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ON SHARP SOLUTIONS TO MAJORIZATION AND FEKETE-SZEGŐ PROBLEMS FOR STARLIKE FUNCTIONS

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Abstract. This paper is concerned with sharp solutions to majorization and Fekete-Szegő problems for several relatively new normalized subfamilies of starlike functions. Regarding starlike functions which are subordinated to Sine and Cosine functions, a number of new results are established. The results provide improvements to some recent results. As applications, some interesting deductions are also listed. The findings of this study are thoroughly supported by examples wherever necessary.

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1. INTRODUCTION

Analytic function theory of one complex variable is a very rich area of study and an elegant subject of classical mathematics. Univalent and multivalent functions appear as fascinating aspect of this theory with the interplay of analytic structures and geometric behaviors of analytic functions. Some early results can be considered as the corner stone of this theory that emerged in the beginning of the 20th century. For example, from Koebe's investigation in 1907, the Gronwall's proof of the area theorem in 1914, and the famous Bieberbach's estimate of the second coefficient in 1916. Applications and extensions of univalent and multivalent function theory have been used in fields such as ordinary and partial differential equations, fractional calculus, operator theory, mathematical physics, and differential subordinations (for details, see [7]).

After the settlement of Bieberbach conjecture by de-Branges [11] concerning modulus of nth coefficients of normalized univalent functions, (see e.g. equation (1.1)), this field has observed exponential growth. A number of basic subfamilies of univalent functions such as starlike, convex, close-to-convex and spiral-like functions were thoroughly investigated. Results such as coefficient bounds, growth, distortion

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and covering theorems, majorization and radius problems were reported for many of these and other closely related subfamilies [1,3–6,8,9,12–14,17,25]. For some other related results on these subfamilies, the reader can refer to [15, 16].

Generally speaking, analytic (geometric) function theory observes close connection with geometry and analysis, and therefore has attracted the attention of many function theorists since the beginning of 20th century to recent times. Most of the results of analytic function theory are available over the open unit disk of the complex plane $\mathbb{C}$ that are characterized by the fact that such a function provides one-to-one mapping onto its image domain.

Let us take $H(\mathcal{E})$ as the family of all analytic functions defined on the open unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$. Take $\mathcal{A} \subset H(\mathcal{E})$ as the family of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{E}) \tag{1.1}$$

that satisfy the conditions

$$f(0) = f'(0) - 1 = 0.$$

Also, we let $\mathcal{U}$ denote the subfamily of all functions $f \in \mathcal{A}$ that are univalent in $\mathcal{E}$. As usual, let $\mathcal{ST}$ denote the familiar subfamily of all starlike functions (see [13]), defined as

$$\mathcal{ST} = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{E}) \right\}.$$

Furthermore, by $\mathcal{ST}(\alpha)$, we mean the subfamily of starlike functions of order $\alpha \in [0, 1)$, which is given as follows:

$$\mathcal{ST}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{E}) \right\}.$$

It is interesting to point out that by using the Alexander relationship (see [13]), we can define the subfamilies $\mathcal{CV}$ and $\mathcal{CV}(\alpha)$, which are, respectively, the families of all convex and convex functions of order $\alpha \in [0, 1)$. More precisely, let $F(z) = zf'(z)$, where $f'(z) \neq 0$ for $z \in \mathcal{E}$, then

$$f(z) \in \mathcal{CV} \Longleftrightarrow F(z) \in \mathcal{ST},$$

and for $\alpha \in [0, 1)$,

$$f(z) \in \mathcal{CV}(\alpha) \Longleftrightarrow F(z) \in \mathcal{ST}(\alpha).$$

For relevant advances in this direction, one can consult references [2, 9, 15, 16].

In what follows, we recall several definitions involving subordination and majorization.

Definition 1 ([13])**.** If $f, g \in \mathcal{A}$, then f is subordinated to g, written as $f \prec g$, if for some analytic function $w(z)$, the relationship

$$f(z) = g(w(z)) \quad (z \in \mathcal{E})$$

holds, where $w(z)$ is the Schwarz function satisfies $w(0)=0$ and $|w(z)| \leq |z| < 1$ for each $z \in \mathcal{E}$.

