

# Autonomous Unicycle: Modeling, Dynamics, and Control

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## Abstract

This article focuses on modeling and analyzing the dynamics of an autonomous unicycle. The equations of motion of the nonholonomic multibody system are derived using the Appellian approach, which enables the use of minimum number of state variables and results in a system of low complexity. The final equations of motion consist of 8 first-order kinematic differential equations and 6 first-order dynamic differential equations. The stability of the open-loop dynamics is investigated and a PD type controller with 3 input torques is proposed that enables the unicycle to stabilize its upright position and travel along a straight path. The performance of the closed-loop system is demonstrated via numerical simulations.

# 1 Introduction

Agility and maneuverability are highly desired characteristics of mobile robots. One way to achieve this is making the uncontrolled system unstable and utilizing high-performance controllers to stabilize the desired maneuvers. A unicycle platform may be utilized to achieve such behavior. This paper focuses on building a modeling and analysis framework for the corresponding multibody system consisting of a rolling wheel, a body (to be balanced), and two flywheels used for balancing and maneuvering; see Fig. 1. A feedback design is proposed that enables the unicycle to drive along a straight path while maintaining its upright position. With the rise of human-ridden electric unicycles in our cities we envision autonomous unmanned unicycles to be utilized for freight delivery in urban environments.

While modeling the unicycle we assume a rigid wheel rolling without slipping on the ground, that is, we assume that the single contact point of the wheel has zero velocity. This so-called kinematic constraint results in a so-called nonholonomic system. Nonholonomic systems can be modeled by the Lagrangian approach by introducing Lagrange multipliers (one for each kinematic constraint), where the magnitude of the multipliers represents the magnitude of the constraining forces [29, 39]. However, this leads to differential-algebraic equations which are typically difficult to handle. Instead, in this paper, we utilize the Appellian approach [1, 10], which eliminates the constraining forces and represents the system with a minimum number of state variables. Similar methods were also developed in [13, 17, 37] and the relationships between different approaches were discussed in [6–8, 16]. More details on nonholonomic systems can be found in [2, 3, 5, 9, 11, 18, 19, 22, 24, 25].

Nonholonomic models have been developed for different classes of mobility devices including automobiles [4, 27, 36], bicycles [14, 20], snake-like robots [12, 40], and even swimming robots [26, 28, 34]. When it comes to unicycles, a few different designs have been studied in the past [33]. For example, in [31] the unicycle was constructed from a rolling wheel, a body, and an overhead flywheel, where the latter represented the twisting torso of a rider. The same setup was utilized in [21, 38] where linear quadratic regulators were proposed to balance the upright position. A different approach was taken in [41], where rather than using an overhead flywheel, a lateral flywheel was considered for balancing. This configuration was utilized in [15, 30] where controllers were also proposed for path tracking. Finally, efforts were also made to model the dynamics of riders for human-powered unicycles in [23, 32] and the latter publication also included experiments with human subjects.

In this paper, for the first time, we investigate an autonomous unmanned unicycle which includes both a balancing flywheel and an overhead flywheel (the latter we refer to as the steering flywheel from now on). By applying the Appellian approach, we select the minimum number of state variables (i.e., generalized coordinates and pseudo-velocities) necessary to describe the dynamics. We construct the acceleration energy of the multibody system as a whole, calculate the pseudo-forces from the active forces, and derive the

equations of motion via the Appell-Gibbs equations. These result in a system of first order ordinary differential equations. The linearization of the system is used to investigate the stability of the straight running motion analytically and to study the self-stabilizing effects of the speed as a parameter. Then we utilize a proportional-derivative (PD) type controller to stabilize the upright position and control the straight running motion of the unicycle. We also demonstrate via numerical simulations in MATLAB that the controller, which is tuned using the linearized system, is able to reject perturbations at the nonlinear level.

This paper is organized as follows. The mechanical model of the unicycle is presented in Section 2. The equations of motion are derived in Section 3. Stability analysis and control design are performed in Section 4, where numerical simulation results are also shown to verify the theoretical findings. Finally, our work is concluded and future research directions are outlined in Section 5.

## 2 Modeling

In this section, we establish the mechanical model of the unicycle, describe the reference frames used, and provide the rolling constraints. The derivation of the governing equations is left for the next section.

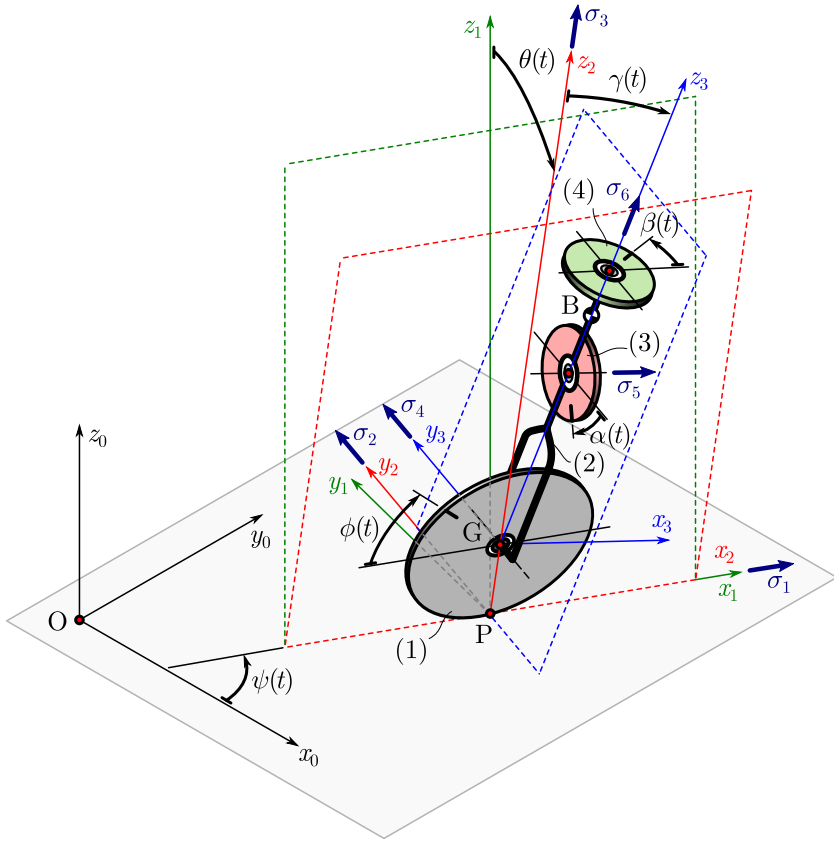
### 2.1 System Components and Parameters

The mechanical model of the unicycle is shown in Fig. 1. Since the unicycle is autonomous, the construction is simpler than the unicycle models with human riders. The system consists of the following components: a rolling wheel (gray); the body (black); a lateral balancing flywheel (red); a steering flywheel (green). The descriptions of these components are summarized in Table 1.

**Table 1:** System Parameter Definitions

Parameter	Description
$m_w$	mass of rolling wheel
$R$	radius of rolling wheel
$m$	mass of body
$h$	distance between wheel center point and the center of mass of the body
$J_x, J_y, J_z$	principal mass moment of inertias of the body
$m_b$	mass of balancing flywheel
$r_b$	radius of balancing flywheel
$m_s$	mass of steering flywheel
$r_s$	radius of steering flywheel

We apply the following modeling assumptions. (i) The driven wheel is modeled by a rigid disc having a single contact point with the rigid flat ground. Rolling without slipping is considered, namely the velocity of the material point P contacting the ground is zero. The rolling resistance is also ignored, as it does not introduce any significant changes beyond a slight shift in equilibrium pitch angle. (ii) The principal axes of the mass moment of inertia tensor coincide with the symmetry axes of the body. Consequently, the matrix of the

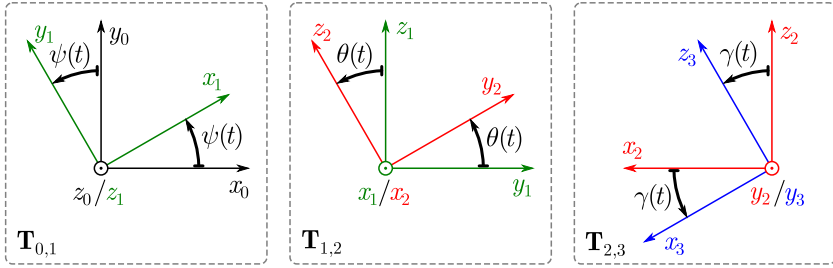


**Fig. 1:** Mechanical model of the unicycle including the rolling wheel, the body, and two flywheels. The reference frames, the angles, and the angular velocity components are indicated in the figure.

inertia tensor is diagonal in the body-fixed frame (when the axis of the frame is aligned with the symmetry axis of the body). (iii) The centers of the lateral balancing and the steering flywheels coincide with the center of mass of body B. (In Fig. 1 these three mass centers are moved apart to increase the readability of the figure). Since the interaction between the body and flywheels is manifested by internal torques, the locations of the flywheels do not influence their effects. These assumptions are introduced to reduce the complexity of the governing equations, while the dynamical behavior of the system remains similar to the behavior of more general configurations.

Finally, the inputs of the system are defined as follows. We apply three electric motors for actuation, hence, an internal driving torque  $M_w$  is applied between the rolling wheel and the body; the lateral balancing torque  $M_b$  is applied between balancing flywheel and the body, and the steering torque  $M_s$  is applied between the steering flywheel and the body.

## 2.2 Configurational Coordinates and Reference Frames



**Fig. 2:** Definitions of frames with rotations.

To describe the motion of the unicycle, one can introduce multiple reference frames. The ground-fixed frame  $F_0$  is spanned by the  $x_0$ ,  $y_0$  and  $z_0$  axes; see Figs. 1 and 2. To determine the spatial position of the unicycle in this frame, we use the coordinates  $x_G$ ,  $y_G$  and  $z_G$  of the center of mass  $G$  of the rolling wheel. In order to describe the spatial orientation of the body of the unicycle, we introduce the yaw  $\psi$ , tilt  $\theta$ , and pitch  $\gamma$  angles. Using these, other frames are introduced as explained below.

Frame  $F_1$  is obtained by rotating frame  $F_0$  with the yaw angle  $\psi$  about the  $z_0$  axis; see Fig. 2(a). A further rotation of frame  $F_1$  by the tilt angle  $\theta$  about the  $x_1$  axis leads to frame  $F_2$ . Finally, the pitch angle  $\gamma$  is used to rotate frame  $F_2$  about  $y_2$  axis to obtain the frame  $F_3$ . The corresponding frame transformation matrices can be written as follows:

$$\mathbf{T}_{0,1} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T}_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \quad \mathbf{T}_{2,3} = \begin{bmatrix} \cos \gamma & 0 & -\sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & \cos \gamma \end{bmatrix}, \quad (1)$$

and

$$\mathbf{T}_{0,2} = \mathbf{T}_{1,2} \mathbf{T}_{0,1}, \quad \mathbf{T}_{0,3} = \mathbf{T}_{2,3} \mathbf{T}_{1,2} \mathbf{T}_{0,1}. \quad (2)$$

These allow us to transform vectors between frames according to

$$\square_{F_j} = \mathbf{T}_{i,j} \square_{F_i}, \quad \square_{F_i} = \mathbf{T}_{i,j}^\top \square_{F_j}, \quad (3)$$

where we exploited that  $\mathbf{T}_{i,j}$  are orthogonal matrices, i.e.,  $\mathbf{T}_{i,j}^{-1} = \mathbf{T}_{i,j}^\top$ .

The angular velocities of the frames (with respect to the ground-fixed frame  $F_0$ ) can also be derived. For example, the angular velocity of frame  $F_2$

represented in frame  $F_2$  is given by

$$\boldsymbol{\Omega}_2 = \begin{bmatrix} \dot{\theta} \\ \dot{\psi} \sin \theta \\ \dot{\psi} \cos \theta \end{bmatrix}_{F_2}. \quad (4)$$

Beyond the above-mentioned rotations, three more angles are necessary to determine the rotational positions of the rolling wheel and the two flywheels. The rotation of the rolling wheel about the  $y_2$  axis (relative to frame  $F_2$ ) is described by the angle  $\phi$ . Similarly, angles  $\alpha$  and  $\beta$  characterize the rotations of the balancing flywheel and the steering flywheel about the  $x_3$  and  $z_3$  axis, respectively (relative to the frame  $F_3$ ).

Thus, without considering the kinematic constraint of rolling, one may use the nine scalar variables  $x_G, y_G, z_G, \psi, \theta, \phi, \gamma, \alpha, \beta$  as configurational coordinates.

