Extremizers and stability of the Betke-Weil inequality

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To the memory of Ulrich Betke and Wolfgang Weil

Abstract

Let K be a compact convex domain in the Euclidean plane. The mixed area A(K, -K) of K and -K can be bounded from above by $1/(6\sqrt{3})L(K)^2$, where L(K) is the perimeter of K. This was proved by Ulrich Betke and Wolfgang Weil (1991). They also showed that if K is a polygon, then equality holds if and only if K is a regular triangle. We prove that among all convex domains, equality holds only in this case, as conjectured by Betke and Weil. This is achieved by establishing a stronger stability result for the geometric inequality $6\sqrt{3}A(K,-K) \le L(K)^2$.

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1 Introduction

For convex domains K, M (compact convex sets with non-empty interior) in \mathbb{R}^2 , let L(K) be the perimeter of K, let A(K) be the area of K, and let A(K, M) denote the mixed area of K and M (see R. Schneider [16] or Section 2). In [5], U. Betke and W. Weil proved the following theorem.

Theorem 1.1 (Betke, Weil (1991)). *If* $K, M \subset \mathbb{R}^2$ *are convex domains, then*

$$L(K)L(M) \ge 8A(K,M) \tag{1.1}$$

with equality if and only if K and M are orthogonal (possibly degenerate) segments.

This result has been generalized to higher dimensions in [6], where also various improvements in the sense of stability results have been obtained. What makes the variational analysis of (1.1) convenient is the fact that K and M can be varied independently of each other and the dependence on K and M is Minkowski linear (in the Euclidean plane). In [5], U. Betke and W. Weil also considered the case where M = -K and found the following sharp geometric inequality.

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Theorem 1.2 (Betke, Weil (1991)). If K is a convex domain in \mathbb{R}^2 , then

$$L(K)^2 \ge 6\sqrt{3}A(K, -K).$$
 (1.2)

In addition, if K is a polygon, then equality holds if and only if K is a regular triangle.

It is clear from the continuity of the involved functionals that it is sufficient to establish this inequality for convex polygons to deduce it for general convex domains in the plane. However, it has remained an open problem to characterize the equality case in (1.2) among all convex domains. We resolve this problem by proving more generally a stability version of Theorem 1.2. We refer to [6] (and in particular to the literature cited there) for a brief introduction to stability improvements of geometric inequalities.

Theorem 1.3. If K is a convex domain in \mathbb{R}^2 and

$$L(K)^2 \le (1+\varepsilon) 6\sqrt{3} A(K, -K)$$

for some $\varepsilon \in [0, 2^{-28}]$, then there exists a regular triangle T with centroid z such that

$$T-z \subset K-z \subset (1+400\sqrt{\varepsilon})(T-z).$$

The optimality of the stability exponent $\frac{1}{2}$ of ε can be seen by considering a regular triangle T of edge length 2. Then we add over each edge E_i of T an isosceles triangle with height $\sqrt{\varepsilon}$ which has the side E_i in common with T (for i=1,2,3). For the resulting hexagon H we have $L(H)^2-6\sqrt{3}A(H,-H)=36\varepsilon$. However, if $d_{\rm tr}(H)$ is the minimal number $\rho\geq 0$ for which there is a regular triangle T_0 with centroid z such that $T_0-z\subset H-z\subset (1+\rho)(T_0-z)$, then it is easy to check that $d_{\rm tr}(H)\geq \sqrt{\varepsilon}$.

Corollary 1.4. Equality holds in (1.2) if and only if K is a regular triangle.

In [5], Betke and Weil also discuss an application of their Theorem 1.2 to an inequality for characteristics of a planar Boolean model. As a consequence of Corollary 1.4, the equality condition for the lower bound provided in [5, Theorem 3] now turns into an 'if and only if' statement.

