$ be a dual 3-net of order $n\geq 4$ which realizes a group G. If either $p=0$ or $p>n$ then one of the following holds.*

- (I) *G is either cyclic or the direct product of two cyclic groups, and $(\Lambda_1,\Lambda_2,\Lambda_3)$ is algebraic.*
- (II) *G is dihedral and $(\Lambda_1,\Lambda_2,\Lambda_3)$ is of tetrahedron type.*
- (III) *G is the quaternion group of order 8.*
- (IV) *G has order 12 and is isomorphic to Alt_4.*
- (V) *G has order 24 and is isomorphic to Sym_4.*
- (VI) *G has order 60 and is isomorphic to Alt_5.*

A computer aided exhaustive search shows that if $p=0$ then (IV) (and hence (V), (VI)) does not occur; see [13]. It has been conjectured that this holds true in any characteristic.

In this paper we focus on 3-nets in $PG(2,\mathbb{K})$ which can be coordinatized by a diassociative loop G different from a group. Recall that a loop G is *diassociative* if any subloop generated by two elements is a group. There are two important classes of diassociative loops: Moufang loops and Steiner loops. Moufang loops are loops satisfying one (hence all) of the following identities.

$$z(x(zy))=((zx)z)y,\qquad x(z(yz))=((xz)y)z,\qquad (zx)(yz)=(z(xy))z.$$

In general, Moufang loops have a rich algebraic structure. This is not the case for *Steiner loops.* Steiner loops are diassociative loops of exponent two. Finite Steiner loops are in one-to-one connection with Steiner triple systems. For other classes of diassociative loops we refer to [9].

Our results consist of three structural theorems on G.

Theorem 1.2. *In the projective plane $PG(2,\mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p\geq 0$, let $(\Lambda_1,\Lambda_2,\Lambda_3)$ be a dual 3-net of order $n\geq 4$ which realizes a diassociative loop G different from a group. Let d be the maximum of the* *orders of the elements in G, and suppose that $d \geq 4$. If either $p = 0$ or $p > n$ then one of the following holds.*

(a) *G has a unique subgroup H of order d. Moreover, each element not in H is an involution, and two such involutions either commute or their product is in H.*

(b) *$d = 4$, and G has a subgroup isomorphic to one of the groups $\mathbf{Q}_8$, Alt_4.*

Theorem 1.3. *With the same hypotheses as in Theorem 1.2, assume further that G contains a subgroup isomorphic to $\mathbf{Q}_8$ but no subgroup isomorphic to Alt_4. Then G has a unique involution and the subgroup generated by any two non-commuting elements is isomorphic to $\mathbf{Q}_8$.*

It may be observed that a loop G as in Theorem 1.3 defines a Steiner triple system in a natural way, namely the points are subgroups of G of order 4 and the blocks are the subgroups isomorphic to $\mathbf{Q}_8$, the point-block incidence being the set theoretic inclusion.

For a commutative loop G, neither (a) nor (b) of Theorem 1.2 can occur, and hence $d \leq 3$. More precisely, the following result holds.

Corollary 1.4. *With the same hypotheses as in Theorem 1.2, assume further that G is commutative. Then every non-trivial element in G has the same order, and G has exponent 2 or 3.*

The quaternion group $\mathbf{Q}_8$ has a counterpart in the class of Moufang loops. Let $\mathbb{O}$ be the division ring of real octonions and let $1, e_1, \ldots, e_7$ be an orthonormal basis. The set

$$\mathbf{O}_{16} = \{\pm 1, \pm e_1, \ldots, \pm e_7\}$$

forms a Moufang loop with a unique involution -1 and 14 elements of order 4. ($\mathbf{O}_{16}$ is also called the *Cayley loop of order 16*.)