### 2.3 Kinematic Constraints of Rolling

The velocity of the wheel-ground contact point  $P$  of the rolling wheel can be derived using the transport formula

$$\mathbf{v}_P = \mathbf{v}_G + \boldsymbol{\omega}_w \times \mathbf{r}_{GP}. \quad (5)$$

The velocity of the center of mass  $G$  of the wheel in the  $F_0$  frame is given by

$$\mathbf{v}_G = \begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{z}_G \end{bmatrix}_{F_0}. \quad (6)$$

The angular velocity of the rolling wheel can be resolved in the  $F_2$  frame as

$$\boldsymbol{\omega}_w = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} + \dot{\psi} \sin \theta \\ \dot{\psi} \cos \theta \end{bmatrix}_{F_2}, \quad (7)$$

and the position vector in the  $F_2$  frame becomes

$$\mathbf{r}_{GP} = \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix}_{F_2}. \quad (8)$$

Transforming the cross product in (5) from the  $F_2$  frame to the  $F_0$  frame with the help of the matrix  $\mathbf{T}_{0,2}^T$  (cf. (1)–(3)) and assuming rolling without slipping, that is,

$$\mathbf{v}_P = \mathbf{0}, \quad (9)$$

one may obtain the constraint equations

$$\begin{cases} \dot{x}_G - \dot{\phi}R \cos \psi - \dot{\psi}R \cos \psi \sin \theta - \dot{\theta}R \sin \psi \cos \theta = 0, & (10) \\ \dot{y}_G - \dot{\phi}R \sin \psi - \dot{\psi}R \sin \psi \sin \theta + \dot{\theta}R \cos \psi \cos \theta = 0, & (11) \\ \dot{z}_G + R\dot{\theta} \sin \theta = 0. & (12) \end{cases}$$

Here (10) and (11) are so-called non-integrable kinematic constraints and they will be utilized in the above differential form in the remaining part of this study. On the other hand, (12) can be integrated with respect to time and rephrased as a geometric constraint:

$$z_G = R \cos \theta. \quad (13)$$

As a consequence, the vertical position  $z_G$  of point G will not be used as a generalized coordinate. That is, the eight generalized coordinates  $x_G, y_G, \psi, \theta, \phi, \gamma, \alpha, \beta$  describe the mechanical system unambiguously.

### 3 Governing Equations

The equations of motion of the unicycle are derived here using the Appellian approach [1, 5, 9, 10]. To obtain more details about the approach, the role of the pseudo-velocities, and their appropriate choice, we refer the reader to [27], as an entry point to the literature. After choosing the pseudo-velocities we construct the acceleration energy of the system which is used to derive the left-hand side of the Appell-Gibbs equations. The pseudo-forces, which show up on the right-hand side of the equations, are obtained by calculating the virtual power of the active forces and torques.

#### 3.1 pseudo-velocities

The angular velocity of the body can be resolved in the  $F_2$  frame as

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta} \\ \dot{\gamma} + \dot{\psi} \sin \theta \\ \dot{\psi} \cos \theta \end{bmatrix}_{F_2}, \quad (14)$$

cf. (7). One may transform this to the  $F_3$  frame using the transformation matrix  $\mathbf{T}_{2,3}$  (cf. (1)–(3)):

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta} \cos \gamma - \dot{\psi} \cos \theta \sin \gamma \\ \dot{\gamma} + \dot{\psi} \sin \theta \\ \dot{\theta} \sin \gamma + \dot{\psi} \cos \theta \cos \gamma \end{bmatrix}_{F_3}. \quad (15)$$

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Similarly, the angular velocities of the balancing flywheel and the steering flywheel can be derived as

$$\boldsymbol{\omega}_b = \begin{bmatrix} \dot{\alpha} + \dot{\theta} \cos \gamma - \dot{\psi} \cos \theta \sin \gamma \\ \dot{\gamma} + \dot{\psi} \sin \theta \\ \dot{\theta} \sin \gamma + \dot{\psi} \cos \theta \cos \gamma \end{bmatrix}_{F_3}, \quad \boldsymbol{\omega}_s = \begin{bmatrix} \dot{\theta} \cos \gamma - \dot{\psi} \cos \theta \sin \gamma \\ \dot{\gamma} + \dot{\psi} \sin \theta \\ \dot{\beta} + \dot{\theta} \sin \gamma + \dot{\psi} \cos \theta \cos \gamma \end{bmatrix}_{F_3}. \quad (16)$$

Since the system has eight generalized coordinates and two kinematic constraints, six pseudo-velocities are required to accomplish the derivation of governing equations and they can be defined as follows. The components of the angular velocity  $\boldsymbol{\omega}_w$  of the wheel (see (7)) are used to define the pseudo-velocities  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$ , respectively. The second component of the angular velocity  $\boldsymbol{\omega}$  of the body, the first component of the angular velocity  $\boldsymbol{\omega}_b$  of the balancing flywheel, and the third component of the angular velocity  $\boldsymbol{\omega}_s$  of the steering flywheel are used as  $\sigma_4$ ,  $\sigma_5$  and  $\sigma_6$ , respectively. That is, the pseudo-velocities are as follows:

$$\begin{aligned} \sigma_1 &:= \dot{\theta}, \\ \sigma_2 &:= \dot{\phi} + \dot{\psi} \sin \theta, \\ \sigma_3 &:= \dot{\psi} \cos \theta, \\ \sigma_4 &:= \dot{\gamma} + \dot{\psi} \sin \theta, \\ \sigma_5 &:= \dot{\alpha} + \dot{\theta} \cos \gamma - \dot{\psi} \cos \theta \sin \gamma, \\ \sigma_6 &:= \dot{\beta} + \dot{\theta} \sin \gamma + \dot{\psi} \cos \theta \cos \gamma. \end{aligned} \quad (17)$$

These definitions together with the kinematic constraints (10),(11) lead to the linear system

$$\begin{bmatrix} 1 & 0 & -R \cos \psi \sin \theta & -R \sin \psi \cos \theta & -R \cos \psi & 0 & 0 & 0 \\ 0 & 1 & -R \sin \psi \sin \theta & R \cos \psi \cos \theta & -R \sin \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin \theta & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin \theta & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos \theta \sin \gamma & \cos \gamma & 0 & 0 & 1 & 0 \\ 0 & 0 & \cos \theta \cos \gamma & \sin \gamma & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}. \quad (18)$$

The determinant of the square matrix above is  $\cos \theta$ , that is, the matrix is only singular for  $\theta = \pm\pi/2$ . which corresponds to the wheel lying on the ground. This indicates that the selection of pseudo-velocities is appropriate. Solving



(18) provides the generalized velocities expressed as functions of the pseudo-velocities and generalized coordinates:

$$\begin{aligned}
 \dot{\theta} &= \sigma_1, \\
 \dot{\gamma} &= \sigma_4 - \sigma_3 \tan \theta, \\
 \dot{\psi} &= \frac{\sigma_3}{\cos \theta}, \\
 \dot{x}_G &= \sigma_2 R \cos \psi + \sigma_1 R \sin \psi \cos \theta, \\
 \dot{y}_G &= \sigma_2 R \sin \psi - \sigma_1 R \cos \psi \cos \theta, \\
 \dot{\phi} &= \sigma_2 - \sigma_3 \tan \theta, \\
 \dot{\alpha} &= \sigma_5 - \sigma_1 \cos \gamma + \sigma_3 \sin \gamma, \\
 \dot{\beta} &= \sigma_6 - \sigma_1 \sin \gamma - \sigma_3 \cos \gamma.
 \end{aligned} \tag{19}$$

Here the equations are ordered while considering the linearization performed further below. We remark that the choice of pseudo-velocities is not unique. For example, one may choose the angular velocity components of the bodies in other frames. The choice made here was found to provide equations of lower complexity while retaining clear physical meaning for each pseudo-velocity.

## 3.2 Acceleration Energy

The total acceleration energy  $S$  of the unicycle can be obtained by combining the acceleration energies of the four rigid bodies:

$$S = S_{\text{wheel}} + S_{\text{body}} + S_{\text{balance}} + S_{\text{steer}}, \tag{20}$$

where  $S_{\text{wheel}}$ ,  $S_{\text{body}}$ ,  $S_{\text{balance}}$  and  $S_{\text{steer}}$  are the acceleration energies of the rolling wheel, the body, the balancing flywheel, and the steering flywheel, respectively. The following subsections will present the calculations of each acceleration energy component.

### 3.2.1 Rolling Wheel

It is convenient to calculate the acceleration energy of the rolling wheel in frame  $F_2$ . The mass moment of inertia tensor at the center of mass  $G$  is

$$\mathbf{J}_w = \frac{m_w R^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{F_2}. \tag{21}$$

Using the definitions (17) of the pseudo-velocities, the angular velocity (7) of the rolling wheel can be represented as

$$\boldsymbol{\omega}_w = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}_{F_2}. \quad (22)$$

Similarly, the angular velocity (4) of frame  $F_2$  can be written as

$$\boldsymbol{\Omega}_2 = \begin{bmatrix} \sigma_1 \\ \sigma_3 \tan \theta \\ \sigma_3 \end{bmatrix}_{F_2}. \quad (23)$$

Finally, using (5),(8),(9),(22) the velocity of the center of mass G of the rolling wheel can be formulated as

$$\mathbf{v}_G = \begin{bmatrix} \sigma_2 R \\ -\sigma_1 R \\ 0 \end{bmatrix}_{F_2}. \quad (24)$$

The time derivatives of the velocity (24) and the angular velocity (22) can be calculated via the frame derivatives. Hence, the acceleration of the center of mass G and the angular acceleration of the rolling wheel can be derived:

$$\mathbf{a}_G = \dot{\mathbf{v}}_G = \begin{bmatrix} R(\dot{\sigma}_2 + \sigma_1 \sigma_3) \\ -R(\dot{\sigma}_1 - \sigma_2 \sigma_3) \\ -R(\sigma_1^2 + \sigma_2 \sigma_3 \tan \theta) \end{bmatrix}_{F_2}, \quad \boldsymbol{\alpha}_w = \dot{\boldsymbol{\omega}}_w = \begin{bmatrix} \dot{\sigma}_1 - \sigma_2 \sigma_3 + \sigma_3^2 \tan \theta \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 + \sigma_1 \sigma_2 - \sigma_1 \sigma_3 \tan \theta \end{bmatrix}_{F_2}, \quad (25)$$

where the dots above the vectors represent the time derivative in the ground-fixed frame  $F_0$ .

Then, the acceleration energy of the rolling wheel can be constructed as

$$\begin{aligned} S_{\text{wheel}} &= \frac{1}{2} m_w \mathbf{a}_G^2 + \frac{1}{2} \boldsymbol{\alpha}_w \cdot \mathbf{J}_w \boldsymbol{\alpha}_w + \boldsymbol{\alpha}_w \cdot (\boldsymbol{\omega}_w \times \mathbf{J}_w \boldsymbol{\omega}_w) + \dots \\ &= m_w R^2 \left( \frac{5}{8} \dot{\sigma}_1^2 + \frac{\sigma_3^2 \tan \theta - 6\sigma_2 \sigma_3}{4} \dot{\sigma}_1 + \frac{3}{4} \dot{\sigma}_2^2 + \sigma_1 \sigma_3 \dot{\sigma}_2 \right. \\ &\quad \left. + \frac{1}{8} \dot{\sigma}_3^2 + \frac{2\sigma_1 \sigma_2 - \sigma_1 \sigma_3 \tan \theta}{4} \dot{\sigma}_3 \right) + \dots, \quad (26) \end{aligned}$$

where  $\cdot$  between vectors denotes the dot product, while  $\dots$  refer to terms that do not depend on the pseudo-accelerations. These terms are not spelled out since they will not play any role when deriving the Appell equations.

### 3.2.2 Body

The mass moment of inertia tensor of the body at the center of mass B can be represented in the  $F_3$  frame as

$$\mathbf{J} = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}_{F_3}. \quad (27)$$

Using definitions of the pseudo-velocities (17), the angular velocity (15) of the body can be expressed in the  $F_3$  frame as

$$\boldsymbol{\omega} = \begin{bmatrix} \sigma_1 \cos \gamma - \sigma_3 \sin \gamma \\ \sigma_4 \\ \sigma_1 \sin \gamma + \sigma_3 \cos \gamma \end{bmatrix}_{F_3}. \quad (28)$$

Note that this also gives the angular velocity of the frame  $F_3$ , which is attached to the body. One may also transform the velocity (24) of the center of mass G of the wheel to the  $F_3$  frame using the matrix  $\mathbf{T}_{2,3}$  (cf. (1)–(3)):

$$\mathbf{v}_G = \begin{bmatrix} \sigma_2 R \cos \gamma \\ -\sigma_1 R \\ \sigma_2 R \sin \gamma \end{bmatrix}_{F_3}, \quad (29)$$

and we have the position vector

$$\mathbf{r}_{GB} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}_{F_3}. \quad (30)$$

Thus, the velocity of the center of mass B of the body can be calculated as

$$\mathbf{v}_B = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{GB} = \begin{bmatrix} \sigma_2 R \cos \gamma + \sigma_4 h \\ -\sigma_1 (R + h \cos \gamma) + \sigma_3 h \sin \gamma \\ \sigma_2 R \sin \gamma \end{bmatrix}_{F_3}. \quad (31)$$

The acceleration of the center of mass B and the angular acceleration can be obtained using frame derivatives:

$$\mathbf{a}_B = \dot{\mathbf{v}}_B = \begin{bmatrix} a_{Bx} \\ a_{By} \\ a_{Bz} \end{bmatrix}_{F_3}, \quad \boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}_{F_3}, \quad (32)$$

and the detailed expressions are given by (A1)–(A2) in Appendix A.