In addition, if $g(z)$ is univalent in $\mathcal{E}$, that is $g \in \mathcal{U}$, then $f \prec g$ can be written equivalently as (see [13])

$$f(0)=g(0) \quad \text{and} \quad f(\mathcal{E}) \subset g(\mathcal{E}) \ (z \in \mathcal{E}).$$

Definition 2 ([19])**.** If $f, g \in \mathcal{A}$, we say that f is majorized by g, written as $f \ll g$, if for some analytic function $\phi(z)$, the relationship

$$f(z)=\phi(z)g(z) \quad (z \in \mathcal{E})$$

holds, where $\phi(z)$ satisfies the condition $|\phi(z)| \leq 1$ for each $z \in \mathcal{E}$.

It should be mentioned that majorization theory was initiated by MacGregor [19], and subsequent results for starlike and convex functions were obtained by the authors [4,5].

In 1992, by using the subordination technique, Ma and Minda [18] presented a unified treatment for the subfamilies of starlike and convex functions. In particular, they took a convex analytic function $\psi(z)$ in $\mathcal{E}$ such that $\psi(0)=1$, $\psi'(0)>0$ and $\Re\big(\psi(z)\big)>0$. In other words, the image domain of $\psi(\mathcal{E})$ is a star-shaped domain with respect to 1, and symmetric with real axis for each $z \in \mathcal{E}$. The following analytic description for the subfamilies of all starlike and convex functions were given by Ma and Minda [18]:

$$\mathcal{ST}(\psi)=\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z)\right\}, \ \mathcal{CV}(\psi)=\left\{f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \psi(z)\right\}.$$

For recent applications of Ma-Minda function classes in complex analysis, we refer the reader to [26,27].

As a special case, if we let

$$\psi(z) := \psi_\alpha(z) = \frac{1+(1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1),$$

then the above mentioned classes reduce to usual subclasses of starlike and convex functions of order α, respectively, see e.g. [2]. Similarly, the choice $\psi_0(z) = (1+z)/(1-z)$ (see Fig. 1 (top left)) gives the usual classes of starlike and convex functions, respectively.

In an analogous manner, by taking $\psi(z)=1+\sin z$ and $\psi(z)=1+\cos z$, the following subfamilies of starlike functions were introduced by Tang et al. [25]:

$$\mathcal{S}_s^* = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1+\sin z\right\}, \ \mathcal{S}_c^* = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1+\cos z\right\}.$$

The same authors in [25] have investigated majorization properties for functions in these families. However, the subfamily S_s^* was earlier introduced and investigated by Cho et al. [10]. Geometrically, by the above function families, we mean that the image domain of $|z| < 1$ under $zf'(z)/f(z)$ in each class represent an eight shaped and circular region as shown in Fig. 1 (top right, bottom).

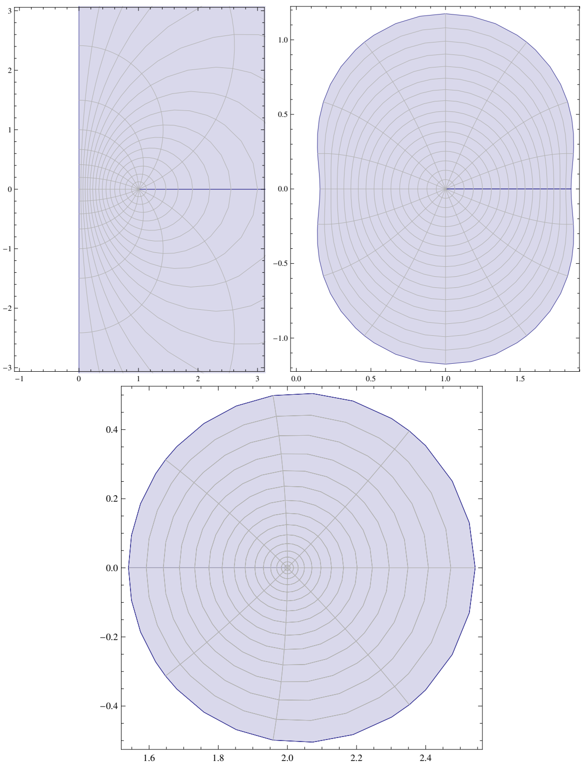

FIGURE 1. Graphs of analytic functions $\psi_0(z) = (1+z)/(1-z)$ (top left), $\psi(z) = 1+\sin z$ (top right), $\psi(z) = 1+\cos z$ (bottom).

