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EXISTENCE OF SOLUTIONS TO MULTIVALUED PROBLEMS IN INCOMPLETE METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we obtain some existence results for multivalued contraction mappings in the context of an O-complete orthogonal metric space (not necessarily complete metric space). Also, we provide a partial solution to Reich's problem for multivalued orthogonal contraction mappings and extend Mizoguchi-Takahashi's point theorem. In addition, we give an example to demonstrate the applicability of our established results. We study the solution of a differential equation and its Ulam's stability as an application of the obtained results.

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1. INTRODUCTION AND PRELIMINARIES

Gordji et al. [12] established the notion of orthogonal sets and subsequently extended the Banach fixed point theorem in the so-called orthogonal-complete metric spaces to substitute the metric space completeness in fixed point results. Several authors have investigated this concept as not necessarily complete metric spaces (see [6, 7, 10, 23]).

With thanks to Nadler [18], Gordji et al. [12], the goal of this study is to prove some results on Mizoguchi-Takahashi type contraction mappings in the framework of an orthogonal metric space (which is not necessarily complete) and to adequate criteria for the existence of fixed points for such class of mappings.

There are many results on the existence of solutions for multivalued problems (see [1, 10, 15, 22, 23]). In 1974, Reich (see [19, 20]) questioned whether we might use a nonempty closed and bounded set instead of a nonempty compact set. Despite the fact that many fixed point theorists have researched this topic, it has not been entirely solved. There are several partial affirmative answers to this problem; for example, recently, Azé and Corvellec [5] give a partial positive answer to the conjecture using a

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simple variational approach and Mizoguchi et al. [17] generalize Nadler's conclusion in the establishment of metric space frameworks.

Daffer et al. [11] proved that $\alpha(t) = 1 - at^{b-1}$, where $a > 0$, for some $b \in (1,2)$ on some interval $[0,s]$, $0 < s < a^{-\frac{1}{b-1}}$, is a class of functions that satisfy all of Reich's conjecture's conditions and used this class to get their results. Suzuki [24] remarked that the corresponding result of [17] is a real generalization of the corresponding result of Nadler [18].

Recently, Gordji et al. [12] established the concept of an orthogonal set (also known as an O-set) and proved several fixed point theorems in the context of orthogonal metric spaces.

Definition 1. Let $\perp \subset X \times X$ be a binary relation, where $X \neq \varnothing$. If $\perp$ satisfies the following condition:
There exists $x_0 \in X$ such that (for all $y \in X$, $y \perp x_0$) or (for all $y \in X$, $x_0 \perp y$), then it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

If d is a metric on O-set $(X, \perp)$, then $(X, \perp, d)$ is an orthogonal metric space (OMS).

Example 1 ([12]). Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = k\,n$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.

Example 2. Let (X, d) be a metric space and $T : X \to X$ be a Picard operator, that is, T has a unique fixed point $x^* \in X$ and $\lim_{n \to +\infty} T_n(y) = x^*$ for all $y \in X$. We define the binary relation $\perp$ on X by $x \perp y$ if

$$\lim_{n \to \infty} d(x,\ T_n(y)) = 0.$$

Then, $(X, \perp)$ is an O-set (see [12]).

Example 3. Define the binary relation $\perp$ on an inner product space $(X, \langle .,. \rangle)$ by $x \perp y$ if $\langle x, y \rangle = 0$. Here, $0 \perp x$ for all $x \in X$. Hence, $(X, \perp)$ is an O-set (see [12]).

For other definitions viz. O-sequence, Cauchy O-sequence, O-complete, $\perp$-continuous, and $\perp$-preserving we refer to [12].

It is easy to see that every complete metric space is O-complete but the converse is not true (see [12]).

Let H be a Hausdorff-Pompeiu metric induced by metric d on a set X. Denote $\mathcal{CB}(X)$ the family of all nonempty, closed and bounded subsets of X.