Thus, the acceleration energy of the body can be calculated as

$$\begin{aligned}
S_{\text{body}} &= \frac{1}{2} m \mathbf{a}_B^2 + \frac{1}{2} \boldsymbol{\alpha} \cdot \mathbf{J} \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot (\boldsymbol{\omega} \times \mathbf{J} \boldsymbol{\omega}) + \dots \\
&= \frac{1}{2} \left( mR^2 + (J_x + mh^2) \cos^2 \gamma + J_z \sin^2 \gamma + 2mRh \cos \gamma \right) \dot{\sigma}_1^2 \\
&\quad + \left( \sigma_3^2 \left( (J_x + mh^2) \cos^2 \gamma + J_z \sin^2 \gamma + mRh \cos \gamma \right) \tan \theta \right. \\
&\quad \quad + \sigma_1 \sigma_3 \left( (J_x + mh^2 - J_z) \cos \gamma + mRh \right) \tan \theta \sin \gamma \\
&\quad \quad - 2\sigma_1 \sigma_4 \left( (J_x + mh^2 - J_z) \cos \gamma + mRh \right) \sin \gamma \\
&\quad \quad - \sigma_2 \sigma_3 \left( mR^2 + mRh \cos \gamma \right) \\
&\quad \quad \left. + \sigma_3 \sigma_4 \left( J_x - J_y + J_z (\cos^2 \gamma - \sin^2 \gamma) - 2(J_x + mh^2) \cos^2 \gamma \right. \right. \\
&\quad \quad \quad \left. \left. - 2mRh \cos \gamma \right) \right) \dot{\sigma}_1 \\
&\quad + \frac{1}{2} mR^2 \dot{\sigma}_2^2 + \left( -(\sigma_3^2 + \sigma_4^2) mRh \sin \gamma + \sigma_1 \sigma_3 (mR^2 + mRh \cos \gamma) \right) \dot{\sigma}_2 \\
&\quad + \frac{1}{2} \left( (J_x + mh^2) \sin^2 \gamma + J_z \cos^2 \gamma \right) \dot{\sigma}_3^2 \\
&\quad + \left( -\sigma_3^2 (J_x + mh^2 - J_z) \tan \theta \cos \gamma \sin \gamma \right. \\
&\quad \quad - \sigma_1 \sigma_3 \left( (J_x + mh^2) \sin^2 \gamma + J_z \cos^2 \gamma \right) \tan \theta \\
&\quad \quad + \sigma_1 \sigma_4 \left( -J_x + J_y + J_z (\cos^2 \gamma - \sin^2 \gamma) + 2(J_x + mh^2) \sin^2 \gamma \right) \\
&\quad \quad \left. + \sigma_2 \sigma_3 mRh \sin \gamma + 2\sigma_3 \sigma_4 (J_x + mh^2 - J_z) \cos \gamma \sin \gamma \right) \dot{\sigma}_3 \\
&\quad + \frac{1}{2} (J_y + mh^2) \dot{\sigma}_4^2 \\
&\quad + \left( \sigma_1^2 \left( (J_x + mh^2 - J_z) \cos \gamma + mRh \right) \sin \gamma \right. \\
&\quad \quad - \sigma_3^2 (J_x + mh^2 - J_z) \cos \gamma \sin \gamma \\
&\quad \quad + \sigma_1 \sigma_3 \left( (J_x + mh^2 - J_z) (\cos^2 \gamma - \sin^2 \gamma) + mRh \cos \gamma \right) \\
&\quad \quad \left. + \sigma_2 \sigma_3 mRh \tan \theta \sin \gamma \right) \dot{\sigma}_4 \\
&\quad - \left( (J_x + mh^2 - J_z) \cos \gamma + mRh \right) \sin \gamma \dot{\sigma}_1 \dot{\sigma}_3 + mRh \cos \gamma \dot{\sigma}_2 \dot{\sigma}_4 + \dots .
\end{aligned} \tag{33}$$

### 3.2.3 Flywheels

The mass moment of inertia tensors at the centers of masses of the flywheels are

$$\mathbf{J}_b = \frac{m_b r_b^2}{4} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{F_3}, \quad \mathbf{J}_s = \frac{m_s r_s^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{F_3}. \quad (34)$$

Since the centers of masses of both flywheels coincide with that of the body, their accelerations are equal to  $\mathbf{a}_B$  in (32). The angular velocities of flywheels can be obtained by substituting the definitions of the pseudo-velocities (17) into (16):

$$\boldsymbol{\omega}_b = \begin{bmatrix} \sigma_5 \\ \sigma_4 \\ \sigma_1 \sin \gamma + \sigma_3 \cos \gamma \end{bmatrix}_{F_3}, \quad \boldsymbol{\omega}_s = \begin{bmatrix} \sigma_1 \cos \gamma - \sigma_3 \sin \gamma \\ \sigma_4 \\ \sigma_6 \end{bmatrix}_{F_3}. \quad (35)$$

Similarly to the previous sections, the angular accelerations of the flywheels can be calculated using frame derivatives:

$$\boldsymbol{\alpha}_b = \begin{bmatrix} \alpha_{bx} \\ \alpha_{by} \\ \alpha_{bz} \end{bmatrix}_{F_3}, \quad \boldsymbol{\alpha}_s = \begin{bmatrix} \alpha_{sx} \\ \alpha_{sy} \\ \alpha_{sz} \end{bmatrix}_{F_3}, \quad (36)$$

and the detailed expressions are given by (A3)-(A4) in Appendix A. Hence, the acceleration energies of the flywheels are:

$$\begin{aligned} S_{\text{balance}} &= \frac{1}{2} m_b \mathbf{a}_B^2 + \frac{1}{2} \boldsymbol{\alpha}_b \cdot \mathbf{J}_b \boldsymbol{\alpha}_b + \boldsymbol{\alpha}_b \cdot (\boldsymbol{\omega}_b \times \mathbf{J}_b \boldsymbol{\omega}_b) + \dots \\ &= \frac{1}{2} m_b \left( (R + h \cos \gamma)^2 + \frac{r_b^2}{4} \sin^2 \gamma \right) \dot{\sigma}_1^2 \\ &\quad + m_b \left( \sigma_3^2 \left( h^2 \cos^2 \gamma + \frac{r_b^2}{4} \sin^2 \gamma + Rh \cos \gamma \right) \tan \theta \right. \\ &\quad \quad + \sigma_1 \sigma_3 \left( \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma + Rh \right) \tan \theta \sin \gamma \\ &\quad \quad - 2 \sigma_1 \sigma_4 \left( \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma + Rh \right) \sin \gamma \\ &\quad \quad - \sigma_2 \sigma_3 (R^2 + Rh \cos \gamma) \\ &\quad \quad \left. - 2 \sigma_3 \sigma_4 \left( h^2 \cos^2 \gamma + \frac{r_b^2}{4} \sin^2 \gamma + Rh \cos \gamma \right) \right. \\ &\quad \quad \left. - \frac{1}{2} \sigma_4 \sigma_5 r_b^2 \sin \gamma \right) \dot{\sigma}_1 \\ &\quad + \frac{1}{2} m_b R^2 \dot{\sigma}_2^2 - m_b \left( (\sigma_3^2 + \sigma_4^2) Rh \sin \gamma - \sigma_1 \sigma_3 (R^2 + Rh \cos \gamma) \right) \dot{\sigma}_2 \\ &\quad + \frac{1}{2} m_b \left( h^2 \sin^2 \gamma + \frac{r_b^2}{4} \cos^2 \gamma \right) \dot{\sigma}_3^2 \end{aligned}$$

$$\begin{aligned}
& + m_b \left( -\sigma_3^2 \left( h^2 - \frac{r_b^2}{4} \right) \tan \theta \cos \gamma \sin \gamma \right. \\
& \quad - \sigma_1 \sigma_3 \left( h^2 \sin^2 \gamma + \frac{r_b^2}{4} \cos^2 \gamma \right) \tan \theta \\
& \quad + 2\sigma_1 \sigma_4 \left( h^2 \sin^2 \gamma + \frac{r_b^2}{4} \cos^2 \gamma \right) + \sigma_2 \sigma_3 R h \sin \gamma \\
& \quad \left. + 2\sigma_3 \sigma_4 \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma \sin \gamma - \frac{1}{2} \sigma_4 \sigma_5 r_b^2 \cos \gamma \right) \dot{\sigma}_3 \\
& + \frac{1}{2} m_b \left( h^2 + \frac{r_b^2}{4} \right) \dot{\sigma}_4^2 \\
& + m_b \left( \sigma_1^2 \left( \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma + R h \right) \sin \gamma \right. \\
& \quad - \sigma_3^2 \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma \sin \gamma \\
& \quad + \sigma_1 \sigma_3 \left( \left( h^2 - \frac{r_b^2}{4} \right) (\cos^2 \gamma - \sin^2 \gamma) + R h \cos \gamma \right) \\
& \quad \left. + \frac{1}{2} \sigma_1 \sigma_5 r_b^2 \sin \gamma + \sigma_2 \sigma_3 R h \tan \theta \sin \gamma + \frac{1}{2} \sigma_3 \sigma_5 r_b^2 \cos \gamma \right) \dot{\sigma}_4 \\
& + \frac{1}{4} m_b r_b^2 \dot{\sigma}_5^2 \\
& - m_b \left( \left( h^2 - \frac{r_b^2}{4} \right) \cos \gamma + R h \right) \sin \gamma \dot{\sigma}_1 \dot{\sigma}_3 + m_b R h \cos \gamma \dot{\sigma}_2 \dot{\sigma}_4 + \dots .
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
S_{\text{steer}} &= \frac{1}{2} m_s \mathbf{a}_B^2 + \frac{1}{2} \boldsymbol{\alpha}_s \cdot \mathbf{J}_s \boldsymbol{\alpha}_s + \boldsymbol{\alpha}_s \cdot (\boldsymbol{\omega}_s \times \mathbf{J}_s \boldsymbol{\omega}_s) + \dots \\
&= \frac{1}{2} m_s \left( (R + h \cos \gamma)^2 + \frac{r_s^2}{4} \cos^2 \gamma \right) \dot{\sigma}_1^2 \\
& \quad + m_s \left( \sigma_3^2 \left( h^2 \cos^2 \gamma + \frac{r_s^2}{4} \cos^2 \gamma + R h \cos \gamma \right) \tan \theta \right. \\
& \quad + \sigma_1 \sigma_3 \left( \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma + R h \right) \tan \theta \sin \gamma \\
& \quad - 2\sigma_1 \sigma_4 \left( \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma + R h \right) \sin \gamma \\
& \quad - \sigma_2 \sigma_3 (R^2 + R h \cos \gamma) \\
& \quad - 2\sigma_3 \sigma_4 \left( h^2 \cos^2 \gamma + \frac{r_s^2}{4} \cos^2 \gamma + R h \cos \gamma \right) \\
& \quad \left. + \frac{1}{2} \sigma_4 \sigma_6 r_s^2 \cos \gamma \right) \dot{\sigma}_1
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} m_s R^2 \dot{\sigma}_2^2 - m_s \left( (\sigma_3^2 + \sigma_4^2) Rh \sin \gamma - \sigma_1 \sigma_3 (R^2 + Rh \cos \gamma) \right) \dot{\sigma}_2 \\
& + \frac{1}{2} m_s \left( h^2 + \frac{r_s^2}{4} \right) \sin^2 \gamma \dot{\sigma}_3^2 \\
& + m_s \left( -\sigma_3^2 \left( h^2 + \frac{r_s^2}{4} \right) \tan \theta \cos \gamma \sin \gamma \right. \\
& \quad \left. - \sigma_1 \sigma_3 \left( \left( h^2 + \frac{r_s^2}{4} \right) \sin^2 \gamma \right) \tan \theta \right. \\
& \quad \left. + 2\sigma_1 \sigma_4 \left( \left( h^2 + \frac{r_s^2}{4} \right) \sin^2 \gamma \right) + \sigma_2 \sigma_3 Rh \sin \gamma \right. \\
& \quad \left. + 2\sigma_3 \sigma_4 \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma \sin \gamma - \frac{1}{2} \sigma_4 \sigma_6 r_s^2 \sin \gamma \right) \dot{\sigma}_3 \\
& + \frac{1}{2} m_s \left( h^2 + \frac{r_s^2}{4} \right) \dot{\sigma}_4^2 \\
& + m_s \left( \sigma_1^2 \left( \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma + Rh \right) \sin \gamma \right. \\
& \quad \left. - \sigma_3^2 \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma \sin \gamma \right. \\
& \quad \left. + \sigma_1 \sigma_3 \left( \left( h^2 + \frac{r_s^2}{4} \right) (\cos^2 \gamma - \sin^2 \gamma) + Rh \cos \gamma \right) \right. \\
& \quad \left. - \frac{1}{2} \sigma_1 \sigma_6 r_s^2 \cos \gamma + \sigma_2 \sigma_3 Rh \tan \theta \sin \gamma + \frac{1}{2} \sigma_3 \sigma_6 r_s^2 \sin \gamma \right) \dot{\sigma}_4 \\
& + \frac{1}{4} m_s r_s^2 \dot{\sigma}_6^2 \\
& - m_s \left( \left( h^2 + \frac{r_s^2}{4} \right) \cos \gamma + Rh \right) \sin \gamma \dot{\sigma}_1 \dot{\sigma}_3 + m_s Rh \cos \gamma \dot{\sigma}_2 \dot{\sigma}_4 + \dots .
\end{aligned} \tag{38}$$

### 3.3 Virtual Power and pseudo-forces

To determine the pseudo-forces, which give the right-hand side of the Appell equations, the virtual power of the active forces is required.