Motivated essentially by these facts, and from the reported works in [24, 25], we introduce the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$ of starlike functions of order $\alpha \in [0,1)$ by subordinating each function in the member class to

$$\psi_s(z) = 1+\sin z \quad \text{and} \quad \psi_c(z) = 1+z\cos z,$$

respectively. From onward now, unless otherwise stated, we take $\mathcal{S}_c^*$ as the subfamily subordinated to $\psi_c(z)$. We now give the following definition.

Definition 3. Let $f \in \mathcal{A}$. We say that $f \in \mathcal{ST}_s(\alpha)$, if it satisfies the following condition:

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \prec \psi_s(z).$$

In an analogous manner, we say that $f \in \mathcal{ST}_c(\alpha)$, if the condition

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-\alpha\right) \prec \psi_c(z)$$

is satisfied.

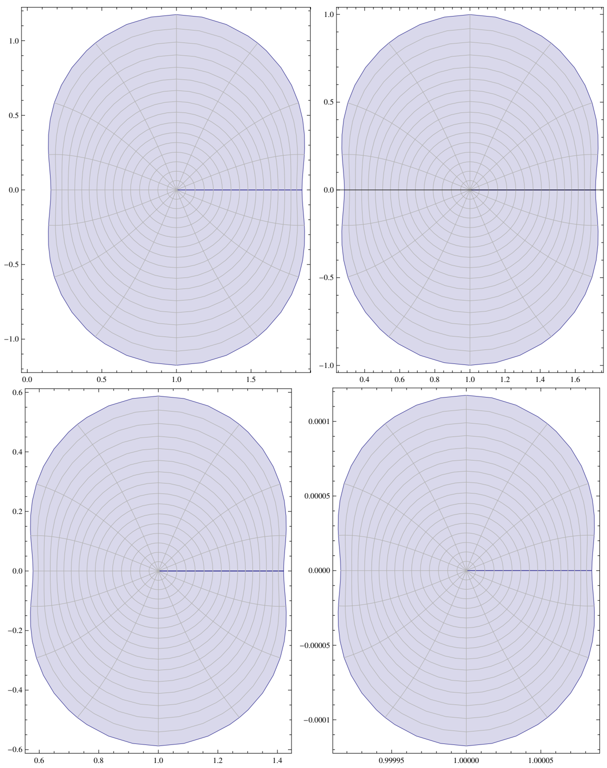

FIGURE 2. Graphs of analytic functions $\psi_s^{\alpha}(z) = 1 + (1-\alpha)\sin z$ for $\alpha = 0.0$ (top left), $\alpha = 0.15$ (top right), $\alpha = 0.5$ (bottom left), and $\alpha \to 1$ (bottom right).

It is now obvious that the function $\psi_c(z) = 1 + z\cos z$ satisfies the requirements as provided by Ma and Minda [18]. In particular, $\psi_c(0) = 1,\ \psi_c'(0) > 0$, and the image domain is star-shaped region with respect to 1.

Alternatively, if one consider the functions $\psi_s^{\alpha}(z) = \alpha + (1-\alpha)\psi_s(z)$ and $\psi_c^{\alpha}(z) = \alpha + (1-\alpha)\psi_c(z)$, that is

$$\psi_s^{\alpha}(z) = 1 + (1-\alpha)\sin z \ \text{ and } \ \psi_c^{\alpha}(z) = 1 + (1-\alpha)z\cos z \ (0 \le \alpha < 1),$$

then the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$ can be defined equivalently as:

$$\mathcal{ST}_s(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi_s^\alpha(z) \right\}, \quad \mathcal{ST}_c(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi_c^\alpha(z) \right\}.$$

As an immediate deduction, one can see that

$$\mathcal{ST}_s(0) \equiv \mathcal{S}_s^* \text{ and } \mathcal{ST}_c(0) \equiv \mathcal{S}_c^*.$$

Remark 1. It can be seen from Figs. 2 and 3 that the image domains of $\psi_s^\alpha(z)$ and $\psi_c^\alpha(z)$ become smaller and smaller as $\alpha \to 1$. In addition, these image domains are star-shaped with respect to 1, and lie in the right half plane implying that $\Re(\psi_s^\alpha(z)) > 0$ and $\Re(\psi_c^\alpha(z)) > 0$ (cf. Fig 1 (top left)). Moreover, it can be seen that the following inclusion relationships

$$\psi_s^\alpha(\mathcal{E}) \subset \psi_s(\mathcal{E}) \subset \psi(\mathcal{E}) \text{ and } \psi_c^\alpha(\mathcal{E}) \subset \psi_c(\mathcal{E}) \subset \psi(\mathcal{E}) \ (0 \leq \alpha < 1)$$

hold, where

$$\psi(z) := \psi_0(z) = \frac{1+z}{1-z}$$

maps the open unit disk to right-half plane for each $z \in \mathcal{E}$, see Fig. 1 (top left). This in turn implies that

$$\mathcal{ST}_s(\alpha) \subset \mathcal{S}_s^* \subset \mathcal{ST} \text{ and } \mathcal{ST}_c(\alpha) \subset \mathcal{S}_c^* \subset \mathcal{ST} \ \ (0 \leq \alpha < 1).$$