Theorem 1.5. *With the same hypotheses as in Theorem 1.2, assume further that G is a Moufang loop. Then G contains either the octonion loop $\mathbf{O}_{16}$, or it has a subgroup isomorphic to Alt_4.*

An interesting issue which appears to be rather difficult is the existence and construction of 3-nets in the classical projective plane $\mathrm{PG}(2, \mathbb{K})$ realizing a loop different from a group. All such examples available in the literature are 3-nets of order $n = 5, 6$, obtained by computer aided searches; see [16].

2. Proof of Theorem 1.2

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a 3-net of order n coordinatized by a diassociative loop G but not by a group. Let $g \in G$ be an element whose order is d, and let $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ be the 3-net (subnet of Λ) coordinatized by $\langle g \rangle$. Take any element $h \in G$ not lying in the cyclic group generated by g, and consider the subgroup H generated by g and h. Obviously, H is not a cyclic group. Since H is a subloop of G, H also realizes a 3-net $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ in $PG(2, \mathbb{K})$. The classification [10, Theorem 1.1] applies to H and yields one of the cases below, apart from the sporadic cases. Therefore, dismissing (b), we have that either

(i) H is the direct product of two cyclic subgroups, and $\Delta_1 \cup \Delta_2 \cup \Delta_3$ lies on a plane non-singular cubic curve $\mathcal{F}_3$, or

(ii) H is a dihedral group, and Δ is of tetrahedron type.

We first investigate case (i). As G is not a group, it must contain an element $u \notin H$. Replacing h, H by $u, U = \langle g, u\rangle$ in the above argument shows that U realizes a 3-net $\Phi = (\Phi_1 \cup \Phi_2 \cup \Phi_3)$ in $PG(2, \mathbb{K})$, and that U is either the direct product of two cyclic groups, or it is a dihedral group. The latter case here cannot actually occur. To show this, observe that if U is dihedral then the maximality of d implies that u is an (involutory) element lying in some coset of $\langle g\rangle$. Since Φ is of tetrahedron type in this case, we have that Ψ is a triangular 3-net in $PG(2, \mathbb{K})$ of order d. But this is impossible in our case, since the points of Ψ lie on $\mathcal{F}_3$. In fact, $\mathcal{F}_3$ is non-singular while Ψ_1 consists of $d > 3$ collinear points. Therefore, U is a direct product of two cyclic groups and $\Phi_1 \cup \Phi_2 \cup \Phi_3$ lies on a non-singular plane cubic curve $\mathcal{F}_1$. The intersection $\mathcal{F}_3 \cap \mathcal{F}_1$ contains all points of Ψ. Since $d > 3$, this yields $\mathcal{F}_3 = \mathcal{F}_1$. Since u denotes any element of G outside H, it turns out that $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ lies on $\mathcal{F}_3$. But then G itself is a group, the direct product of two cyclic groups. Therefore, (i) cannot actually occur.

In case (ii), we may assume that $H_1 = \langle g, h_1\rangle$ is dihedral for any $h_1 \in G \setminus \langle g\rangle$. Therefore, h_1 is an involution. Moreover, if h_2 is another involution in G, then either h_2 commutes with h_1, or h_2 lies H_1.

3. Proof of Theorem 1.3

From the definition of a dual 3-net, there is a triple of bijective maps from G to $(\Lambda_1, \Lambda_2, \Lambda_3)$, say α, β, γ respectively such that $a \cdot b = c$ in G if and only if $\alpha(a), \beta(b), \gamma(c)$ are three collinear points in $PG(2, \mathbb{K})$, for any $a, b, c \in G$.

Choose two elements $g_1, g_2 \in G$ which generate a subgroup H isomorphic to $\mathbf{Q}_8$. We remark that case (a) in Theorem 1.2 cannot occur since both g and h have order 4. Therefore, $d = 4$. Set $g_3 = g_1 g_2$; then $\langle g_1\rangle$, $\langle g_2\rangle$, $\langle g_3\rangle$ are the three cyclic subgroups of order 4 in H. Take an element $u \in G$ not lying in H.

