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EXISTENCE OF TWO WEAK SOLUTIONS FOR SOME ELLIPTIC PROBLEMS INVOLVING $p(x)$-BIHARMONIC OPERATOR

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Abstract. In this paper, we establish the existence of at least two distinct weak solutions for fourth-order PDEs with variable exponents, subject to Navier boundary conditions in a smooth bounded domain in $\mathbb{R}^N$, under a suitable subcritical growth condition with the classical Ambrosetti-Rabinowitz condition. The approach is based on variational methods and critical point theory.

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1. INTRODUCTION

Differential equations and variational problems with variable exponents growth conditions have been studied more in the last few years. These problems are connected to modeling of nonlinear electrorheological fluids and elastic mechanics. In addition, the study of these problems has become an important subject by progress in physics and other topics. In this sense, we refer the reader to [1, 4, 7, 12, 16, 17, 20]. Fourth-order differential equations become visible in many applications such as, micro-electro-mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [11]. The existence of solutions of $p(x)$-biharmonic problems has been studied by several authors (see [2, 3, 8, 13, 14]).

For instance, El Amrouss *et al.* [8] studied a class of $p(x)$-biharmonic of the form

$$\begin{cases} \Delta^2_{p(x)} u = \lambda |u|^{p(x)-2} u + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \leq 0$, $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$ and $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Using the Mountain Pass Theorem, they obtained the existence of at least one solution and the existence of infinitely many solutions of this problem.

Recently, motivated by this interest, in [2], the authors established the existence and multiplicity of solutions to the following problem

$$\begin{cases} \Delta^2_{p(x)}u+|u|^{p(x)-2}u=\lambda|u|^{q(x)-2}u+\mu|u|^{\gamma(x)-2}u & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega, \end{cases}$$

where where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N\geq 1$, p, q and γ are continuous functions on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x)>1$, $\inf_{x\in\overline{\Omega}} q(x)>1$, $\inf_{x\in\overline{\Omega}}\gamma(x)>1$ and λ and μ are parameters such that $\lambda^2+\mu^2\neq 0$.

In this paper, we want to consider the following fourth-order elliptic equation with Navier boundary conditions

$$\begin{cases} \Delta^2_{p(x)}u+|u|^{p(x)-2}u=\lambda f(x,u) & \text{in } \Omega,\\ u=\Delta u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N\geq 1$, $p(\cdot)\in C(\overline{\Omega})$ such that $1<p^-:=\min_{x\in\overline{\Omega}} p(x)\leq p(x)\leq p^+:=\max_{x\in\overline{\Omega}} p(x)<+\infty$, $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

(f_1) $|f(x,t)|\leq a_1+a_2|t|^{q(x)-1}, \quad \forall(x,t)\in\Omega\times\mathbb{R}$,

for some non-negative constants a_1,a_2, and $q(x)$ is a continuous function on $\overline{\Omega}$ with $1<q(x)<p_2^*(x)$ for each $x\in\overline{\Omega}$, where

$$p_2^*(x):=\begin{cases}\frac{Np(x)}{N-2p(x)}, & 2p(x)<N,\\ +\infty, & 2p(x)\geq N.\end{cases}$$

In this work, our goal is to obtain the existence of two distinct weak solutions for problem (1.1).

Recall that a function $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ is said to be a Carathéodory function, if

(C_1) the function $x\to f(x,t)$ is measurable for every $t\in\mathbb{R}$;
(C_2) the function $t\to f(x,t)$ is continuous for almost every $x\in\Omega$.

2. PRELIMINARIES AND BASIC DEFINITIONS

To study problem (1.1), we need some some theories on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Set

$$C_+(\overline{\Omega}):=\left\{h: h\in C(\overline{\Omega}),\, h(x)>1,\ \ \forall x\in\overline{\Omega}\right\}.$$

For any $h\in C_+(\overline{\Omega})$, we define

$$h^+:=\max\left\{h(x): x\in\overline{\Omega}\right\}, \qquad h^-:=\min\left\{h(x): x\in\overline{\Omega}\right\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) := \left\{ u\colon \Omega \to \mathbb{R} : u \text{ is measurable and} \int_\Omega |u(x)|^{p(x)} dx \, < +\infty \right\}.$$

We can introduce the so-called *Luxemburg norm* on $L^{p(x)}(\Omega)$ by

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

and $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$ becomes a Banach space.