$H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ defined by, for every $A, B \in \mathcal{CB}(X)$,

$$H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$.

Now, we have the following open question:

Let (X,d) be a O-complete OMS (not necessarily complete metric space) and $T: X \to \mathcal{CB}(X)$ satisfies the following

$$H(Tx, Ty) \leq \alpha(d(x,y))\, d(x,y), \tag{1.1}$$

for all $x,y \in X$ where α is a function from $[0,\infty)$ into $[0,1)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$. Does T has a fixed point?

In the context of OMS, we propose a partial solution to Reich's original problem using weaker multivalued orthogonal contraction mappings, that is (1.1).

2. Main results

We get certain findings for multivalued orthogonal Mizoguchi-Takahashi type contractions in the context of O-complete OMS (not necessarily complete metric space) and then construct fixed points of mappings fulfilling these contractions in this section.

We denote $\mathcal{P}$ be the family of all the functions $\alpha : [0,+\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \alpha(r) < 1$, for all $r \in [0,+\infty)$.

Here, we define the orthogonal relation between two nonempty subsets of an orthogonal set.

Definition 2. Let $(X,\perp)$ an orthogonal set. Suppose that A and B are two nonempty subsets of X. The set A is orthogonal to set B given by symbol $\perp_1$ and defined as follows: $A \perp_1 B$, if for every $a \in A$ and $b \in B$, $a \perp b$.

We are now ready to provide our first outcome.

Theorem 1. *Let $T: X \to \mathcal{CB}(X)$ be a multivalued mapping on an O-complete OMS $(X,\perp,d)$. Assume that the following conditions hold:*

- (i) *for all $x,y \in X$, $x \perp y$ implies $Tx \perp_1 Ty$,*
- (ii) *there exists $x_0 \in X$ such that $\{x_0\} \perp_1 Tx_0$ or $Tx_0 \perp_1 \{x_0\}$,*
- (iii) *if $\{x_n\}$ is O-sequence in X such that $x_n \to x^* \in X$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $x_{n(k)} \perp x^*$ or $x^* \perp x_{n(k)}$ for all $n \in \mathbb{N}$,*
- (iv) *there exists $\alpha \in \mathcal{P}$ such that*

$$H(Tx,Ty) \leq \alpha(d(x,y))\, d(x,y),$$

for all $x,y \in X$ with $x \perp y$.

Then there exists $x \in X$ such that $x \in Tx$, that is, a fixed point of T.

Proof. By assumption (i), there exists $x_1 \in Tx_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. By assumption (ii), we get $Tx_0 \perp_1 Tx_1$, that is there exists $x_2 \in Tx_1$ such that $x_1 \perp x_2$ or $x_2 \perp x_1$.

Define a function $\beta \in \mathcal{P}$ such that $\beta(t) = \frac{\alpha(t)+1}{2}$. Then $\beta(t) < \alpha(t)$ and for all $t \in [0,\infty)$ $\limsup_{r\to t+0} \beta(r) < 1$. Therefore, using (iv), we get

$$d(x_1,x_2) \leq H(Tx_0, Tx_1) \leq \beta(d(x_0,x_1))\, d(x_0,x_1). \tag{2.1}$$

Continuing this process inductively, we can construct an orthogonal sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}\cup\{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N}\cup\{0\}$.

$$d(x_{n+1},x_{n+2}) \leq \beta(d(x_n,x_{n+1}))\, d(x_n,x_{n+1}), \tag{2.2}$$

since $\beta(t) < 1$, for all $t \in [0,\infty)$ and $\{d(x_{n+1},x_n)\}$ is strictly non-increasing sequence. Suppose that $t_n = d(x_{n+1},x_n) \to t$, for some $t \geq 0$.
Since $\limsup_{s\to t+0} \beta(s) < 1$ and $\beta(t) < 1$, there exist $r \in [0,1)$ and $\varepsilon > 0$ such that $\beta(s) \leq r$ for all $s \in [t, t+\varepsilon]$. Now, we can choose some $m \in \mathbb{N}\cup\{0\}$ such that $t \leq t_n \leq t+\varepsilon$ for all $n \in \mathbb{N}\cup\{0\}$ with $n \geq m$. Here it is to note that

$$t_{n+1} \leq \beta(t_n)\, t_n \leq r\, t_n,$$

and thus by ratio test, we get

$$\sum_{n=1}^{\infty} t_n < \infty.$$

This shows that $\{x_n\}$ is a Cauchy orthogonal sequence. Since X is an O-complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$.