For the rolling wheel, the gravitational force points to the negative  $z_0$  direction in the  $F_0$  frame. To transform this to the  $F_2$  frame one may use the rotation matrix  $\mathbf{T}_{0,2}$  (cf. (1)–(3)):

$$\mathbf{G}_w = \begin{bmatrix} 0 \\ -m_w g \sin \theta \\ -m_w g \cos \theta \end{bmatrix}_{F_2} . \tag{39}$$

The driving torque applied at the rolling wheel can be written as

$$\mathbf{M}_w = \begin{bmatrix} 0 \\ M_w \\ 0 \end{bmatrix}_{F_2}. \quad (40)$$

These, together with  $\boldsymbol{\omega}_w$  in (22) and  $\mathbf{v}_G$  in (24), allow the calculation of the virtual power

$$\begin{aligned} \delta P_{\text{wheel}} &= \mathbf{G}_w \cdot \delta \mathbf{v}_G + \mathbf{M}_w \cdot \delta \boldsymbol{\omega}_w \\ &= m_w g R \sin \theta \delta \sigma_1 + M_w \delta \sigma_2, \end{aligned} \quad (41)$$

where  $\delta$  refers to virtual quantities.

Virtual power of the forces acting on the lateral balancing and steering flywheels can be derived in a similar manner. The gravitational forces acting on the flywheels also point to the negative  $z_0$  direction in the  $F_0$  frame and they can be transformed to the  $F_3$  frame using the rotation matrix  $\mathbf{T}_{0,3}$  (cf. (1)–(3)):

$$\mathbf{G}_b = \begin{bmatrix} m_b g \cos \theta \sin \gamma \\ -m_b g \sin \theta \\ -m_b g \cos \theta \cos \gamma \end{bmatrix}_{F_3}, \quad \mathbf{G}_s = \begin{bmatrix} m_s g \cos \theta \sin \gamma \\ -m_s g \sin \theta \\ -m_s g \cos \theta \cos \gamma \end{bmatrix}_{F_3}, \quad (42)$$

while the balancing and steering torques are given by

$$\mathbf{M}_b = \begin{bmatrix} M_b \\ 0 \\ 0 \end{bmatrix}_{F_3}, \quad \mathbf{M}_s = \begin{bmatrix} 0 \\ 0 \\ M_s \end{bmatrix}_{F_3}, \quad (43)$$

respectively.

Using  $\boldsymbol{\omega}_b$  and  $\boldsymbol{\omega}_s$  in (35), as well as  $\mathbf{v}_B$  in (31), the virtual powers of the forces acting on the flywheels can be written as

$$\begin{aligned} \delta P_{\text{balance}} &= \mathbf{G}_b \cdot \delta \mathbf{v}_B + \mathbf{M}_b \cdot \delta \boldsymbol{\omega}_b \\ &= (m_b g R \sin \theta + m_b g h \sin \theta \cos \gamma) \delta \sigma_1 - m_b g h \sin \theta \sin \gamma \delta \sigma_3 \\ &\quad + m_b g h \cos \theta \sin \gamma \delta \sigma_4 + M_b \delta \sigma_5 \end{aligned} \quad (44)$$

and

$$\begin{aligned} \delta P_{\text{steer}} &= \mathbf{G}_s \cdot \delta \mathbf{v}_B + \mathbf{M}_s \cdot \delta \boldsymbol{\omega}_s \\ &= (m_s g R \sin \theta + m_s g h \sin \theta \cos \gamma) \delta \sigma_1 - m_s g h \sin \theta \sin \gamma \delta \sigma_3 \\ &\quad + m_s g h \cos \theta \sin \gamma \delta \sigma_4 + M_s \delta \sigma_6. \end{aligned} \quad (45)$$

Finally, the virtual power of the forces that act on the body has to be derived. The gravitational force acting on the body can be transformed from



the  $F_0$  frame to the  $F_3$  frame using the rotation matrix  $\mathbf{T}_{0,3}$  (cf. (1)–(3)):

$$\mathbf{G} = \begin{bmatrix} mg \cos \theta \sin \gamma \\ -mg \sin \theta \\ -mg \cos \theta \cos \gamma \end{bmatrix}_{F_3}. \quad (46)$$

Note that the driving torque  $M_w$ , the balancing torque  $M_b$ , and the steering torque  $M_s$  are all internal torques. Namely, they induce reaction torques on the body with the same magnitudes but in opposite directions. Thus, the virtual power related to active forces acting on the body can be constructed as

$$\begin{aligned} \delta P_{\text{body}} &= \mathbf{G} \cdot \delta \mathbf{v}_B - (\mathbf{M}_w + \mathbf{M}_b + \mathbf{M}_s) \cdot \delta \boldsymbol{\omega} \\ &= (mgR \sin \theta + mgh \sin \theta \cos \gamma - M_b \cos \gamma - M_s \sin \gamma) \delta \sigma_1 \\ &\quad + (-mgh \sin \theta \sin \gamma + M_b \sin \gamma - M_s \cos \gamma) \delta \sigma_3 \\ &\quad + (mgh \cos \theta \sin \gamma - M_w) \delta \sigma_4. \end{aligned} \quad (47)$$

Then the total virtual power of the active forces is given by the sum of (41), (44), (45) and (47):

$$\delta P = \delta P_{\text{wheel}} + \delta P_{\text{body}} + \delta P_{\text{balance}} + \delta P_{\text{steer}}, \quad (48)$$

and hence, the pseudo-forces  $\Pi_i$  can be extracted from the coefficients of the virtual pseudo-velocities  $\delta \sigma_i$  as

$$\begin{aligned} \Pi_1 &= (m_w + m + m_b + m_s)gR \sin \theta + (m + m_b + m_s)gh \sin \theta \cos \gamma \\ &\quad - M_b \cos \gamma - M_s \sin \gamma, \\ \Pi_2 &= M_w, \\ \Pi_3 &= -(m + m_b + m_s)gh \sin \theta \sin \gamma + M_b \sin \gamma - M_s \cos \gamma, \\ \Pi_4 &= (m + m_b + m_s)gh \cos \theta \sin \gamma - M_w, \\ \Pi_5 &= M_b, \\ \Pi_6 &= M_s. \end{aligned} \quad (49)$$

### 3.4 Equations of Motion

Based on the previous sub-sections, the Appell equations

$$\frac{\partial S}{\partial \dot{\sigma}_i} = \Pi_i, \quad i = 1, \dots, 6, \quad (50)$$

can be derived. These first-order differential equations depend on the pseudo-accelerations  $\dot{\sigma}_i$  linearly, and thus, they can be arranged them into the form:

$$\begin{aligned}\dot{\sigma}_1 &= f_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \theta, \gamma, M_b, M_s), \\ \dot{\sigma}_2 &= f_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \theta, \gamma, M_w), \\ \dot{\sigma}_3 &= f_3(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \theta, \gamma, M_b, M_s), \\ \dot{\sigma}_4 &= f_4(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \theta, \gamma, M_w) \\ \dot{\sigma}_5 &= f_5(M_b) \\ \dot{\sigma}_6 &= f_6(M_s)\end{aligned}\tag{51}$$

where  $f_i, i = 1, \dots, 4$  can be displayed by running the MATLAB code given in Appendix E, while  $f_5$  and  $f_6$  depend only on the control torques and they read

$$f_5(M_b) = \frac{2M_b}{m_b r_b^2}, \quad f_6(M_s) = \frac{2M_s}{m_s r_s^2}.\tag{52}$$

In summary, the governing equations of the system can be obtained by combining the Appell equations (51) and the kinematic equations (19). This leads to fourteen first order differential equations, that is, the system evolves in a fourteen-dimensional state space  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \theta, \gamma, \psi, x_G, y_G, \phi, \alpha, \beta$ , where again the variables are ordered considering the linearization performed further below. The motion is controlled by the input torques  $M_w, M_b, M_s$ . The goal of control design is to create feedback laws which determine the torques as a function of the states so that the control objectives are achieved.

## 4 Control Design

The model (51),(19) constructed in the previous section can be formulated as a control affine system

$$\dot{x} = f(x) + g(x)u,\tag{53}$$

where the state vector  $x \in \mathbb{R}^{14}$  and the input vector  $u \in \mathbb{R}^3$  are defined as

$$x := [\sigma_1 \ \sigma_2 \ \sigma_3 \ \sigma_4 \ \sigma_5 \ \sigma_6 \ \theta \ \gamma \ \psi \ x_G \ y_G \ \phi \ \alpha \ \beta]^\top,\tag{54}$$

and

$$u := [M_w \ M_b \ M_s]^\top,\tag{55}$$

respectively, and we have  $f : \mathbb{R}^{14} \rightarrow \mathbb{R}^{14}$  and  $g : \mathbb{R}^{14} \rightarrow \mathbb{R}^3$ .

One may show that for zero input

$$u^* := [0 \ 0 \ 0]^\top,\tag{56}$$

the system (53) possesses the equilibrium

$$x^* := \left[0 \ \frac{v^*}{R} \ 0 \ 0 \ \omega_b^* \ \omega_s^* \ 0 \ 0 \ 0 \ 0 \ v^* t \ 0 \ \frac{v^*}{R} t \ \omega_b^* t \ \omega_s^* t\right]^\top,\tag{57}$$

which corresponds to the unicycle running along the  $x_0$  axis with constant speed  $v^*$  while the body is standing in an upright position. In this case, the constant rotational speed of the rolling wheel is  $\frac{v^*}{R}$  while the constant rotational speeds  $\omega_b^*$  and  $\omega_s^*$  of the flywheels are considered to be nonzero for generality (cf. (17)). The control objective is set to maintain this motion.

The control objective will be achieved by establishing feedback laws of the form

$$u = k(x), \quad (58)$$

where  $k : \mathbb{R}^{14} \rightarrow \mathbb{R}^3$ . This assigns the torques in (55) as functions of the state variables in (54). Substituting (58) into (53) we obtain the closed-loop system

$$\dot{x} = f(x) + g(x)k(x), \quad (59)$$

where the right-hand side only depends on the state.

Below we first analyze the stability of the equilibrium (57) without control and explain the self-stabilizing effect in the tilt motion. This is followed by designing a controller which can stabilize the yaw, tilt, and pitch motions simultaneously.

## 4.1 Open-loop Dynamics

Here we study the stability of the equilibrium by linearizing the open-loop dynamics and investigate the effects of the equilibrium speed on stability.

The linearization of the open-loop system (53) about the equilibrium (57) can be written as

$$\dot{\tilde{x}} = A \tilde{x} + B u, \quad (60)$$

where

$$A = \begin{bmatrix} 0 & 0 & A_{13} & A_{14} & 0 & 0 & A_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & 0 & A_{23} & 0 & 0 & 0 & 0 & A_{28} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31} & 0 & 0 & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{41} & 0 & A_{43} & 0 & 0 & 0 & 0 & A_{48} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & B_{12} & 0 \\ B_{21} & 0 & 0 \\ 0 & 0 & B_{33} \\ B_{41} & 0 & 0 \\ 0 & \frac{2}{m_b r_b^2} & 0 \\ 0 & 0 & \frac{2}{m_s r_s^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (61)$$

and  $A_{13}, A_{14}, A_{17}, A_{21}, A_{23}, A_{28}, A_{31}, A_{34}, A_{41}, A_{43}, A_{48}, B_{12}, B_{21}, B_{33}, B_{41}$  are given in Appendix B. In order to evaluate stability we compute the eigenvalues of the matrix  $A$ .

First, we discuss the case when  $\omega_b^* = \omega_s^* = 0$ , that is, the case when flywheels are not spinning in equilibrium (cf. (57)). Then we have  $A_{14} = A_{21} = A_{23} = A_{34} = A_{41} = A_{43} = 0$  (see Appendix B) and the eigenvalues are given by the characteristic equation

$$\det(\lambda I - A) = (\lambda^4 - (A_{13}A_{31} + A_{17} + A_{48})\lambda^2 + (A_{13}A_{31} + A_{17})A_{48})\lambda^{10} = 0. \quad (62)$$

That is, the matrix  $A$  has four nonzero eigenvalues and ten zero eigenvalues. The nonzero eigenvalues can be calculated analytically as

$$\lambda_{1,2} = \pm\sqrt{A_{48}}, \quad \lambda_{3,4} = \pm\sqrt{A_{13}A_{31} + A_{17}}, \quad (63)$$

while the corresponding eigenvectors can be found in Appendix C.