Proposition 1 (Theorems 1.6 and 1.10 of [10])**.** *The space* $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\right)$ *is separable, uniformly convex, reflexive Banach space and its conjugate space is* $L^{q(x)}(\Omega)$*, where*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \forall x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$*, we have*

$$\left| \int_\Omega uv \, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), \, |\alpha| \le k \right\},$$

where

$$D^\alpha u := \frac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \partial x_2{}^{\alpha_2} \cdots \partial x_N{}^{\alpha_N}} u,$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| := \sum_{i=1}^{N} \alpha_i$.

The space $W^{k,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^{p(x)}(\Omega)}$$

also becomes a separable and reflexive Banach space (Theorem 2.1 of [10]). For more details, we refer the reader to [9, 10, 15, 22].

Denote

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)}, & kp(x) < N, \\ +\infty, & kp(x) \ge N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \ge 1$.

Proposition 2 (Theorem 2.3 of [10]). *For $p,r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

By $W_0^{k,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Further, denote by X the space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\| := \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)} \right) dx \leq 1 \right\}. \tag{2.1}$$

Remark 1.

(1) According to [23], the norm $\|.\|_{2,p(x)}$ is equivalent to the norm $\|\Delta\cdot\|_{L^{p(x)}(\Omega)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\Delta\cdot\|_{L^{p(x)}(\Omega)}$ are equivalent.

(2) By the above remark and Proposition 2, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q \in C(\overline{\Omega})$ and $1 \leq q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

In the sequel, we will denote by c_q the best constant for which one has

$$\|u\|_{L^{q(x)}(\Omega)} \leq c_q \|u\| \tag{2.2}$$

for all $u \in X$.

Proposition 3 (Proposition 3.2 of [8]). *If we denote*

$$\rho(u) := \int_\Omega \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)} \right) dx,$$

then, for $u, u_n \in X$, we have

(1) $\|u\| < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(respectively* $= 1; > 1$*);*

(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) $\|u_n\| \to 0$ *(respectively* $\to +\infty$ *)* $\Leftrightarrow \rho(u_n) \to 0$ *(respectively* $\to +\infty$*).*

Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$ for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover, we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1),

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$, where

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)} \right) dx, \qquad \Psi(u) := \int_\Omega F(x,u(x))dx. \tag{2.3}$$

Fixing the real parameter λ, a function $u\colon \Omega \to \mathbb{R}$ is said to be a *weak solution* of (1.1) if $u \in X$ and

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx \\ -\lambda \int_\Omega f(x,u(x))v(x)dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of (1.1).

Definition 1. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palise-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ in X such that

(a) $\{I(u_n)\}$ is bounded,
(b) $\lim\limits_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Our main tool is the following critical points theorem.

Theorem 1 (Theorem 3.2 of [5]). *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\{\Phi(u)<r\}} \Psi(u) < +\infty$ and assume that, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies* (PS)*-condition and it is unbounded from below. Then, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional I_λ admits two distinct critical points.*

3. Main results

In this section we establish the main abstract result of this paper. We recall that c_q is the constant of the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ for each $q \in C(\overline{\Omega})$ and $1 \le q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, and c_1 stands for c_q with $q = 1$ (see (2.2)).

Before introducing our result, we observe that putting

$$[\alpha]^h := \max\left\{\alpha^{h^-}, \alpha^{h^+}\right\}, \qquad [\alpha]_h := \min\left\{\alpha^{h^-}, \alpha^{h^+}\right\},$$

It is easy to verify that

$$[\alpha]^{\frac{1}{h}} := \max\left\{\alpha^{\frac{1}{h^-}}, \alpha^{\frac{1}{h^+}}\right\}, \qquad [\alpha]_{\frac{1}{h}} := \min\left\{\alpha^{\frac{1}{h^-}}, \alpha^{\frac{1}{h^+}}\right\}.$$

Theorem 2. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition* (f_1) *holds. Moreover, assume that*

(f_2) *there exist $\theta > p^+$ and $M > 0$ such that*

$$0 < \theta F(x,t) \le t f(x,t)$$

for each $x \in \Omega$ and $|t| \geq M$. Then, for each $\lambda \in]0, \lambda^*[$, problem (1.1) admits at least two distinct weak solutions, where

$$\lambda^* := \frac{q^-}{q^- a_1 c_1 (p^+)^{\frac{1}{p^-}} + a_2 [c_q]^q (p^+)^{\frac{q^+}{p^-}}}.$$