Assume that u is an involution. For $i = 1, 2, 3$, the group $U_i = \langle u, g_i\rangle$ contains at least two distinct involutions, and hence it is not isomorphic to $\mathbf{Q}_8$. From Theorem 1.1 applied to U_i, we deduce that either U_i is dihedral, or abelian.

We first investigate the case when all U_i's are abelian. Clearly, $U_i = \langle g_i\rangle \times \langle u\rangle \cong C_4 \times C_2$, and case (a) of Theorem 1.2 holds for the sub 3-net $(\Delta_1^i, \Delta_2^i, \Delta_3^i)$ realizing U_i. Let $\mathcal{F}_i$ be the cubic curve containing $\Delta_1^i \cup \Delta_2^i \cup \Delta_3^i$. Since U_i is not cyclic, $\mathcal{F}_i$ is nonsingular. These three sub 3-nets of order 8 share a sub 3-net of order 4, say $(\Omega_1, \Omega_2, \Omega_3)$, realizing the group $T = \langle u, g_1^2 = g_2^2\rangle$. Since $|\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3| \geq 12 > 9$, this yields that $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$. But then the sub 3-net realizing H lies on $\mathcal{F}_1$ a contradiction since $\mathbf{Q}_8$ is not abelian.

Assume now that U_1, U_2 are abelian and U_3 dihedral. With the same argument, the dual sub 3-nets realizing U_1, U_2 are contained in the nonsingular cubic curve $\mathcal{F}$. The dual sub 3-net realizing U_3 is of tetrahedron type, which means that the four points of $\alpha(\langle g_3\rangle)$ are collinear. The triple $(\alpha(\langle g_3\rangle), \beta(H \setminus \langle g_3\rangle), \gamma(H \setminus \langle g_3\rangle))$ is a dual 3-net realizing $\langle g_3\rangle$. On the one hand, the 8 points of $\beta(H \setminus \langle g_3\rangle) \cup \gamma(H \setminus \langle g_3\rangle)$ are contained in a (possibly degenerate) conic $\mathcal{C}$, see [2, Theorem 5.1]. On the other hand, $H \setminus \langle g_3\rangle \subset \langle g_1\rangle \cup \langle g_2\rangle \subset U_1 \cup U_2$. This implies $\beta(H \setminus \langle g_3\rangle) \cup \gamma(H \setminus \langle g_3\rangle) \subset \mathcal{F}$ and $|\mathcal{F} \cap \mathcal{C}| \geq 8$, a contradiction.

Assume that U_1, U_2 are dihedral. Hence the dual sub 3-nets realizing U_1, U_2 are of tetrahedron type yielding that the four points of $\alpha(\langle g_1\rangle)$ and the four points of $\alpha(\langle g_2\rangle)$ are contained in the lines ℓ_1, ℓ_2, respectively. However, $\alpha(1), \alpha(g_1^2 = g_2^2) \in \ell_1 \cap \ell_2$, thus, $\ell_1 = \ell_2$. Similarly, the six points of $\beta(\langle g_1\rangle \cup \langle g_2\rangle)$ and the six points of

$\gamma(\langle g_1\rangle \cup \langle g_2\rangle)$ are contained in the lines m, m', respectively. If U_3 is dihedral, then the dual sub 3-net realizing H is contained in $\ell_1 \cup m \cup m'$, which is impossible since H is not cyclic. If U_3 is abelian, then the sub 3-net realizing it is contained in the nonsingular cubic curve $\mathcal{F}$. The second component of its sub 3-net $(\alpha(\langle g_3\rangle), \beta(H \setminus \langle g_3\rangle), \gamma(H \setminus \langle g_3\rangle))$ is contained in m, hence $|m \cap \mathcal{F}| \geq 4$, a contradiction.

Assume that u has order 3. Then U is neither a dihedral group nor isomorphic to $\mathbf{Q}_8$. From Theorem 1.1, $U = \langle g\rangle \times \langle u\rangle$ and hence U is a cyclic group of order 12 contradicting the remark at the beginning about case (a) in Theorem 1.2.