Remark 2. We would like to point out that, it is an open question to establish inclusion relationships of the classes $\mathcal{ST}_s(\alpha)$, $\mathcal{ST}_c(\alpha)$ and $\mathcal{ST}(\alpha)$. More precisely, we pose the following problems:

$$\mathcal{ST}_s(\alpha) \subset \mathcal{ST}_c(\alpha) \subset \mathcal{ST}(\alpha) \text{ or } \mathcal{ST}_c(\alpha) \subset \mathcal{ST}_s(\alpha) \subset \mathcal{ST}(\alpha) \ \ (0 \leq \alpha < 1).$$

The main objective of this study is to solve majorization and Fekete-Szegő problems for functions in the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. This is due to the recent studies conducted so far in this direction, see e.g. [3,6,9,14–17,20–23]. The obtained results thus comprehend and improve the already existed results in a unified manner, especially those in references [24,25], and will also continue to hold for various subfamilies of analytic functions. We point out that these results are obtained without involving any linear/nonlinear differential or integral operators, as compared to those in [14–17] and the references cited therein.

The rest of discussions are arranged as follows. In Section 2, we recall some useful results as lemmas. Section 3 is devoted to the main results and their demonstrations. Section 4 describes several applications of the main results. In Section 5, we end the discussion with a concise conclusion.

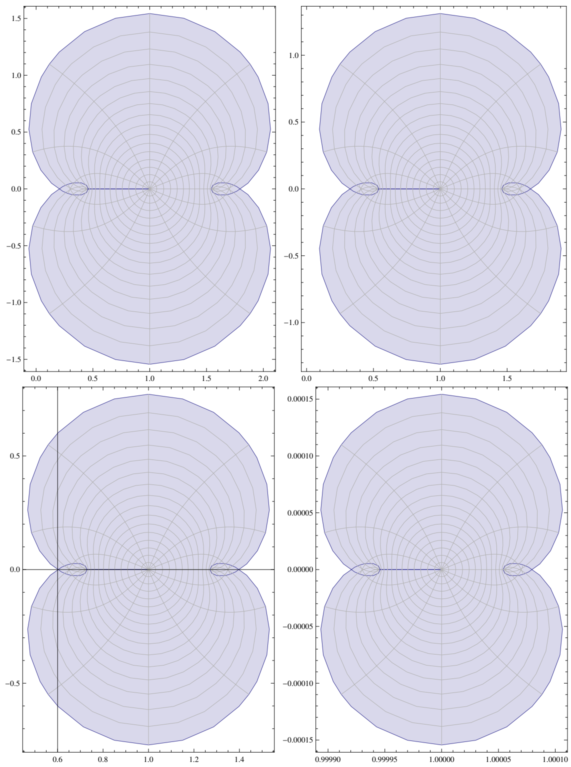

FIGURE 3. Graphs of analytic functions $\psi_c^\alpha(z) = 1 + (1-\alpha)z\cos z$ for $\alpha = 0.0$ (top left), $\alpha = 0.15$ (top right), $\alpha = 0.5$ (bottom left), and $\alpha \to 1$ (bottom right)

2. PRELIMINARY RESULTS

In this section, we present some helpful results that are essential in the proof of main results.

Lemma 1 ([10]). *The analytic function $\phi(z)$ for each $|z| = r < 1$ satisfies the following inequality*

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} = \frac{1-|\phi(z)|^2}{1-r^2} \quad (z \in \mathcal{E}).$$

For some recent applications of this lemma, see [24, 25].