Since  $A_{48} > 0$ , the pair eigenvalues  $\lambda_{1,2}$  are real. As one of these eigenvalues is located in the right half complex plane, the equilibrium is unstable. The elements of the eigenvectors  $v_{1,2}$  in (C10) indicate that the eigenvalue pair  $\lambda_{1,2}$  is associated with the pitch dynamics of the unicycle body, and that the instability corresponds to the body falling in this direction. Moreover, since  $A_{13} > 0$ ,  $A_{31} < 0$  and  $A_{17} > 0$ , the pair of eigenvalues  $\lambda_{3,4}$  are either real or form a purely imaginary complex conjugate pair. The non-zero elements of the eigenvectors  $v_{3,4}$  in (C10) indicate that the eigenvalue pair  $\lambda_{3,4}$  is associated with the tilt dynamics of the unicycle.

For additional insights, the transfer functions from the input torques to the pitch, tilt, and yaw angles are calculated and included in Appendix D. It can be observed that the nonzero transfer function corresponding to the pitch angle output has its poles at  $\lambda_{1,2}$ . Moreover, the nonzero transfer functions corresponding to the tilt and yaw angle outputs have poles at  $\lambda_{3,4}$ . We also remark that among all nonzero transfer functions, only the one from steering torque to the yaw angle has non-minimum phase zeros. We will utilize these observations when designing controllers in the next section.

Using the formulas in Appendix B, one may observe that the eigenvalue pair  $\lambda_{1,2}$  is independent of the speed  $v^*$ , while the speed enters the eigenvalues pair  $\lambda_{3,4}$  via  $A_{13}A_{31} \propto -(v^*)^2$ . That is, for small speed the tilt and yaw directions are unstable, but once the velocity reaches a critical limit, i.e.,  $v^* > v_{\text{cr}}^*$ , they become (neutrally) stable. Solving the equation  $A_{17} + A_{13}A_{31} = 0$ , after some algebraic manipulation, we obtain the critical speed

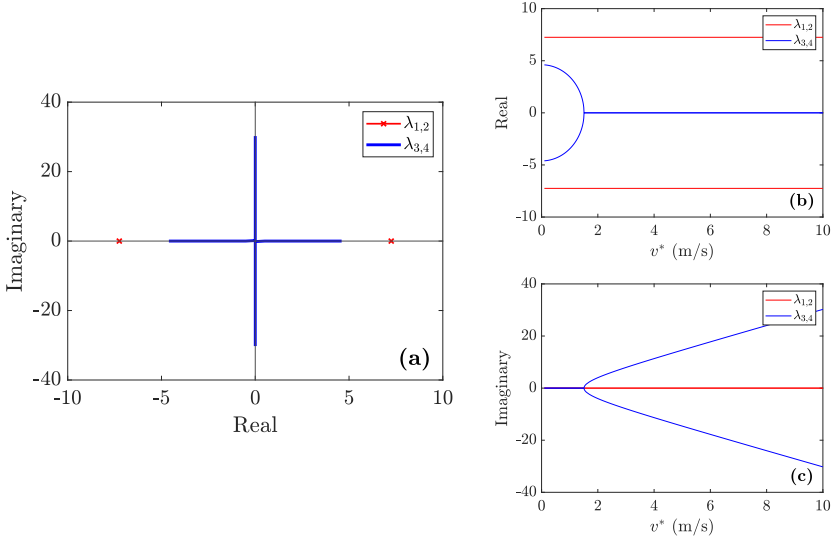
$$v_{\text{cr}}^* = \sqrt{\frac{(m_w R^2 + m_b r_b^2 + 4J_z)(m_w R + \hat{m}(R + h))g}{(3m_w R + 2\hat{m}(R + h))m_w R}}, \quad (64)$$

where

$$\hat{m} = m + m_b + m_s. \quad (65)$$

Substituting the values in Table 2 and we obtain  $v_{\text{cr}}^* \approx 1.51$  [m/s].

The root locus plot in Fig. 3(a) depicts how the nonzero eigenvalues (63) move in the complex plane as equilibrium speed  $v^*$  is varied between 0 and

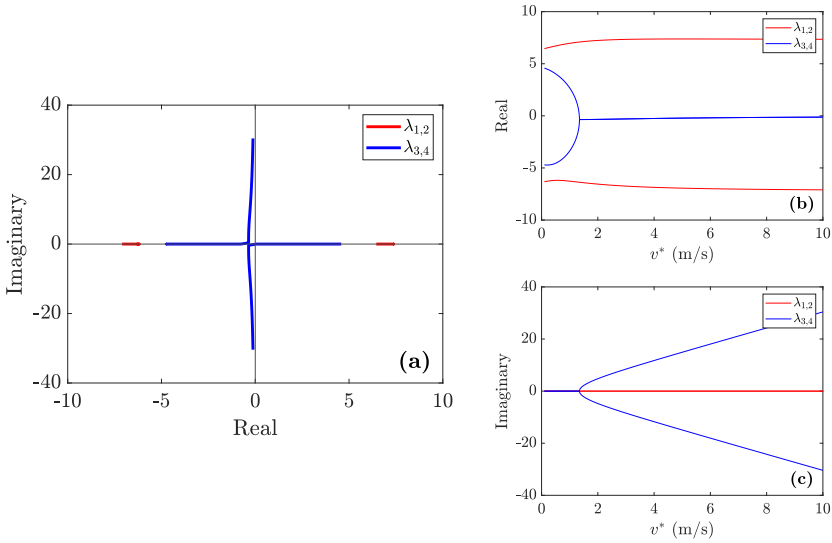


**Fig. 3:** (a) Root locus of nonzero eigenvalues of the linearized open-loop system when varying the speed of the rolling wheel. (b,c) Real and imaginary parts of the nonzero eigenvalues as a function of speed. The equilibrium flywheel spinning rates are  $\omega_b^* = \omega_s^* = 0$  [rad/s].

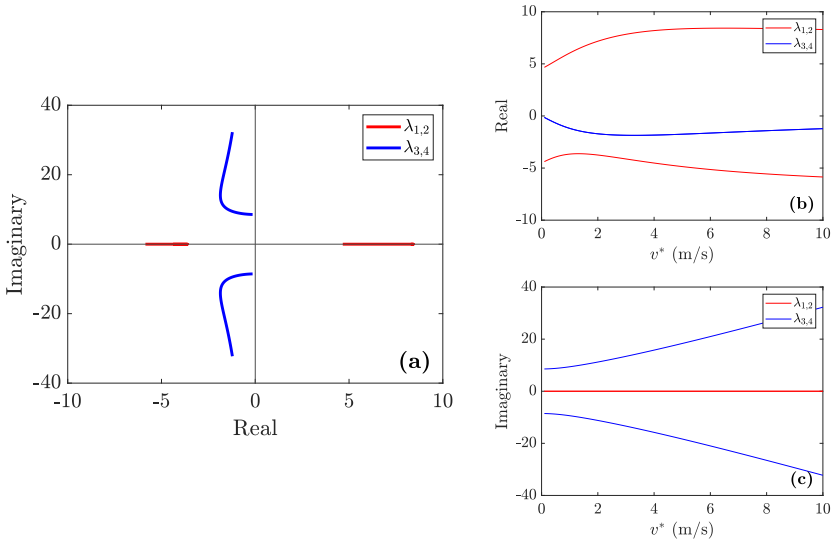
10 [m/s]. The eigenvalue pair  $\lambda_{1,2}$  stays at the same location independent of the speed. The eigenvalues of the pair  $\lambda_{3,4}$  move toward each other along the real axis as  $v^*$  is increased, and after they “meet” at the origin, they become a complex conjugate pair and move away from each other along the imaginary axis. The real and imaginary parts of the eigenvalues are plotted in Fig. 3(b) and (c), respectively. Note that the tilt is stabilized by increasing the speed despite there no feedback control being applied to the system at this point. Also note that the vibration frequency increases from zero up to cc. 4.8 [Hz] as the speed is increased from  $v_{cr}^* \approx 1.5$  [m/s] up to 10 [m/s].

**Table 2:** Parameter Values

Parameter	Value
$m_w$	2 kg
$R$	0.15 m
$m$	3 kg
$h$	0.3 m
$m_b$	1 kg
$r_b$	0.15 m
$m_s$	1 kg
$r_s$	0.15 m
$J_x$	0.1 kg m <sup>2</sup>
$J_y$	0.1 kg m <sup>2</sup>
$J_z$	0.02 kg m <sup>2</sup>



**Fig. 4:** (a) Root locus of nonzero eigenvalues of the linearized open-loop system when varying the speed of the rolling wheel. (b,c) Real and imaginary parts of the nonzero eigenvalues as a function of speed. The equilibrium flywheel spinning rates are  $\omega_b^* = \omega_s^* = 30$  [rad/s].



**Fig. 5:** (a) Root locus of nonzero eigenvalues of the linearized open-loop system when varying the speed of the rolling wheel. (b,c) Real and imaginary parts of the nonzero eigenvalues as a function of speed. The equilibrium flywheel spinning rates are  $\omega_b^* = \omega_s^* = 100$  [rad/s].

Such a self-stabilization effect in the tilt dynamics can also be observed in the general case when  $\omega_b^* \neq 0$  and  $\omega_s^* \neq 0$ . Then the characteristic equation becomes

$$\begin{aligned} \det(\lambda I - A) = & (\lambda^4 - (A_{13}A_{31} + A_{14}A_{41} + A_{34}A_{43} + A_{17} + A_{48})\lambda^2 \\ & - (A_{13}A_{34}A_{41} + A_{14}A_{31}A_{43})\lambda \\ & + A_{13}A_{31}A_{48} + A_{17}A_{34}A_{43} + A_{17}A_{48})\lambda^{10} = 0, \end{aligned} \quad (66)$$

cf. (62). Notice that the elements  $A_{21}$ ,  $A_{23}$  and  $A_{28}$  do not appear in the characteristic equation. The matrix  $A$  still has four nonzero eigenvalues and ten zero eigenvalues. While the nonzero eigenvalues cannot be calculated analytically in this general case, the qualitative picture remains similar for moderate values of flywheel spinning rates, as illustrated in Fig. 4. The eigenvalue pair  $\lambda_{1,2}$  (corresponding to the pitch dynamics) remains real, while the eigenvalue pair  $\lambda_{3,4}$  (corresponding to the tilt dynamics) changes from real to complex conjugate (with negative real part) once the equilibrium speed exceeds the critical value  $v_{cr}^* \approx 1.34$  [m/s]. For higher values of flywheel spinning rates, the critical speed becomes zero, that is, the eigenvalue pair  $\lambda_{3,4}$  remains complex conjugate (with negative real part) in the whole range of equilibrium speed as illustrated in Fig. 5.

We remark that for nonzero flywheel spinning rates, the eigenvalues are not symmetric to the origin. Nevertheless, utilizing Liouville's formula [35] and that  $\text{tr}(A) = 0$ , one may show that the nonzero eigenvalues satisfy  $\sum_{j=1}^4 \lambda_j = 0$ .

## 4.2 Control Design

The stability investigation above revealed that the pitch dynamics is unstable without control while the tilt dynamics can be made stable (or at least neutrally stable) once the unicycle travels fast enough. From the transfer functions in Appendix D, it can be observed how the tilt, pitch, and yaw angles are affected by the three torque inputs. As a result, by constructing a feedback law one can make the pitch, the tilt, and the yaw motions asymptotically stable by moving the corresponding six eigenvalues to the left half complex plane. This allows one to achieve the control goal of making the unicycle travel along a straight line with constant speed (cf. (57)).

For simplicity, we use proportional-derivative (PD) type control to maintain zero pitch angle  $\gamma$ , tilt angle  $\theta$ , and yaw angle  $\psi$ :

$$\begin{aligned} M_w &= p_w \gamma + d_w \dot{\gamma}, \\ M_b &= p_b \theta + d_b \dot{\theta}, \\ M_s &= p_s \psi + d_s \dot{\psi}. \end{aligned} \quad (67)$$

When constructing this feedback law one may consider the practical feasibility of the controller. The states used in (67) can be obtained via the use of inertial measurement units (IMU). IMUs can directly measure accelerations and

angular velocity components, and by means of integration and appropriate filtering, one may obtain the yaw, tilt, and pitch angles. Alternatively, the yaw angle can be obtained directly with differential GPS or magnetic sensors.

Substituting  $\dot{\theta}, \dot{\gamma}, \dot{\psi}$  from (19), the control laws can be written as functions of the state:

$$\begin{bmatrix} M_w \\ M_b \\ M_s \end{bmatrix} = \begin{bmatrix} p_w \gamma + d_w (\sigma_4 - \sigma_3 \tan \theta) \\ p_b \theta + d_b \sigma_1 \\ p_s \psi + d_s \frac{\sigma_3}{\cos \theta} \end{bmatrix}, \quad (68)$$

cf. (54),(55).