Therefore, G contains just one involution v, and if $u \neq v$ then u has order 4. Let $u_1, u_2 \in G$ be any two distinct elements other than v. Since U contains no element of order 3, $U = \langle u_1, u_2\rangle$ is a 2-group of exponent 4 containing a unique involution. Since U has order bigger than 4, the only possibility is $U \cong \mathbf{Q}_8$.

Remark 3.1. Let S be the Steiner loop of order 10 corresponding to the Steiner triple system $AG(2,3)$. S has a central extension Q of order 20 all proper subloops are isomorphic to $C_2, C_4, C_2 \times C_4$, or $\mathbf{Q}_8$. In particular, Q is diassociative. By Theorem 1.3, Q has no projective realization despite all its subloops have.

4. Proof of Theorem 1.5

We start with three important facts on Moufang loops of small exponent. First, as diassociative loops of exponent 2 are commutative, Moufang loops of exponent 2 are elementary abelian groups. Second, by [6, Corollary 1] finite Moufang loops of exponent 3 are nilpotent. This implies that any proper finite Moufang loop of exponent 3 contains a subloop of order 27. The classification of small Moufang loops [8] shows that Moufang loops of order 27 are groups. Thus, if G is a Moufang loop of exponent 3 then it contains a subgroup H of order 27. Since no such group H has a realization in $\mathrm{PG}(2,\mathbb{K})$ by Theorem 1.1, we have a contradiction.

Let us assume that G has an element g of order $d > 4$. Put $U = \langle g\rangle$. By Theorem 1.2, any $h \in G \setminus U$ has order 2 and $\langle U, h\rangle$ is a dihedral group of order $2d$. In particular, on the one hand, $hU = Uh$, and on the other hand, the involutions generate G. [7, Theorem 1] implies that U is a normal subloop of G. For any subset X of G, let $\Lambda_i(X)$ denote the points of Λ_i, indexed by the elements of X. [10, Proposition 22] implies that any of the sets $\Lambda_i(U)$, $\Lambda_i(Uh)$ is contained in a line.

Choose an element $h \in G \setminus U$. Then we have four points P, Q, R, S such that the point sets $\Lambda_1(U), \Lambda_2(U), \Lambda_3(U), \Lambda_1(Uh), \Lambda_2(Uh), \Lambda_3(Uh)$ are contained in the lines QR, RS, PR, SP, SQ, PQ. In fact, the points P, Q, R, S are the vertices of the tetrahedron type dual 3-net, corresponding to the dihedral group $\langle U, h\rangle$. Only the vertex S depends on the choice of h; $S = S_h$.

Choose elements $h_1, h_2 \in G \setminus U$ such that $\langle U, h_1, h_2\rangle$ is a non-associative subloop of G. Let $P, Q, R, S_{h_1}, S_{h_2}, S_{h_1h_2}$ be the vertices of the tetrahedron type dual nets of $\langle U, h_1\rangle$, $\langle U, h_2\rangle$ and $\langle U, h_1h_2\rangle$. The sets

$$\Lambda_1(Uh_1), \Lambda_2(Uh_2), \Lambda_3(Uh_1h_2)$$

of points form a triangular dual 3-net. [10, Proposition 10] implies $S_{h_1} = S_{h_2} = S_{h_1h_2}$, a contradiction.

Finally, assume that G has no subgroup isomorphic to Alt_4. By Theorem 1.3, G has a unique (central) involution u. As two non-commuting elements generate a subloop isomorphic to $\mathbf{Q}_8$, the factor $G/\langle u\rangle$ is an elementary abelian 2-group.

Thus, G contains a non-associative subloop S of order 16 with a unique involution. Using the classification of small Moufang loops in [8], $S \cong \mathbf{O}_{16}$ follows.

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