Now, we claim that $x^* \in Tx^*$. Assume to the contrary that $x^* \notin Tx^*$. Hence there exists $n_1 \in \mathbb{N}$ such that $x^* \notin \{x_n\}_{n\geq n_1}$ and $d(x_{n(k)}, Tx^*) > 0$. By using our assumption (iii), we have $x_{n(k)} \perp x^*$ or $x^* \perp x_{n(k)}$. Using (iv), we get

$$\begin{aligned} d(x_{n(k)+1}, Tx^*) &\leq H(Tx_{n(k)}, Tx^*) \\ &\leq \beta(d(x_{n(k)},x^*))\, d(x_{n(k)},x^*). \end{aligned}$$

Taking $k \to +\infty$, we get $x^* \in \overline{Tx^*} = Tx^*$. Hence we get the result. $\square$

Remark 1. In 2012, Karapinar and Samet [16] gave some results using α-admissible mappings. Later various generalizations have done using this (see [15,22]). It is interesting to see the main result without using orthogonality on α-admissible mappings for multivalued operators in the setting of orthogonal metric space.

3. CONSEQUENCES

3.1. *Single valued result*

As a consequence of Theorem 1, we have the following result for single valued mappings by replacing condition (iii) with T is $\perp$-continuous.

Theorem 2. *Let $T : X \to X$ be a self mapping on an O-complete OMS $(X,\perp,d)$. Assume that the following conditions hold:*

(i) *for all* $x,y \in X$, $x \perp y$ *implies* $Tx \perp Ty$,
(ii) *there exists* $x_0 \in X$ *such that* $x_0 \perp Tx_0$ *or* $Tx_0 \perp x_0$,
(iii) T *is* $\perp$*-continuous,*
(iv) *there exists* $\alpha \in \mathcal{P}$ *such that*

$$d(Tx, Ty) \leq \alpha(d(x,y))\, d(x,y),$$

for all $x,y \in X$ *with* $x \perp y$.

Then T *has a fixed point.*

Proof. In this case, we may choose T as a multivalued mapping by assuming that Tx is a singleton set for every $x \in X$. Also, $\{x_n\}$ is a Cauchy orthogonal sequence and $\lim_{n\to\infty} x_n = x^*$, by arguing on the same lines of Theorem 1. As T is $\perp$-continuous, we have

$$d(x^*, Tx^*) = \lim_{n\to+\infty} d(Tx_n, Tx^*) = 0,$$

that is, x^* is a fixed point of T. □

3.2. *Coupled fixed point*

Now we'll explain how our findings enable us to construct coupled fixed point theorems in O-complete OMS. We begin by recalling the following definition.

Let $G: X \times X \to X$ be a given mapping. We say that $(x,y) \in X \times X$ is a coupled fixed point of G if $G(x,y) = x$ and $G(y,x) = y$.

Our result is based on the following simple lemma which tells us when a coupled fixed point is a fixed point (see Samet [21]).

Lemma 1. *Let* $G: X \times X \to X$ *be a given mapping, where* $X \neq \varnothing$. *Define the mapping* $T: Y = X \times X \to Y = X \times X$ *by* $T(x,y) = (G(x,y), G(y,x))$, *for all* $(x,y) \in X \times X$. *Then,* (x,y) *is a fixed point of* T *if and only if* (x,y) *is a coupled fixed point of* G.