Lemma 2 ([3]). *Let*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathcal{E})$$

be a Schwarz function. Then for any real number β,

$$\left|w_2 - \beta w_1^2\right| \leq \begin{cases} -\beta & (\beta < -1), \\ 1 & (-1 \leq \beta \leq 1), \\ \beta & (\beta > 1). \end{cases}$$

These estimates are sharp and attain for $\beta > 1$ *or* $\beta < -1$, *iff* $w(z) = z$ *or one of its rotations. If* $-1 < \beta < 1$, *then equality occurs iff* $w(z) = z^2$ *or one of its rotations. Equality also occurs for* $\beta = -1$, *iff*

$$w(z) = \frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations, while for $\beta = 1$, *iff*

$$w(z) = -\frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations.

Lemma 3 ([3]). *Let*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathcal{E})$$

be a Schwarz function. Then for any complex number β,

$$\left|w_2 - \beta w_1^2\right| \leq \max\{1, |\beta|\}.$$

This estimate is sharp and attains equality by $w(z) = z$ *or* $w(z) = z^2$.

3. Main results

In this section, we investigate the majorization problems for functions in the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. The Fekete-Szegő type problems are also solved for each function in $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. Throughout the discussion, it is assumed that $0 \leq \alpha < 1$.

3.1. *Majorization problems*

We begin by stating the following result.

Theorem 1. *Let* $f \in \mathcal{A}$ *and* $g \in \mathcal{ST}_s(\alpha)$. *If* f *is majorized by* g *for each* $z \in \mathcal{E}$, *then for* $|z| = r \leq r_s$,

$$|f'(z)| \leq |g'(z)|,$$

where r_s *is the least positive root of the following equation*

$$[1 - (1-\alpha)\sin r](1 - r^2) - 2r = 0. \tag{3.1}$$

Proof. For $g \in \mathcal{ST}_s(\alpha)$, by using the subordination principle, we can write

$$\frac{zg'(z)}{g(z)} = 1 + (1-\alpha)\sin(w(z)) \quad (z \in \mathcal{E}), \tag{3.2}$$

where $w(z)$ is a Schwarz function. Now, by setting $w(z) = \rho\exp(it)$ such that $\rho \leq |z| = r < 1$ and $t \in [-\pi,\pi]$. Then for $\alpha < 1$, it can be seen that

$$\begin{aligned}|\sin(w(z))|^2 &= |\sin(\rho\exp(it))|^2 \\ &= \cos^2(\rho\cos t)\sinh^2(\rho\sin t) + \sin^2(\rho\cos t)\cosh^2(\rho\sin t).\end{aligned}$$

Thus, for the function

$$\vartheta(t) = \cos^2(\rho\cos t)\sinh^2(\rho\sin t) + \sin^2(\rho\cos t)\cosh^2(\rho\sin t),$$

one can readily verify that

$$\vartheta'(t) = \sinh(2\rho\sin t) - \sin(2\rho\cos t) = 0$$

admits five real roots within the interval $[-\pi,\pi]$, namely, $0, \pm\frac{\pi}{2}, \pm\pi$. In addition, we see that $\vartheta(-t) = \vartheta(t)$, so we consider $t \in [0,\pi]$ to simplify the analysis. A simple exercise reveals that

$$\max\left\{\vartheta(0), \vartheta\left(\frac{\pi}{2}\right), \vartheta(\pi)\right\} = \vartheta\left(\frac{\pi}{2}\right) = \sinh^2(\rho).$$

Now, it follows that

$$|\sin(w(z))| = |\sin(\rho\exp(it))| \leq \sinh\rho \leq \sinh r. \tag{3.3}$$

Moreover, we note that

$$\sin r \leq \sin\rho \leq |\sin(w(z))|. \tag{3.4}$$

Combining (3.3) and (3.4), then (3.2) gives

$$\left|\frac{g(z)}{g'(z)}\right| = \frac{|z|}{|1+(1-\alpha)\sin(w(z))|} \leq \frac{|z|}{1-(1-\alpha)|\sin(w(z))|} \leq \frac{r}{1-(1-\alpha)\sin r}. \tag{3.5}$$

From the assumption, on the other hand, we have $f(z) \ll g(z)$ for $z \in \mathcal{E}$, which is equivalent to the statement

$$f(z) = \phi(z)g(z) \quad (z \in \mathcal{E}).$$

Then by differentiation, we obtain

$$f'(z) = \phi'(z)g(z) + \phi(z)g'(z) = \left(\phi(z) + \phi'(z)\frac{g(z)}{g'(z)}\right)g'(z),$$

which implies that

$$|f'(z)| \leq \left(|\phi(z)| + |\phi'(z)|\left|\frac{g(z)}{g'(z)}\right|\right)|g'(z)|. \tag{3.6}$$