The control law (68) is in the form (58) and substituting this to the system (51),(19), which fits the form (53), we obtain the closed-loop system as (59). Here we analyze the closed-loop dynamics by linearizing (51),(19),(68) about the equilibrium (57). This lead to

$$\dot{\tilde{x}} = \bar{A} \tilde{x}, \quad (69)$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & 0 & A_{13} & A_{14} & 0 & 0 & \bar{A}_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & 0 & A_{23} & \bar{A}_{24} & 0 & 0 & 0 & \bar{A}_{28} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31} & 0 & \bar{A}_{33} & A_{34} & 0 & 0 & 0 & 0 & \bar{A}_{39} & 0 & 0 & 0 & 0 & 0 \\ A_{41} & 0 & A_{43} & \bar{A}_{44} & 0 & 0 & 0 & \bar{A}_{48} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2d_b}{m_b r_b^2} & 0 & 0 & 0 & 0 & 0 & \frac{2p_b}{m_b r_b^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2d_s}{m_s r_s^2} & 0 & 0 & 0 & 0 & 0 & \frac{2p_s}{m_s r_s^2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (70)$$

Here the elements  $A_{13}, A_{14}, A_{21}, A_{23}, A_{31}, A_{34}, A_{41}, A_{43}$  do not change compared to (61), while the elements  $\bar{A}_{17}, \bar{A}_{28}, \bar{A}_{48}$  are altered by the gains  $p_w, d_w, p_b, d_b, p_s, d_s$ , and the new elements  $\bar{A}_{11}, \bar{A}_{24}, \bar{A}_{33}, \bar{A}_{39}, \bar{A}_{44}$  appear for nonzero gains; see Appendix B.

One may show that the characteristic equation takes the form

$$\det(\lambda I - \bar{A}) = p(\lambda) \lambda^8 = 0, \quad (71)$$

where  $p(\lambda)$  is a sixth order polynomial given in Appendix B. Notice that the elements  $A_{21}, A_{23}, \bar{A}_{24}$  and  $\bar{A}_{28}$  do not appear in the characteristic equation. The matrix  $\bar{A}$  has six nonzero eigenvalues (corresponding to the pitch, tilt, and yaw motions) and eight zero eigenvalues. The pole-placement technique



is used to obtain an appropriate set of control gains  $p_w, d_w, p_b, d_b, p_s, d_s$  such that the six nonzero eigenvalues can be placed to the open left-half complex plane to achieve stability. We tuned the gains to be

$$(p_w, d_w, p_b, d_b, p_s, d_s) = (15, 15, 30, 15, 15, 15), \quad (72)$$

which, together with the parameter choice in Table 2 and with the choice of  $\omega_b^* = \omega_s^* = 0$  [rad/s] and  $v^* = 5$  [m/s], yields the eigenvalues  $-0.35, -0.57, -0.93, -13.09, -119.97, -405.18$ .

We remark that, as shown by the remaining zero eigenvalues, the simple controller designed above cannot stabilize the whole dynamics. In fact, one may show that the system is not controllable at the linear level which demands a more sophisticated nonlinear control design. Nevertheless, the designed controller enables the unicycle to travel along a straight line in a stable manner as will be shown below by simulations.

### 4.3 Simulation Results

Here we use numerical simulations to illustrate the behavior of the nonlinear closed-loop system (51), (19), (68). We test how the system responds to perturbations of different sizes around the equilibrium (57). Simulations are executed using a 4-th order Runge-Kutta method with fixed time step, to ensure small simulation errors despite the lack of damping in the system. We observed that varying time step algorithms (that are typically built into commercial packages) failed to ensure the required accuracy.

Simulations are performed using three sets of initial conditions where the angles  $\theta$ ,  $\gamma$ , and  $\psi$  are perturbed at different levels. More precisely we use the initial condition

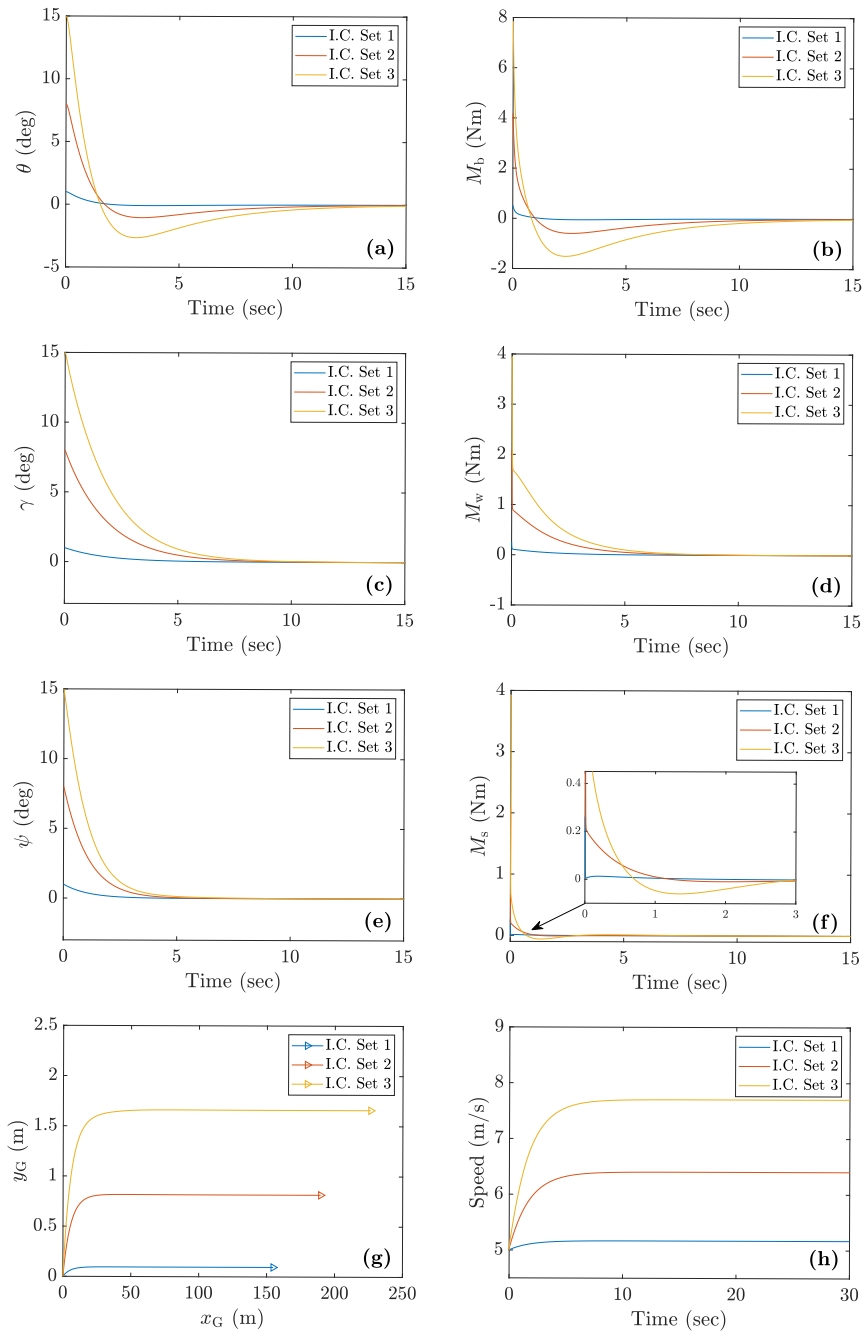
$$x(0) = [0 \quad \frac{v^*}{R} \quad 0 \quad 0 \quad 0 \quad 0 \quad \theta_0 \quad \gamma_0 \quad \psi_0 \quad 0 \quad 0 \quad 0 \quad 0]^T, \quad (73)$$

and the initial angles  $\theta_0$ ,  $\gamma_0$  and  $\psi_0$  are listed in Table 3. The parameter values are selected according to Table 2 while we use  $v^* = 5$  [m/s].

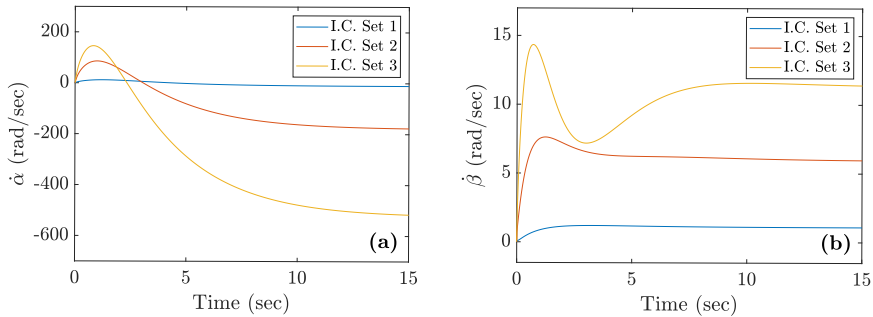
The simulation results are displayed in Fig. 6. It can be observed that when initial perturbations in  $\theta$ ,  $\gamma$ , and  $\psi$  are larger, the overshoot becomes larger and it takes longer time for the system to settle back to the equilibrium. The required torques  $M_b$ ,  $M_w$ , and  $M_s$  are also larger for larger perturbations. Nevertheless, the body is able to eventually return to the upright position and head toward the positive  $x_0$  direction, confirming the functionality of the controller. We remark that since speed tracking is not included in the control

**Table 3:** Initial Conditions

Set No.	1	2	3
$\theta_0$	1°	8°	15°
$\gamma_0$	1°	8°	15°
$\psi_0$	1°	8°	15°



**Fig. 6:** Simulation results. (a-f) Time histories of the body angles and input torques. (g-h) Trajectory and speed of the center of mass G of the rolling wheel.



**Fig. 7:** Time histories of rotational speeds of (a) lateral balancing flywheel and (b) steering flywheel.

objective, the speed  $\dot{\phi}R$  settles at levels different from the initial value. Similar behavior can be observed in the rotational speeds  $\dot{\alpha}$  and  $\dot{\beta}$  of the balancing and steering flywheels in Fig. 7.

## 5 Conclusion

In this article, the dynamics of an autonomous unmanned unicycle was studied. The equations of motion were derived by using the Appellian approach which enabled us to eliminate the kinematic constraints and describe the nonholonomic dynamics using the minimal number of state variables. The acceleration energy of the multibody system as a whole was constructed and a set of nonlinear differential equations was derived. The linear stability of the upright, straight running motion was investigated and it was shown that above a critical speed, the tilt motion is stabilized even without using feedback. Then a PD-type controller was designed which was able to stabilize the pitch, tilt, and yaw motions. It was shown by numerical simulations of the nonlinear systems that the controlled unicycle was able to balance itself and travel along a straight path while rejecting perturbations.

This work made the first step toward the creation of an autonomous unicycle. Still, there are many challenges to overcome before a working prototype can be built. Most importantly, in order to enable the unicycle to maneuver along an arbitrary path, the position variables need to be incorporated into the control design. On one hand, this will require one to analyze the nonlinear dynamics of the system. On the other hand, the control design also needs to be performed at the nonlinear level by applying more complex nonlinear controllers. The resulting controllers need to be evaluated on higher fidelity unicycle models which can be obtained by relaxing some of the modelling assumptions used in this work (like rolling without slipping) and/or incorporating suspension and/or tire dynamics.

## Declarations

**Ethical Approval.** Not applicable.

**Competing interests.** No, we declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

**Authors' contributions.** X. C. and D. C. B. carried out the analytical calculations as well as the numerical simulations. D. T. created some of the illustrations. D. T. and G. O. supervised the research and all authors participated in the writing of the manuscript.

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**Availability of data and materials.** The codes used to generate the data were included in the appendix.