Proof. Our aim is to apply Theorem 1 to problem (1.1) in the case $r = 1$ to the space X with the norm $\|\cdot\|$ defined in (2.1) and to the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ defined in (2.3) for all $u \in X$. Clearly, $\Phi(0) = \Psi(0) = 0$. The functional Φ is in $C^1(X, \mathbb{R})$ and $\Phi' \colon X \to X^*$ is a homeomorphism (see Theorem 3.4 of [8]). Moreover, thanks to condition (f_1) and to the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, Ψ is in $C^1(X, \mathbb{R})$ and has compact derivative and

$$\Psi'(u)(v) = \int_\Omega f(x, u(x)) v(x) dx,$$

for every $v \in X$. Now we prove that $I_\lambda = \Phi - \lambda \Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$d := \sup_n I_\lambda(u_n) < +\infty, \qquad \left\| I'_\lambda(u_n) \right\|_{X^*} \to 0, \tag{3.1}$$

contains a convergent subsequence. Thus it is sufficient to verify that $\{\|u_n\|\}$ is bounded. Assume $\|u_n\| > 1$ for convenience. For n large enough, we have by (3.1)

$$d \geq I_\lambda(u_n) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \lambda \int_\Omega F(x, u_n(x)) dx,$$

then, by (f_2) and Proposition 3,

$$\begin{aligned}
I_\lambda(u_n) &\geq \frac{1}{p^+} \int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \frac{\lambda}{\theta} \int_\Omega f(x, u_n(x)) u_n(x) dx \\
&= \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx \\
&\quad + \frac{1}{\theta} \left[\int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \lambda \int_\Omega f(x, u_n(x)) u_n(x) dx \right] \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} + \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle.
\end{aligned}$$

Due to (3.1), we can actually assume that $\left| \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \right| \leq \|u_n\|$. Thus

$$d + \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-}.$$

Since $\theta > p^+$ and $p^- > 1$, it follows from this quadratic inequality that $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, then we gain the following convergence

$$\Phi'(u_n) \to \lambda\Psi'(u).$$

Since Φ' is a homeomorphism, then $u_n \to u$ and so I_λ satisfies (PS)-condition.

At this step we prove that there is a positive constant C such that

$$F(x,t) \geq C|t|^\theta \tag{3.2}$$

for all $x \in \Omega$ and $|t| > M$. For this, setting $a(x) := \min_{|\xi|=M} F(x,\xi)$ and

$$\varphi_t(s) := F(x,st) \quad \forall s > 0, \tag{3.3}$$

by (f_2), for every $x \in \Omega$ and $|t| > M$ one has

$$0 < \theta\varphi_t(s) = \theta F(x,st) \leq st \cdot f(x,st) = s\varphi'_t(s) \quad \forall s > \frac{M}{|t|}.$$

Therefore,

$$\int_{M/|t|}^1 \frac{\varphi'_t(s)}{\varphi_t(s)} ds \geq \int_{M/|t|}^1 \frac{\theta}{s} ds.$$

Then

$$\varphi_t(1) \geq \varphi_t\left(\frac{M}{|t|}\right) \frac{|t|^\theta}{M^\theta}.$$

Taking into account of (3.3), we obtain

$$F(x,t) \geq F\left(x, \frac{M}{|t|}t\right) \frac{|t|^\theta}{M^\theta} \geq a(x)\frac{|t|^\theta}{M^\theta} \geq C|t|^\theta,$$

where $C > 0$ is a constant. Thus, (3.2) is proved.

Fixed $u_0 \in X\backslash\{0\}$ for each $t > 1$ one has

$$I_\lambda(tu_0) \leq \frac{1}{p^-} t^{p^+} \|u_0\|^p - \lambda C t^\theta \int_\Omega |u_0(x)|^\theta dx.$$

Since $\theta > p^+$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0,\lambda^*[$ for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty,1[)$, thanks to Proposition 3, one has

$$\|u\| \leq \left[p^+\Phi(u)\right]^{\frac{1}{p}} < \left[p^+\right]^{\frac{1}{p}} = \left(p^+\right)^{\frac{1}{p^-}}. \tag{3.4}$$

By Theorem 1.3 of [10] and from the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we have

$$\int_\Omega |u(x)|^{q(x)} dx \leq \left[\|u\|_{L^{q(x)}(\Omega)}\right]^q \leq [c_q\|u\|]^q, \tag{3.5}$$

for each $u \in X$. Moreover, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1), (3.4) and (3.5) imply that for each $u \in \Phi^{-1}(]-\infty,1[)$, we have

$$\Psi(u) \leq a_1 \int_\Omega |u(x)|dx + \frac{a_2}{q^-}\int_\Omega |u(x)|^{q(x)} \leq a_1c_1\|u\| + \frac{a_2}{q^-}[c_q\|u\|]^q$$

$$\leq a_1 c_1 \left(p^+\right)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[c_q]^q \left(p^+\right)^{\frac{q^+}{p^-}},$$

and so

$$\sup_{\Phi(u)<1} \Psi(u) \leq a_1 c_1 \left(p^+\right)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[c_q]^q \left(p^+\right)^{\frac{q^+}{p^-}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{3.6}$$