By setting $|\phi(z)| = \tau$ $(\tau \in [0,1])$, and using Lemma 2 together with (3.5), inequality (3.6) yields

$$|f'(z)| \leq \left(\tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)\sin r](1-r^2)}\right)|g'(z)|.$$

In what follows, we define the following function

$$\Lambda_\alpha(r,\tau) := \tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)\sin r](1-r^2)}.$$

Then we will reach to the required result, if

$$\Lambda_\alpha(r,\tau) \leq 1 \quad (\forall \tau \in [0,1];\, \alpha \in [0,1)),$$

i.e.,

$$r_s = \max\{r, \alpha \in [0,1) : \Lambda_\alpha(r,\tau) \leq 1;\, \forall \tau \in [0,1]\}.$$

This is equivalent to the following statement

$$r_s = \max\{r, \alpha \in [0,1) : \Pi_\alpha(r,\tau) := 1 - \Lambda_\alpha(r,\tau) \geq 0 \quad (\forall \tau \in [0,1])\},$$

where

$$\Pi_\alpha(r,\tau) = [1-(1-\alpha)\sin r](1-r^2) - (1+\tau)r.$$

Now, following the proof of Tang et al. [25], we can verify that the minima of $\Pi_\alpha(r,\tau)$ occurs at $\tau = 1$, that is

$$\min\{\Pi_\alpha(r,\tau),\ \forall \tau \in [0,1]\} = \Pi_\alpha(r,1) =: \Upsilon_\alpha(r),$$

where

$$\Upsilon_\alpha(r) = [1-(1-\alpha)\sin r](1-r^2) - 2r.$$

By using the fact that

$$\Upsilon_\alpha(0) = 1 > 0 \ \text{ and } \ \Upsilon_\alpha(1) = -2 < 0,$$

we arrive at the conclusion that there exists r_s for which

$$\Upsilon_\alpha(r) \geq 0 \quad (\forall r \in [0, r_s]),$$

where r_s is the least positive root of equation (3.1). □

Remark 3. A search for the least positive root immediately yields $r_s = 0.372342$ at $\alpha = 0.627658$ whenever $0 < \alpha < 1$.

Remark 4. We would like to mention that the obtained results are sharper than those earlier reported ones in [25]. Indeed, the estimate in Theorem 1 is sharper than [25, Theorem 2.1]. This is due to the fact that

$$\sin r \leq \sinh r \quad (r \in [0,1]).$$

Theorem 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_c(\alpha)$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \leq r_c$,*

$$|f'(z)| \leq |g'(z)|,$$

where r_c is the least positive root of the following equation

$$[1-(1-\alpha)r\cos r](1-r^2) - 2r = 0. \tag{3.7}$$

Proof. For $g \in \mathcal{ST}_c(\alpha)$, by using the subordination principle, we can write

$$\frac{zg'(z)}{g(z)} = 1 + (1-\alpha)z\cos(w(z)) \quad (z \in \mathcal{E}), \tag{3.8}$$

where $w(z)$ is a Schwarz function. By setting $w(z) = \rho\exp(it)$ such that $\rho \leq |z| = r < 1$ and $t \in [0,\pi]$, we get the following estimates:

$$\cos r \leq \cos\rho \leq |\cos(w(z))| \leq |\cos(\rho\exp(it))| \leq \cosh\rho \leq \cosh r.$$

Now, following the proof of Theorem 1 and (3.8), for $0 \leq \alpha < 1$, we obtain

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{r}{1-(1-\alpha)r\cos r}. \tag{3.9}$$

Also, for $f(z) \ll g(z)$, we can write

$$f(z) = \phi(z)g(z) \quad (z \in \mathcal{E}),$$

which implies that

$$|f'(z)| \leq \left(|\phi(z)| + |\phi'(z)|\left|\frac{g(z)}{g'(z)}\right|\right)|g'(z)|. \tag{3.10}$$

By setting $|\phi(z)| = \tau$ ($\tau \in [0,1]$), and using Lemma 2 together with (3.9), inequality (3.10) becomes

$$|f'(z)| \leq \left(\tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)r\cos r](1-r^2)}\right)|g'(z)|.$$

For the function

$$\Xi_\alpha(r,\tau) = \tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)r\cos r](1-r^2)},$$

we reach to the required result if $\Xi_\alpha(r,\tau) \leq 1$ for $\forall\tau \in [0,1]$, i.e.,

$$r_s = \max\{r, \alpha \in [0,1) : \Xi_\alpha(r,\tau) \leq 1, \ \forall\tau \in [0,1]\}.$$