## Appendix A Detailed Expressions of Accelerations

The components of the acceleration of the center of mass B and the angular acceleration of the body in (32) are

$$\begin{aligned}
 a_{Bx} &= \dot{\sigma}_2 R \cos \gamma + \dot{\sigma}_4 h + \sigma_1^2 (R \sin \gamma + h \cos \gamma \sin \gamma) - \sigma_3^2 h \cos \gamma \sin \gamma \\
 &\quad + \sigma_1 \sigma_3 (R \cos \gamma + h (\cos^2 \gamma - \sin^2 \gamma)) + \sigma_2 \sigma_3 R \tan \theta \sin \gamma, \\
 a_{By} &= -\dot{\sigma}_1 (R + h \cos \gamma) + \dot{\sigma}_3 h \sin \gamma - \sigma_3^2 h \tan \theta \cos \gamma \\
 &\quad - \sigma_1 \sigma_3 h \tan \theta \sin \gamma + \sigma_2 \sigma_3 R + 2\sigma_1 \sigma_4 h \sin \gamma + 2\sigma_3 \sigma_4 h \cos \gamma, \\
 a_{Bz} &= \dot{\sigma}_2 R \sin \gamma - \sigma_1^2 (R \cos \gamma + h \cos^2 \gamma) - \sigma_3^2 h \sin^2 \gamma - \sigma_4^2 h \\
 &\quad + \sigma_1 \sigma_3 (R \sin \gamma + 2h \cos \gamma \sin \gamma) - \sigma_2 \sigma_3 R \tan \theta \cos \gamma,
 \end{aligned} \tag{A1}$$

and

$$\begin{aligned}
 \alpha_x &= \dot{\sigma}_1 \cos \gamma - \dot{\sigma}_3 \sin \gamma + \sigma_3^2 \tan \theta \cos \gamma \\
 &\quad + \sigma_1 \sigma_3 \tan \theta \sin \gamma - \sigma_1 \sigma_4 \sin \gamma - \sigma_3 \sigma_4 \cos \gamma, \\
 \alpha_y &= \dot{\sigma}_4, \\
 \alpha_z &= \dot{\sigma}_1 \sin \gamma + \dot{\sigma}_3 \cos \gamma + \sigma_3^2 \tan \theta \sin \gamma \\
 &\quad - \sigma_1 \sigma_3 \tan \theta \cos \gamma + \sigma_1 \sigma_4 \cos \gamma - \sigma_3 \sigma_4 \sin \gamma.
 \end{aligned} \tag{A2}$$

The components of the angular accelerations of the flywheels in (36) are

$$\begin{aligned}
 \alpha_{bx} &= \dot{\sigma}_5, \\
 \alpha_{by} &= \dot{\sigma}_4 - \sigma_1^2 \cos \gamma \sin \gamma + \sigma_3^2 \cos \gamma \sin \gamma \\
 &\quad - \sigma_1 \sigma_3 (\cos^2 \gamma - \sin^2 \gamma) + \sigma_1 \sigma_5 \sin \gamma + \sigma_3 \sigma_5 \cos \gamma, \\
 \alpha_{bz} &= \dot{\sigma}_1 \sin \gamma + \dot{\sigma}_3 \cos \gamma + \sigma_3^2 \tan \theta \sin \gamma \\
 &\quad - \sigma_1 \sigma_3 \tan \theta \cos \gamma + 2\sigma_1 \sigma_4 \cos \gamma - 2\sigma_3 \sigma_4 \sin \gamma - \sigma_4 \sigma_5,
 \end{aligned} \tag{A3}$$

and

$$\begin{aligned}
 \alpha_{sx} &= \dot{\sigma}_1 \cos \gamma - \dot{\sigma}_3 \sin \gamma + \sigma_3^2 \tan \theta \cos \gamma \\
 &\quad + \sigma_1 \sigma_3 \tan \theta \sin \gamma - 2\sigma_1 \sigma_4 \sin \gamma - 2\sigma_3 \sigma_4 \cos \gamma + \sigma_4 \sigma_6, \\
 \alpha_{sy} &= \dot{\sigma}_4 + \sigma_1^2 \cos \gamma \sin \gamma - \sigma_3^2 \cos \gamma \sin \gamma \\
 &\quad + \sigma_1 \sigma_3 (\cos^2 \gamma - \sin^2 \gamma) - \sigma_1 \sigma_6 \cos \gamma + \sigma_3 \sigma_6 \sin \gamma, \\
 \alpha_{sz} &= \dot{\sigma}_6.
 \end{aligned} \tag{A4}$$

## Appendix B Elements of Matrices in the Linearized Systems

The components of matrix  $A$  in (61):

$$\begin{aligned}
 A_{13} &= \frac{2(3m_w R + 2\hat{m}(R+h))v^*}{Q_1}, & A_{14} &= -\frac{2m_s r_s^2 \omega_s^*}{Q_1}, \\
 A_{17} &= \frac{4(m_w R + \hat{m}(R+h))g}{Q_1}, \\
 A_{21} &= -\frac{4\hat{m}m_s h r_s^2 \omega_s^*}{RQ_2}, & A_{23} &= \frac{4\hat{m}m_b h r_b^2 \omega_b^*}{RQ_2}, & A_{28} &= -\frac{8\hat{m}^2 h^2 g}{RQ_2}, \\
 A_{31} &= -\frac{2m_w R v^*}{Q_3}, & A_{34} &= \frac{2m_b r_b^2 \omega_b^*}{Q_3}, \\
 A_{41} &= \frac{2(3m_w + 2\hat{m})m_s r_s^2 \omega_s^*}{Q_2}, & A_{43} &= -\frac{2(3m_w + 2\hat{m})m_b r_b^2 \omega_b^*}{Q_2} \\
 A_{48} &= \frac{4(3m_w + 2\hat{m})\hat{m}hg}{Q_2},
 \end{aligned} \tag{B5}$$

the components of matrix  $B$  in (61):

$$\begin{aligned}
 B_{12} &= -\frac{4}{Q_1}, & B_{21} &= \frac{2(4\hat{m}(R+h)h + m_b r_b^2 + m_s r_s^2 + 4J_y)}{R^2 Q_2}, \\
 B_{33} &= -\frac{4}{Q_3}, & B_{41} &= -\frac{4(3m_w R + 2\hat{m}(R+h))}{RQ_2},
 \end{aligned} \tag{B6}$$

the components of matrix  $\bar{A}$  in (70):

$$\begin{aligned}
\bar{A}_{11} &= -\frac{4d_b}{Q_1}, & \bar{A}_{17} &= \frac{4(m_w R + \hat{m}(R+h))g - 4p_b}{Q_1}, \\
\bar{A}_{24} &= \frac{2(4\hat{m}(R+h)h + m_b r_b^2 + m_s r_s^2 + 4J_y)d_w}{R^2 Q_2}, \\
\bar{A}_{28} &= -\frac{8\hat{m}^2 R h^2 g - 2(4\hat{m}(R+h)h + m_b r_b^2 + m_s r_s^2 + 4J_y)p_w}{R^2 Q_2}, \\
\bar{A}_{33} &= -\frac{4d_s}{Q_3}, & \bar{A}_{39} &= -\frac{4p_s}{Q_3}, \\
\bar{A}_{44} &= -\frac{4(3m_w R + 2\hat{m}(R+h))d_w}{R Q_2}, \\
\bar{A}_{48} &= \frac{4\hat{m}(3m_w + 2\hat{m})R h g - 4(3m_w R + 2\hat{m}(R+h))p_w}{R Q_2},
\end{aligned} \tag{B7}$$

where

$$\begin{aligned}
\hat{m} &= m + m_b + m_s, \\
Q_1 &= 5m_w R^2 + 4\hat{m}(R+h)^2 + m_s r_s^2 + 4J_x, \\
Q_2 &= (3m_w + 2\hat{m})(m_b r_b^2 + m_s r_s^2 + 4J_y) + 12\hat{m}m_w h^2, \\
Q_3 &= m_w R^2 + m_b r_b^2 + 4J_z.
\end{aligned} \tag{B8}$$

The polynomial in (71):

$$\begin{aligned}
p(\lambda) &= \lambda^6 - (\bar{A}_{11} + \bar{A}_{33} + \bar{A}_{44})\lambda^5 \\
&+ (\bar{A}_{11}\bar{A}_{33} + \bar{A}_{11}\bar{A}_{44} - A_{13}A_{31} - A_{14}A_{41} + \bar{A}_{33}\bar{A}_{44} - A_{34}A_{43} \\
&\quad - \bar{A}_{17} - \bar{A}_{39} - \bar{A}_{48})\lambda^4 \\
&+ (-\bar{A}_{11}\bar{A}_{33}\bar{A}_{44} + \bar{A}_{11}A_{34}A_{43} + A_{13}A_{31}\bar{A}_{44} \\
&\quad - A_{13}A_{34}A_{41} - A_{14}A_{31}A_{43} + A_{14}\bar{A}_{33}A_{41} \\
&\quad + \bar{A}_{11}\bar{A}_{39} + \bar{A}_{11}\bar{A}_{48} + \bar{A}_{17}\bar{A}_{33} + \bar{A}_{17}\bar{A}_{44} + \bar{A}_{33}\bar{A}_{48} + \bar{A}_{39}\bar{A}_{44})\lambda^3 \\
&+ (-\bar{A}_{11}\bar{A}_{33}\bar{A}_{48} - \bar{A}_{11}\bar{A}_{39}\bar{A}_{44} + A_{13}A_{31}\bar{A}_{48} \\
&\quad + A_{14}\bar{A}_{39}A_{41} - \bar{A}_{17}\bar{A}_{33}\bar{A}_{44} + \bar{A}_{17}A_{34}A_{43} \\
&\quad + \bar{A}_{17}\bar{A}_{39} + \bar{A}_{17}\bar{A}_{48} + \bar{A}_{39}\bar{A}_{48})\lambda^2 \\
&- (\bar{A}_{11}\bar{A}_{39}\bar{A}_{48} + \bar{A}_{17}\bar{A}_{33}\bar{A}_{48} + \bar{A}_{17}\bar{A}_{39}\bar{A}_{44})\lambda - \bar{A}_{17}\bar{A}_{39}\bar{A}_{48}.
\end{aligned} \tag{B9}$$

## Appendix C Eigenvectors of the Open-loop System

When  $\omega_b^* = \omega_s^* = 0$ , the eigenvectors corresponding to the non-zero eigenvalues (63) are

$$v_{1,2} = \begin{bmatrix} 0 \\ \pm\sqrt{A_{48}} \\ 0 \\ \frac{\pm(\sqrt{A_{48}})^3}{A_{28}} \\ 0 \\ 0 \\ 0 \\ \frac{A_{48}}{A_{28}} \\ 0 \\ R \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_{3,4} = \begin{bmatrix} -\frac{A_{13}A_{31}+A_{17}}{A_{31}} \\ 0 \\ \mp\sqrt{A_{13}A_{31}+A_{17}} \\ 0 \\ 0 \\ 0 \\ \mp\sqrt{A_{13}A_{31}+A_{17}} \\ \frac{A_{31}}{A_{31}} \\ 0 \\ -1 \\ 0 \\ \pm\left(\frac{R\sqrt{A_{13}A_{31}+A_{17}}}{A_{31}} - \frac{v^*}{\sqrt{A_{13}A_{31}+A_{17}}}\right) \\ 0 \\ \pm\sqrt{A_{13}A_{31}+A_{17}} \\ \frac{A_{31}}{A_{31}} \\ 1 \end{bmatrix}, \quad (C10)$$

where  $A_{ij}, B_{ij}$  are defined in Appendix B.

## Appendix D Transfer Functions for the Open-Loop System

When  $\omega_b^* = \omega_s^* = 0$ , the transfer function from input torques  $M_w, M_b$  and  $M_s$  to angles  $\gamma, \theta$  and  $\psi$  are:

$$\begin{aligned} T_{M_w \rightarrow \gamma}(\lambda) &= \frac{B_{41}}{\lambda^2 - A_{48}}, \\ T_{M_b \rightarrow \theta}(\lambda) &= \frac{B_{12}}{\lambda^2 - (A_{13}A_{31} + A_{17})}, \\ T_{M_s \rightarrow \psi}(\lambda) &= \frac{B_{33}(\lambda^2 - A_{17})}{\lambda^2(\lambda^2 - (A_{13}A_{31} + A_{17}))}, \\ T_{M_b \rightarrow \psi}(\lambda) &= \frac{A_{31}B_{12}}{\lambda(\lambda^2 - (A_{13}A_{31} + A_{17}))}, \\ T_{M_s \rightarrow \theta}(\lambda) &= \frac{A_{13}B_{33}}{\lambda(\lambda^2 - (A_{13}A_{31} + A_{17}))}, \\ T_{M_w \rightarrow \theta}(\lambda) &= T_{M_b \rightarrow \gamma}(\lambda) = T_{M_s \rightarrow \gamma}(\lambda) = T_{M_w \rightarrow \psi}(\lambda) = 0, \end{aligned} \quad (D11)$$

where  $A_{ij}, B_{ij}$  can be found in Appendix B.