From (3.6) one has

$$\lambda \in \left]0, \lambda^*\right[\subseteq \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}} \Psi(u)}\right[.$$

So, all hypotheses of Theorem 1 are verified. Therefore for each $\lambda \in \left]0, \lambda^*\right[$ the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). □

Remark 2. We observe that, if f is non-negative and $f(x,0) \neq 0$ in Ω, then Theorem 2 ensures the existence of two positive weak solutions for problem (1.1) (see, e.g., Theorem 11.1 of [18]).

A special case of Theorem 2 reads as follows.

Theorem 3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with $f(0) \neq 0$, satisfying for some $q \in (p, p_2^*)$,*

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^{q-1}} = 0,$$

where $p > 1$ and

$$p_2^* := \begin{cases} \frac{Np}{N-2p}, & 2p < N, \\ +\infty, & 2p \geq N. \end{cases}$$

Then, there exists $\lambda^* > 0$, such that, for any $\lambda \in \left]0, \lambda^*\right[$ the following problem

$$\begin{cases} \Delta_p^2 u + |u|^{p-2}u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions.

Remark 3. Thanks to Talenti's inequality, it is possible to obtain an estimate of the embedding's constants c_1, c_q. By the Sobolev embedding theorem, there exists a positive constant c such that (see Proposition B.7 of [19])

$$\|u\|_{L^{p^{-*}}(\Omega)} \leq c\,\|u\| \quad (\forall u \in X). \tag{3.7}$$

The best constant that appears in (3.7) is (see [21])

$$c := \frac{1}{N\sqrt{\pi}} \left(\frac{N!\,\Gamma\left(\frac{N}{2}\right)}{2\Gamma\left(\frac{N}{p^-}\right)\Gamma\left(N+1-\frac{N}{p^-}\right)} \right)^{\frac{1}{N}} \eta^{1-\frac{1}{p^-}}, \tag{3.8}$$

where

$$\eta := \frac{N(p^- - 1)}{N - p^-}.$$

Due to (3.8), as a simple consequence of Hölder's inequality, it follows that

$$c_q \leq \frac{\operatorname{meas}(\Omega)^{\frac{p^{-*} - q^+}{p^{-*} q^+}}}{N\sqrt{\pi}} \left(\frac{N!\Gamma\left(\frac{N}{2}\right)}{2\Gamma\left(\frac{N}{p^-}\right)\Gamma\left(N+1-\frac{N}{p^-}\right)} \right)^{\frac{1}{N}} \eta^{1-\frac{1}{p^-}},$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω.

In conclusion, we present a concrete example of application of Theorem 2 whose construction is motivated by Example 4.1 of [6].

Example 1. We consider the function f defined by

$$f(x,t) := \begin{cases} c + dqt^{q(x)-1} & \text{if } x \in \Omega,\ t \geq 0, \\ c - dq(-t)^{q(x)-1} & \text{if } x \in \Omega,\ t < 0. \end{cases}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where $p, q \in C(\overline{\Omega})$ verify the condition $1 < p^+ < q^- \leq q(x) < p^*(x)$ for each $x \in \Omega$ and c, d are two positive constants. Fixed $p^+ < \theta < q^-$ and

$$r > \max\left\{ \left[\frac{(\theta-1)c}{d(q^- - \theta)}\right]^h, \left[\frac{c}{d}\right]^h \right\},$$

with $h(\cdot) = \frac{1}{q(\cdot)-1}$. We prove that f verifies the assumptions requested in Theorem 2. condition (f_1) of Theorem 2 is easily verified. We observe that

$$F(x,t) = ct + d|t|^{q(x)},$$

for each $(x,t) \in \Omega \times \mathbb{R}$. Taking into account that, condition (f_2) is verified (see Example 4.1 of [6]) and clearly $f(x,0) \neq 0$ in Ω, problem (1.1) has at least two non-trivial weak solutions for every $\lambda \in]0, \lambda^*[$, where λ^* is the constant introduced in the statement of Theorem 2.

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References

[1] M. I. Abbas and M. A. Ragusa, "On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function." *Symmetry-Basel*, vol. 13, no. 2, article no. 264, 2021, doi: 10.3390/sym13020264.