Alternatively, we need to show that

$$r_s = \max\{r, \alpha \in [0,1) : \Omega_\alpha(r,\tau) \geq 0, \ \forall\tau \in [0,1]\},$$

where

$$\Omega_\alpha(r,\tau) = (1-r^2)[1-(1-\alpha)r\cos r] - (1+\tau)r.$$

As in the proof of Theorem 1, it is easy to verify that $\Omega_\alpha(r,\tau)$ takes minima at $\tau = 1$, namely,

$$\min\{\Omega_\alpha(r,\tau), \ \forall\tau \in [0,1]\} = \Omega_\alpha(r,1) := \Theta_\alpha(r),$$

where

$$\Theta_\alpha(r) = (1-r^2)[1-(1-\alpha)r\cos r] - 2r.$$

Also, we can see that

$$\Theta_\alpha(0) = 1 > 0 \ \text{ and } \ \Theta_\alpha(1) = -2 < 0,$$

so there exists r_c for which

$$\Theta_\alpha(r) \geq 0 \quad (\forall r \in [0, r_c]),$$

where r_c is the least positive root of equation (3.7). □

Remark 5. As illustrated earlier, a search for the least positive root yields $r_c = 0.374041$ at $\alpha = 0.625959$ whenever $0 < \alpha < 1$.

3.2. *Fekete-Szegő type problems for the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$*

In this subsection, the Fekete-Szegő type problems are solved for functions in the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$.

Theorem 3. *Let $f \in \mathcal{ST}_s(\alpha)$ and $0 \leq \alpha < 1$. Then for any real number β,*

$$|a_3 - \beta a_2^2| \leq \begin{cases} \left(\frac{\alpha-1}{2}\right)\sigma & (\beta < \sigma_1), \\ \frac{1-\alpha}{2} & (\sigma_1 \leq \beta \leq \sigma_2), \\ \left(\frac{1-\alpha}{2}\right)\sigma & (\beta > \sigma_2), \end{cases} \tag{3.11}$$

where

$$\sigma = (2\beta - 1)(1-\alpha), \ \ \sigma_1 = \frac{\alpha(\alpha-1)}{2}, \ \ \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}. \tag{3.12}$$

Proof. By substituting

$$w(z) = 1 + w_1 z + w_2 z^2 + \cdots$$

and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

into (3.2), we see that

$$a_2 z + (2a_3 - a_2^2)z^2 + \cdots = (1-\alpha)w_1 z + (1-\alpha)w_2 z^2 + \cdots. \tag{3.13}$$

Equating coefficients of the same powers of z in (3.13), we find that

$$a_2 = (1-\alpha)w_1, \ \ a_3 = \frac{1}{2}\left[(1-\alpha)w_2 + (1-\alpha)^2 w_1^2\right].$$

Now, it is a simple exercise to get

$$|a_3 - \beta a_2^2| \leq \left(\frac{1-\alpha}{2}\right)|w_2 - \sigma w_1^2|, \tag{3.14}$$

where σ is given by (3.12).

Hence, from Lemma 2, the first inequality in (3.11) is established, whenever

$$(2\beta - 1)(1-\alpha) < -1 \Longleftrightarrow \beta < \sigma_1 = \frac{\alpha(\alpha-1)}{2}.$$

Similarly, an application of Lemma 2 gives the third inequality in (3.11), whenever

$$(2\beta - 1)(1-\alpha) > 1 \Longleftrightarrow \beta > \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}.$$

Finally, the second inequality in (3.11) follows immediately by Lemma 2 for

$$\sigma_1 \leq \beta \leq \sigma_2.$$

□

Theorem 4. *Let $f \in \mathcal{ST}_s(\alpha)$ and $0 \leq \alpha < 1$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \frac{1-\alpha}{2} \max\{1, |\sigma|\}, \tag{3.15}$$

where σ is given by (3.12).