## Appendix E MATLAB Code for the EoM Derivation

```

%% Define Parameter Symbols
syms m_w R m h J_x J_y J_z m_b r_b m_s r_s g M_w M_b M_s delta...
x_G y_G z_G psi theta phi gamma alpha beta x_dot_G y_dot_G z_dot_G...
psi_dot theta_dot phi_dot gamma_dot alpha_dot beta_dot...
sigma1 sigma2 sigma3 sigma4 sigma5 sigma6...
sigma_dot_1 sigma_dot_2 sigma_dot_3 sigma_dot_4 sigma_dot_5 sigma_dot_6;

%% Frame Transformation
T_02=[cos(psi),sin(psi),0;...
-cos(theta)*sin(psi),cos(psi)*cos(theta),sin(theta);...
sin(psi)*sin(theta),-cos(psi)*sin(theta),cos(theta)];
T_23=[cos(gamma),0,-sin(gamma);...
0,1,0;...
sin(gamma),0,cos(gamma)];
T_03=T_23*T_02;

%% Constraints Derivation
vG_F0=[x_dot_G;y_dot_G;z_dot_G];
omega_w_F2=T_02*[0;0;psi_dot]+[theta_dot;phi_dot;0];
rGP_F2=[0;0;-R];
vP_F0=vG_F0+inv(T_02)*cross(omega_w_F2,rGP_F2);
vP_F0=simplify(vP_F0);
constraints=vP_F0==[0;0;0];
% kinematic constraints
KC=[vP_F0(1);vP_F0(2)]==[0;0];
% geometric constraints
GC=z_G==R*cos(theta);

%% Pseudo Velocities
omega_F3=T_03*[0;0;psi_dot]+T_23*[theta_dot;0;0]+[0;gamma_dot;0];
omega_b_F3=omega_F3+[alpha_dot;0;0];
omega_s_F3=omega_F3+[0;0;beta_dot];
% pseudo velocities
pseudo_vel=[sigma1;sigma2;sigma3;sigma4;sigma5;sigma6]==...
[omega_w_F2;omega_F3(2);omega_b_F3(1);omega_s_F3(3)];

%% Kinematics Equations
A=[1,0,-R*cos(psi)*sin(theta),-R*cos(theta)*sin(psi),-R*cos(psi),0,0,0;...
0,1,-R*sin(psi)*sin(theta),R*cos(psi)*cos(theta),-R*sin(psi),0,0,0;...
0,0,0,1,0,0,0,0;...
0,0,sin(theta),0,1,0,0,0;...
0,0,cos(theta),0,0,0,0,0;...
0,0,sin(theta),0,0,1,0,0;...
0,0,-cos(theta)*sin(gamma),cos(gamma),0,0,1,0;...
0,0,cos(gamma)*cos(theta),sin(gamma),0,0,0,1];
det(A);
B=[x_dot_G;y_dot_G;psi_dot;theta_dot;phi_dot;gamma_dot;alpha_dot;beta_dot];
C=[0;0;sigma1;sigma2;sigma3;sigma4;sigma5;sigma6];
A*B=C;
B_soln=inv(A)*C;
% kinematic equations
EOM_p1=B==B_soln;

%% Acceleration Energy of Wheel
vG_F2=[0;0;0]-cross(omega_w_F2,rGP_F2);
vG_F2=subs(vG_F2,B,B_soln);
omega_20_F2=T_02*[0;0;psi_dot]+[theta_dot;0;0];
omega_20_F2=subs(omega_20_F2,B,B_soln);
aG_F2=[R*sigma_dot_2;-R*sigma_dot_1;0]+cross(omega_20_F2,vG_F2);
S_wheel_T1=(1/2)*m_w*(aG_F2(1)^2+aG_F2(2)^2+aG_F2(3)^2);

omega_w_F2=[sigma1;sigma2;sigma3];
alpha_w_F2=[sigma_dot_1;sigma_dot_2;sigma_dot_3]...

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+cross(omega_20_F2,omega_w_F2);
J_w_F2=[(1/4)*m_w*R^2,0,0;...
0,(1/2)*m_w*R^2,0;...
0,0,(1/4)*m_w*R^2];
S_wheel_T2=(1/2)*[alpha_w_F2(1),alpha_w_F2(2),alpha_w_F2(3)]...
*J_w_F2*alpha_w_F2;

H_w_F2=J_w_F2*omega_w_F2;
S_wheel_T3=det([alpha_w_F2(1),alpha_w_F2(2),alpha_w_F2(3);...
omega_w_F2(1),omega_w_F2(2),omega_w_F2(3);...
H_w_F2(1),H_w_F2(2),H_w_F2(3)]);

S_wheel=S_wheel_T1+S_wheel_T2+S_wheel_T3;

%% Acceleration Energy of Body
vG_F3=T_23*vG_F2;
omega_F3=subs(omega_F3,B,B_soln);
rGB_F3=[0;0;h];
vB_F3=vG_F3+cross(omega_F3,rGB_F3);
vB_F3_local_der(1,1)=h*sigma_dot_4+R*(sigma_dot_2*cos(gamma)...
-sigma_2*gamma_dot*sin(gamma));
vB_F3_local_der(2,1)=-R*sigma_dot_1-h*(sigma_dot_1*cos(gamma)...
-sigma_1*gamma_dot*sin(gamma)-sigma_dot_3*sin(gamma)...
-sigma_3*gamma_dot*cos(gamma));
vB_F3_local_der(3,1)=R*(sigma_dot_2*sin(gamma)+sigma_2*gamma_dot*cos(gamma));
vB_F3_local_der=subs(vB_F3_local_der,B,B_soln);
omega_30_F3=omega_F3;
aB_F3=vB_F3_local_der+cross(omega_30_F3,vB_F3);
S_body_T1=(1/2)*m*(aB_F3(1)^2+aB_F3(2)^2+aB_F3(3)^2);

omega_F3_local_der(1,1)=sigma_dot_1*cos(gamma)-sigma_1*gamma_dot*sin(gamma)...
-sigma_dot_3*sin(gamma)-sigma_3*gamma_dot*cos(gamma);
omega_F3_local_der(2,1)=sigma_dot_4;
omega_F3_local_der(3,1)=sigma_dot_3*cos(gamma)-sigma_3*gamma_dot*sin(gamma)...
+sigma_dot_1*sin(gamma)+sigma_1*gamma_dot*cos(gamma);
omega_F3_local_der=subs(omega_F3_local_der,B,B_soln);
alpha_F3=omega_F3_local_der+cross(omega_30_F3,omega_F3);
J_F3=[J_x,0,0;0,J_y,0;0,0,J_z];
S_body_T2=(1/2)*[alpha_F3(1),alpha_F3(2),alpha_F3(3)]*J_F3*alpha_F3;

H_F3=J_F3*omega_F3;
S_body_T3=det([alpha_F3(1),alpha_F3(2),alpha_F3(3);...
omega_F3(1),omega_F3(2),omega_F3(3);...
H_F3(1),H_F3(2),H_F3(3)]);

S_body=S_body_T1+S_body_T2+S_body_T3;

%% Acceleration Energy of Balancing Flywheel
S_balance_T1=(1/2)*m_b*(aB_F3(1)^2+aB_F3(2)^2+aB_F3(3)^2);

omega_b_F3=subs(omega_b_F3,B,B_soln);
omega_b_F3_local_der(1,1)=sigma_dot_5;
omega_b_F3_local_der(2,1)=sigma_dot_4;
omega_b_F3_local_der(3,1)=sigma_dot_3*cos(gamma)...
-sigma_3*gamma_dot*sin(gamma)+sigma_dot_1*sin(gamma)...
+sigma_1*gamma_dot*cos(gamma);
omega_b_F3_local_der=subs(omega_b_F3_local_der,B,B_soln);
alpha_b_F3=omega_b_F3_local_der+cross(omega_30_F3,omega_b_F3);
J_b_F3=[(1/2)*m_b*r_b^2,0,0;0,(1/4)*m_b*r_b^2,0;0,0,(1/4)*m_b*r_b^2];
S_balance_T2=(1/2)*[alpha_b_F3(1),alpha_b_F3(2),alpha_b_F3(3)]...
*J_b_F3*alpha_b_F3;

H_b_F3=J_b_F3*omega_b_F3;
S_balance_T3=det([alpha_b_F3(1),alpha_b_F3(2),alpha_b_F3(3);...
omega_b_F3(1),omega_b_F3(2),omega_b_F3(3);...
H_b_F3(1),H_b_F3(2),H_b_F3(3)]);

S_balance=S_balance_T1+S_balance_T2+S_balance_T3;

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%% Acceleration Energy of Steering Flywheel
S_steer_T1=(1/2)*m_s*(aB_F3(1)^2+aB_F3(2)^2+aB_F3(3)^2);

omega_s_F3=subs(omega_s_F3,B,B_soln);
omega_s_F3_local_der(1,1)=sigma_dot_1*cos(gamma)...
    -sigma1*gamma_dot*sin(gamma)-sigma_dot_3*sin(gamma)...
    -sigma3*gamma_dot*cos(gamma);
omega_s_F3_local_der(2,1)=sigma_dot_4;
omega_s_F3_local_der(3,1)=sigma_dot_6;
omega_s_F3_local_der=subs(omega_s_F3_local_der,B,B_soln);
alpha_s_F3=omega_s_F3_local_der+cross(omega_30_F3,omega_s_F3);
J_s_F3=[(1/4)*m_s*r_s^2,0,0;0,(1/4)*m_s*r_s^2,0;0,0,(1/2)*m_s*r_s^2];
S_steer_T2=(1/2)*[alpha_s_F3(1),alpha_s_F3(2),alpha_s_F3(3)]...
    *J_s_F3*alpha_s_F3;

H_s_F3=J_s_F3*omega_s_F3;
S_steer_T3=det([alpha_s_F3(1),alpha_s_F3(2),alpha_s_F3(3);...
    omega_s_F3(1),omega_s_F3(2),omega_s_F3(3);...
    H_s_F3(1),H_s_F3(2),H_s_F3(3)]);

S_steer=S_steer_T1+S_steer_T2+S_steer_T3;

%% Total Acceleration Energy
S=S_wheel+S_body+S_balance+S_steer;

%% Virtual Power
% virtual power of rolling wheel
G_w_F2=T_02*[0;0;-m_w*g];
M_w_F2=[0;M_w;0];
M_w_F3=T_23*M_w_F2;
delta_P_wheel=[G_w_F2(1),G_w_F2(2),G_w_F2(3)]...
    *[delta*vG_F2(1);delta*vG_F2(2);delta*vG_F2(3)]+...
    [M_w_F2(1),M_w_F2(2),M_w_F2(3)]...
    *delta*omega_w_F2(1);delta*omega_w_F2(2);delta*omega_w_F2(3)];

% virtual power of balancing flywheel
G_b_F3=T_03*[0;0;-m_b*g];
M_b_F3=[M_b;0;0];
delta_P_balance=[G_b_F3(1),G_b_F3(2),G_b_F3(3)]...
    *[delta*vB_F3(1);delta*vB_F3(2);delta*vB_F3(3)]+...
    [M_b_F3(1),M_b_F3(2),M_b_F3(3)]...
    *delta*omega_b_F3(1);delta*omega_b_F3(2);delta*omega_b_F3(3)];

% virtual power of steering flywheel
G_s_F3=T_03*[0;0;-m_s*g];
M_s_F3=[0;0;M_s];
delta_P_steer=[G_s_F3(1),G_s_F3(2),G_s_F3(3)]...
    *[delta*vB_F3(1);delta*vB_F3(2);delta*vB_F3(3)]+...
    [M_s_F3(1),M_s_F3(2),M_s_F3(3)]...
    *delta*omega_s_F3(1);delta*omega_s_F3(2);delta*omega_s_F3(3)];

% virtual power of body
G_F3=T_03*[0;0;-m*g];
delta_P_body=[G_F3(1),G_F3(2),G_F3(3)]...
    *[delta*vB_F3(1);delta*vB_F3(2);delta*vB_F3(3)]+...
    [-M_w_F3(1),-M_w_F3(2),-M_w_F3(3)]...
    *[delta*omega_F3(1);delta*omega_F3(2);delta*omega_F3(3)]+...
    [-M_b_F3(1),-M_b_F3(2),-M_b_F3(3)]...
    *[delta*omega_F3(1);delta*omega_F3(2);delta*omega_F3(3)]+...
    [-M_s_F3(1),-M_s_F3(2),-M_s_F3(3)]...
    *delta*omega_F3(1);delta*omega_F3(2);delta*omega_F3(3)];

% total virtual power
delta_P=delta_P_wheel+delta_P_balance+delta_P_steer+delta_P_body;

%% Pseudo forces
Pi_1=R*g*(m+m_w+m_b+m_s)*sin(theta)...

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+g*h*(m+m_b+m_s)*cos(gamma)*sin(theta)-M_b*cos(gamma)-M_s*sin(gamma);
Pi_2=M_w;
Pi_3=M_b*sin(gamma)-M_s*cos(gamma)-g*h*(m+m_b+m_s)*sin(gamma)*sin(theta);
Pi_4=g*h*(m+m_b+m_s)*cos(theta)*sin(gamma)-M_w;
Pi_5=M_b;
Pi_6=M_s;

%% Appell Equations
Appell_1=simplify(diff(S,sigma_dot_1))==Pi_1;
Appell_2=simplify(diff(S,sigma_dot_2))==Pi_2;
Appell_3=simplify(diff(S,sigma_dot_3))==Pi_3;
Appell_4=simplify(diff(S,sigma_dot_4))==Pi_4;
Appell_5=simplify(diff(S,sigma_dot_5))==Pi_5;
Appell_6=simplify(diff(S,sigma_dot_6))==Pi_6;

[sigma_dot_1_soln,sigma_dot_2_soln,sigma_dot_3_soln,...
 sigma_dot_4_soln,sigma_dot_5_soln,sigma_dot_6_soln]=...
 solve([Appell_1,Appell_2,Appell_3,Appell_4,Appell_5,Appell_6],...
 [sigma_dot_1,sigma_dot_2,sigma_dot_3,sigma_dot_4,sigma_dot_5,sigma_dot_6]);

%% EOM
% Appell equations
EOM_p2=...
 [sigma_dot_1==simplify(expand(sigma_dot_1_soln),'Steps',50);... % f_1
 sigma_dot_2==simplify(expand(sigma_dot_2_soln),'Steps',50);... % f_2
 sigma_dot_3==simplify(expand(sigma_dot_3_soln),'Steps',50);... % f_3
 sigma_dot_4==simplify(expand(sigma_dot_4_soln),'Steps',50);... % f_4
 sigma_dot_5==simplify(expand(sigma_dot_5_soln),'Steps',50);... % f_5
 sigma_dot_6==simplify(expand(sigma_dot_6_soln),'Steps',50)]; % f_6

% EOM
EOM=[EOM_p2;EOM_p1];

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