[2] G. A. Afrouzi, M. Mirzapour, and N. T. Chung, "Existence and non-existence of solutions for a $p(x)$-biharmonic problem." *Electron. J. Differential Equations*, vol. 2015, article no. 158, 2015.

[3] G. A. Afrouzi and S. Shokooh, "Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator." *Electron. J. Differential Equations*, vol. 2015, article no. 317, 2015.

[4] Y. Benia and A. Scapellato, "Existence of solution to Korteweg-de Vries equation in a non-parabolic domain." *Nonlinear Anal.*, vol. 195, article no. 111758, 2020, doi: 10.1016/j.na.2020.111758.

[5] G. Bonanno, "Relations between the mountain pass theorem and local minima." *Adv. Nonlinear Anal.*, vol. 1, no. 3, pp. 205–220, 2012, doi: 10.1515/anona-2012-0003.

[6] G. Bonanno and A. Chinnì, "Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent." *J. Math. Anal. Appl.*, vol. 418, no. 2, pp. 812–827, 2014, doi: 10.1016/j.jmaa.2014.04.016.

[7] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents.* Heidelberg: Springer-Verlag, 2011. doi: 10.1007/978-3-642-18363-8.

[8] A. El Amrouss, F. Moradi, and M. Moussaoui, "Existence of solutions for fourth-order PDEs with variable exponents." *Electron. J. Differential Equations*, vol. 2009, article no. 153, 2009.

[9] X.-L. Fan and X. Fan, "A Knobloch-type result for $p(t)$-Laplacian systems." *J. Math. Anal. Appl.*, vol. 282, no. 2, pp. 453–464, 2003, doi: 10.1016/S0022-247X(02)00376-1.

[10] X.-L. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$." *J. Math. Anal. Appl.*, vol. 263, no. 2, pp. 424–446, 2001, doi: 10.1006/jmaa.2000.7617.

[11] A. Ferrero and G. Warnault, "On solutions of second and fourth order elliptic equations with power-type nonlinearities." *Nonlinear Anal.*, vol. 70, no. 8, pp. 2889–2902, 2009, doi: 10.1016/j.na.2008.12.041.

[12] C. S. Goodrich, M. A. Ragusa, and A. Scapellato, "Partial regularity of solutions to $p(x)$-Laplacian PDEs with discontinuous coefficients." *J. Differential Equations*, vol. 268, no. 9, pp. 5440–5468, 2020, doi: 10.1016/j.jde.2019.11.026.

[13] L. Li, L. Ding, and W.-W. Pan, "Existence of multiple solutions for a $p(x)$-biharmonic equation." *Electron. J. Differential Equations*, vol. 2013, article no. 139, 2013.

[14] L. Li and C. Tang, "Existence and multiplicity of solutions for a class of $p(x)$-biharmonic equations." *Acta Math. Sci. Ser. B (Engl. Ed.)*, vol. 33, no. 1, pp. 155–170, 2013, doi: 10.1016/S0252-9602(12)60202-1.

[15] M. Mihăileseu, "Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator." *Nonlinear Anal.*, vol. 67, no. 5, pp. 1419–1425, 2007, doi: 10.1016/j.na.2006.07.027.

[16] J. Musielak, *Orlicz Spaces and Modular Spaces.* Berlin: Springer, 1983. doi: 10.1007/BFb0072210.

[17] N. S. Papageorgiou and A. Scapellato, "Nonlinear Robin problems with general potential and crossing reaction." *Rend. Lincei-Mat. Appl.*, vol. 30, no. 1, pp. 1–29, 2019, doi: 10.4171/RLM/833.

[18] P. Pucci and J. Serrin, "The strong maximum principle revisited." *J. Differential Equations*, vol. 196, no. 1, pp. 1–66, 2004, doi: 10.1016/j.jde.2003.05.001.

[19] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* Providence: Amer. Mat. Soc., 1986. doi: 10.1090/cbms/065.

[20] M. Ružička, *Electrorheological fluids: Modeling and Mathematical Theory.* Berlin: Springer-Verlag, 2000. doi: 10.1007/BFb0104029.

[21] G. Talenti, "Best constants in Sobolev inequality," *Ann. Math. Pura Appl.*, vol. 110, pp. 353–372, 1976, doi: 10.1007/BF02418013.

[22] J. Yao, "Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators." *Nonlinear Anal.*, vol. 68, no. 5, pp. 1271–1283, 2008, doi: 10.1016/j.na.2006.12.020.

[23] A. Zang and Y. Fu, "Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces." *Nonlinear Anal.*, vol. 69, no. 10, pp. 3629–3636, 2008, doi: 10.1016/j.na.2007.10.001.

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