Theorem 3. *Let* $(X,\perp,d)$ *be an O-complete OMS and* $G: X \times X \to X$ *be a self mapping on* X. *Assume that the following conditions hold:*

(i) *there exists* $\alpha \in \mathcal{P}$ *such that for all* $x,y,u,v \in X$ *with* $x \perp y, u \perp v$,

$$d(G(x,y), G(u,v)) \leq \alpha(d((x,y),(u,v)))\, d((x,y),(u,v)), \tag{3.1}$$

(ii) G *is* $\perp$*-preserving,*
(iii) *there exists* $(x_0,y_0) \in X$ *such that* $x_0 \perp G(x_0,y_0)$ *or* $G(x_0,y_0) \perp x_0$ *and* $y_0 \perp G(y_0,x_0)$ *or* $G(y_0,x_0) \perp y_0$,
(iv) G *is* $\perp$*-continuous;*

Then G *has a coupled fixed point.*

Proof. Here take $(Y = X \times X, d)$ is O-complete orthogonal metric space. Define the mapping $T : Y \to Y$ by $T(x,y) = (G(x,y), G(y,x))$, for all $(x,y) \in X \times X$. From (3.1), we have

$$d(T(\xi,\eta)) \leq \alpha(d(\xi,\eta))\, d(\xi,\eta),$$

for all $\xi = (\xi_1,\xi_2), \eta = (\eta_1,\eta_2) \in Y$. So using Theorem 2, we get the result. □

Remark 2. On the same lines of Theorem 3, we can prove other coupled fixed point results.

3.3. *Illustration*

In this section, we provide an example to show the usability of our obtained results.

Example 4. Let

$$X = \{-\frac{1}{2},\dots,-\frac{1}{2^n},\dots\} \cup \{0\} \cup \{\frac{1}{2},\dots,\frac{1}{2^n},\dots\}$$

and $d : X \times X \to [0,\infty)$ be a mapping defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a relation $\perp$ on X by $x \perp y$ if and only if $xy \in \{x,y\} \subseteq X$.

Thus $(X,\perp,d)$ is an O-complete orthogonal metric space. Now, we define a mapping $T : X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \{0\}, & x = 0 \\ \{\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\}, & x = -\frac{1}{2^n}, \frac{1}{2^n}, n \geq 1. \end{cases}$$

Here T satisfies all the hypothesis of Theorem 1 for

$$\alpha(t) = \begin{cases} \frac{1}{2}, t > 0, \\ 0, t = 0 \end{cases}.$$

Hence T has a fixed point.

4. Applications

Many researchers worked on different problems and obtained the solution using fixed point approach, see [2, 3, 23]. Ulam [14, 25] stability has attracted attention of several authors in fixed point theory, see [4, 8, 9, 13, 23]. In this section, we investigate the existence of a solution to a differential equation and its Ulam stability as an application of the results presented in previous sections.

4.1. *Existence of solution for differential equations*

The purpose of this section is to find the existence of solution for a first order boundary value problem using the results proved in the paper.

In this section, we examine the following boundary value problem:

$$\begin{cases} u'(t) = \mathscr{T}(t,u(t)), t \in I = [0,T], T > 0 \\ u(0) = u(T), \end{cases} \tag{4.1}$$

where $u \in C(I,\mathbb{R})$ and $N : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X := C([0,1],\mathbb{R})$ be the set of continuous real valued functions defined on I, endowed with the metric $d : X \times X \to [0,\infty)$ defined as $d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$, for all $u,v \in X$. Define an orthogonal relation $u \perp v$ if and only if $uv \geq 0$, for all $u,v \in X$. Then $(X,\perp,d)$ is OMS.