Proof. By applying Lemma 3 to inequality (3.14) for the complex number β, the desired inequality (3.15) follows. □

Theorem 5. *Let $f \in \mathcal{ST}_c(\alpha)$ and $0 \leq \alpha < 1$. Then for any real number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \begin{cases} \frac{\alpha-1}{2}\sigma & (\beta < \sigma_1), \\ \frac{1-\alpha}{2} & (\sigma_1 \leq \beta \leq \sigma_2), \\ \frac{1-\alpha}{2}\sigma & (\beta > \sigma_2), \end{cases}$$

where

$$\sigma = (2\beta-1)(1-\alpha), \quad \sigma_1 = \frac{\alpha(\alpha-1)}{2}, \quad \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}.$$

Proof. The proof is similar to that of Theorem 3. □

Theorem 6. *Let $f \in \mathcal{ST}_c(\alpha)$ and $0 \leq \alpha < 1$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \frac{1-\alpha}{2} \max\{1, |\sigma|\},$$

where σ is given by (3.12).

Proof. The proof is similar to that of Theorem 4. □

Remark 6. The results provided in Theorems 3, 4, 5 and 6 are sharp. Indeed, in Theorems 3 and 4, equalities occur for the function $f_s(z)$ given by

$$f_s(z) = z\exp\left(\int_0^z \frac{\psi_s^\alpha(t)-1}{t}\right) \mathrm{d}t = z + (1-\alpha)z^2 + \frac{(1-\alpha)^2}{2}z^3 + \cdots.$$

Also, in Theorems 5 and 6, equalities occur for the function $f_c(z)$ given by

$$f_c(z) = z\exp\left(\int_0^z \frac{\psi_c^\alpha(t)-1}{t}\right) \mathrm{d}t = z + (1-\alpha)z^2 + \frac{(1-\alpha)^2}{2}z^3 + \cdots.$$

4. Applications

In this section, our aim is to highlight some interesting applications of the main results (Theorems 1, 2, 3, 4, 5, 6) in the form of the following corollaries and remarks.

Corollary 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_s(0) \equiv S_s^*$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \leq r_s$,*

$$|f'(z)| \leq |g'(z)|$$

where r_s is the least positive root of the following equation

$$(1-r^2)(1-\sin r) - 2r = 0.$$

Remark 7. Corollary 1 is sharper than the one obtained in [25, Theorem 2.1].

Corollary 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_c(0) \equiv S_c^*$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \leq r_c$,*

$$|f'(z)| \leq |g'(z)|$$

where r_c is the least positive root of the following equation

$$\left(1-r^2\right)(1 - r\cos r) - 2r = 0.$$

Remark 8. In view of Theorems 3, 4, 5, 6, one can make the following deductions easily.

Corollary 3. *Let $f \in \mathcal{ST}_s(0) \equiv S_s^*$ or $f \in \mathcal{ST}_c(0) \equiv S_c^*$. Then for any real number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \begin{cases} -\frac{2\beta-1}{2} & (\beta < 0), \\ \frac{1}{2} & (0 \leq \beta \leq 1), \\ \frac{2\beta-1}{2} & (\beta > 1). \end{cases}$$

Corollary 4. *Let $f \in \mathcal{ST}_s(0) \equiv S_s^*$ or $f \in \mathcal{ST}_c(0 \equiv S_c^*$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \max\left\{\frac{1}{2}, \frac{|2\beta-1|}{2}\right\}.$$

Remark 9. A direct application of Lemma 2 also shows that (see (3.13))

$$|a_2| \leq 1-\alpha, \quad |a_3| \leq \frac{(1-\alpha)^2}{2}, \tag{4.1}$$

and

$$|2a_3 - a_2^2| \leq 1-\alpha \Longrightarrow \left|a_3 - \frac{a_2^2}{2}\right| \leq \frac{1-\alpha}{2}, \tag{4.2}$$

which holds wherever $f \in \mathcal{ST}_s(\alpha)$ or $f \in \mathcal{ST}_c(\alpha)$. The last inequality is the same as, if one take $\beta = 1/2$ in Theorems 3 and 5 (cf. inequality (3.11)). We note that the coefficient bounds for a_2 and a_3 are sharp, and equality occurs for the functions defined in Remark 6.

5. Conclusion

In this paper, we introduce some new subfamilies of starlike functions of order $\alpha \in [0,1)$, by subordinated to sine and cosine functions. For functions in these subfamilies, sharp solutions of majorization and Fekete-Szegő problems have been derived. Applications of the obtained results for some new and known families of analytic functions have also been demonstrated for further illustration. The obtained results have unified and improved some recent results in the literature, especially those in [24, 25]. Based on Remark 2, it will be an interesting research problem to find conditions on α $(0 \le \alpha < 1)$ or to find radius of a certain disk within $|z| < 1$ so that the inclusion results hold.

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