Clearly, (4.1) is equivalent to the following linear first order equation

$$u'(t) + \mu u(t) = L(t,u(t)), t \in I, u(0) = u(T), \tag{4.2}$$

where $L(t,u(t)) = \mathscr{T}(t,u(t)) + \mu u(t)$ and $\mu > 0$. Also, a solution of the equation (4.2) is a fixed point of an integral equation

$$\begin{aligned} \mathscr{T}u(t) = {} & \frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ & + \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds, \quad \text{for some } \mu > 0. \end{aligned} \tag{4.3}$$

Also, $u \mapsto L(t,u(t))$ is $\perp$-continuous.

Theorem 4. *Assume that there is $\mu > 0$ such that for all $u,v \in C(I,\mathbb{R})$ with $u \perp v$,*

$$|L(s,u(s)) - L(s,v(s))| \leq e^{-\mu} \mu\, |u(s) - v(s)|,$$

for each $s \in [0,T]$. Then equation (4.1) with given boundary conditions has a solution in $C(I,\mathbb{R})$.

Proof. Define $\mathscr{T} : X \to X$ as in (4.3). So $\mathscr{T}$ is $\perp$-continuous. First, we show that $\mathscr{T}$ is $\perp$-preserving, let $u(t) \perp v(t)$ for all $t \in [0,1]$. Now, from (4.3) we have

$$\begin{aligned} \mathscr{T}u(t) = {} & \frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ & + \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds > 0, \end{aligned}$$

for some $\mu > 0$. It implies that $\mathscr{T}u \perp \mathscr{T}v$.

Now, we have to show that $\mathscr{T}$ satisfies (iv) of Theorem 2 for $\alpha(r) = e^{-r}, r > 0$. For all $t \in [0,1]$, $u(t) \perp v(t)$, we have

$$\begin{aligned} |\mathscr{T}u(t) - \mathscr{T}v(t)| = {} & |\frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ & + \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&-\Big(\frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,v(s))\, ds \\
&\quad+\frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,v(s))\, ds\Big)| \\
&\leq \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}|\, L(s,u(s))-\, L(s,v(s))|\, ds \\
&\quad+\frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}|\, L(s,u(s))-\, L(s,v(s))|\, ds \\
&\leq \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}e^{-\mu}\,\mu \sup_{s\in[0,1]}|u(s)-v(s)|]\, ds \\
&\quad+\frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}e^{-\mu}\,\mu \sup_{s\in[0,1]}|u(s)-v(s)|]\, ds \\
&\leq [e^{-\mu}\mu \sup_{s\in[0,1]}|u(s)-v(s)|]\times \sup_{t\in[0,1]}\{\frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, ds \\
&\quad+\int_t^T e^{\mu(s-t)}\, ds\} \\
&\leq e^{-\mu} \sup_{s\in[0,1]}|u(s)-v(s)|] \\
&= e^{-\mu}d(u,v),
\end{aligned}
$$

for all $u,v\in X$. Therefore, the condition (iv) of Theorem 2 holds. Accordingly all axioms of Theorem 2 are verified and $\mathscr{T}$ has a unique fixed point. It yields that the differential equation (4.1) possesses a unique solution. □

Remark 3. Theorem 4 will used to analyse some of the following real life applications:

1. To study the flow of current in the electric circuit in the engineering problems, for example: $LI'(t)+RI(t)=E$, where $I(t)$ is the current in the circuit at time t, L is inductance, R is resistance, E is electromotive force.

2. To study the velocity of the sky driver at any time. Once the sky diver jumps from an airplane, there are two forces that determine his motion : The pull of the earth's gravity acting down ward and the opposing force of air resistance acting upward. At high speeds, the strength of the air resistance force (the drag force) is proportional to the square of velocity, so the upward resistance force due to the air resistance can be expressed as $v'(t)=g-\frac{k}{m}v^2$, where $v(t)$ is velocity of sky-driver at any time t, g is acceleration due to gravity, the air resistance force has strength kv.

4.2. *Ulam stability of differential equation*

In this section, we discuss the Ulam stability of the differential equation (4.1). Equation (4.1) is called Ulam stable if it satisfies the following condition:

(AI) there is a constant $\delta > 0$, for each $\varepsilon > 0$ and for every solution $u \in X$ satisfying

$$|u'(t) - L(t,u(t))| \leq \varepsilon, \tag{4.4}$$

that is,

$$|\mathscr{T}u(t) - \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds - \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)} L(s,u(s))\, ds| \leq \varepsilon,$$

there exists some $v \in X$ satisfying $v \perp x$ and

$$\mathscr{T}v(t) = \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)} L(s,v(s))\, ds + \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)} L(s,v(s))\, ds, \tag{4.5}$$

such that

$$|v - x| \leq \delta\varepsilon. \tag{4.6}$$

Theorem 5. *Under the hypothesis of Theorem 4, the differential equation (4.1), is Ulam stable.*

Proof. On the account of Theorem 4, we guarantee a unique $v^* \in X$ such that $\mathscr{T}v^*(t) = \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,v^*(s))\, ds + \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,v^*(s))\, ds$, that is, $v^* \in X$ forms a solution of differential equation (4.1). Let $\varepsilon > 0$ and $u^* \in X$ be an ε-solution, that is,

$$\begin{aligned} |u^* - \mathscr{T}u^*| = |u^*(t) &- \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)} L(s,u^*(s))\, ds \\ &- \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)} L(s,u^*(s))\, ds| \\ \leq \varepsilon. \end{aligned}$$

Using Theorem 4, for $v^* \perp u^*$ we have

$$\begin{aligned} |v^* - u^*| &= |\mathscr{T}v^* - u^*| \\ &\leq |\mathscr{T}v^* - \mathscr{T}u^*| + |\mathscr{T}u^* - u^*| \\ &\leq e^{-\mu}|v^* - u^*| + \varepsilon. \end{aligned}$$

Hence, $|v^* - u^*| \leq \frac{1}{1-e^{-\mu}}\varepsilon = \delta\varepsilon$, where $\delta = \frac{1}{1-e^{-\mu}} > 0$. Therefore, equation (4.1) is Ulam stable.

□

4.3. *Ulam stability of fixed point problem*

On OMS $(X, \perp, d)$, $T : X \to X$, we investigate the fixed point equation

$$Tv = v \tag{4.7}$$

and the inequality (for $\varepsilon > 0$)

$$d(Tx, x) \leq \varepsilon. \tag{4.8}$$

Equation (4.7) is called Ulam stable if it satisfies the following condition:
(A) there is a constant $\delta > 0$, for each $\varepsilon > 0$ and for every solution x^* of the inequality (4.8), there is a solution $v^* \in X$ of the equation (4.7) with $v^* \perp x^*$ such that

$$d(v^*, x^*) \leq \delta\varepsilon. \tag{4.9}$$

Theorem 6. *Under the hypothesis of Theorem 2, the fixed point equation (4.7) is Ulam stable.*

Proof. On the account of Theorem 2, we guarantee a unique $v^* \in X$ such that $v^* = Tv^*$, that is, $v^* \in X$ forms a solution of (4.7). Let $\varepsilon > 0$ and $x^* \in X$ be an ε-solution, that is,

$$d(Tx^*, x^*) \leq \varepsilon.$$

Using Theorem 2, for $v^* \perp x^*$ we have

$$\begin{aligned} d(v^*, x^*) = d(Tv^*, x^*) &\leq d(Tv^*, Tx^*) + d(Tx^*, x^*) \\ &\leq \alpha(d(v^*, x^*))\, d(v^*, x^*) + \varepsilon. \end{aligned}$$

Hence, $d(v^*, x^*) \leq \frac{1}{1-\alpha(d(v^*,x^*))}\varepsilon = k\varepsilon$, where $k = \frac{1}{1-\alpha(d(v^*,x^*))} > 0$. Therefore, equation (4.7) is Ulam stable. □

Conclusions. In this manuscript, we provide a partial answer to Reich's problem in the context of incomplete metric spaces.

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