For the proof of the inequality (1.2), it is sufficient to consider (convex) polygons with at most k vertices, for any fixed $k \ge 3$. This task was accomplished by Betke and Weil and we add the observation (extracted from an adaptation of their argument) that for a polygon P which is not a regular k-gon with an odd number of sides, there exist polygons P' with at most k vertices arbitrarily close to P such that

$$\frac{L(P')^2}{A(P', -P')} < \frac{L(P)^2}{A(P, -P)},$$

see Proposition 5.1. While this can be used to prove the inequality, it does not give control over the equality cases. To determine all extremal sets, we show that if K is a convex domain which is not too far from a regular triangle, then the inequality (1.2) can be strengthened to a stability result, that is, we show that if K also satisfies

$$L(K)^2 \le (1+\varepsilon)6\sqrt{3}A(K, -K),$$

then K is ε close to a regular triangle (if $\varepsilon > 0$ is small enough). This local stability result is stated and proved in Section 4 (see Proposition 4.1). An outline of the proof of Proposition 4.1, which is divided into six steps, is given at the beginning of the proof. A major geometric idea,

underlying the argument is to approximate K from inside by a triangle $T \subset K$ with maximal area. With T and K we associate hexagons H_1 , H_2 with $H_2 \supset K$ and $T \subset H_1 \subset K$. Another hexagon H_0 is derived from H_1 so that the perimeter is minimized. Then we show that

$$L(K)^{2} - 6\sqrt{3}A(K, -K) \ge L(H_{0})^{2} - 6\sqrt{3}A(H_{2}, -H_{2}) \ge 0.$$

The fact that the right side is nonnegative is far from obvious. More generally, we use a variational argument and validated numerics to establish a lower bound which involves five parameters which determine the shapes of T and H_2 (see Lemma 3.1). In the course of the proofs, we have to determine various mixed areas of polygons. These mixed areas are obtained by a classical formula due to Minkowski and by a more recent one which is due to Betke [4] and was first applied in [5].

2 Notation and mixed area

For $p_1,\ldots,p_\ell\in\mathbb{R}^2$, $\ell\in\mathbb{N}$, we write $[p_1,\ldots,p_\ell]$ to denote the convex hull of the point set $\{p_1,\ldots,p_\ell\}$. In particular, $[p_1,p_2]$ is the segment connecting p_1 and p_2 , and if p_1,p_2,p_3 are not collinear, then $[p_1,p_2,p_3]$ is the triangle with vertices p_1,p_2,p_3 . In addition, the positive hull of $p_1,p_2\in\mathbb{R}^2$ is given by $\mathrm{pos}\{p_1,p_2\}=\{\alpha_1p_1+\alpha_2p_2:\alpha_1,\alpha_2\geq 0\}$. The scalar product of $x,y\in\mathbb{R}^2$ is denoted by $\langle x,y\rangle$, and the corresponding Euclidean norm of x is $\|x\|=\langle x,x\rangle^{1/2}$. In addition, the determinant of the 2×2 matrix with columns $x,y\in\mathbb{R}^2$ is denoted by $\det(x,y)$. The space of compact convex sets in \mathbb{R}^2 is equipped with the Hausdorff metric.

In the following, by a polygon we always mean a convex set. For a (convex) polygon P in \mathbb{R}^2 , let $\mathcal{U}(P)$ denote the finite set of exterior unit normals to the sides of P. For $u \in \mathcal{U}(P)$, we write $S_P(u)$ to denote the length of the side of P with exterior normal u. As usual, the support function h_K of a compact convex set $K \subset \mathbb{R}^2$ is defined by $h_K(x) = h(K, x) = \max\{\langle x, y \rangle : y \in K\}$ for $x \in \mathbb{R}^2$.

We recall and will use repeatedly two formulas which allow us to calculate and analyze the mixed area A(P,Q) of two polygons P and Q. The first is due to Minkowski (see [16, 12]) and states that

$$A(P,Q) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_Q(u) S_P(u).$$
 (2.1)

Since $h_{-P}(u) = h_P(-u)$ for $u \in \mathbb{R}^2$, (2.1) implies that

$$A(P, -P) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_P(-u) S_P(u).$$
 (2.2)

Since A(P, -P) = A(-P, P) (see also below) and $S_{-P}(u) = S_P(-u)$, we also have

$$A(P, -P) = \frac{1}{2} \sum_{u \in \mathcal{U}(P)} h_P(u) S_P(-u).$$

For instance, if P is a triangle it follows from (2.2) that A(P, -P) = 2A(P).

Another useful formula was established much later by Betke [4]. If w is a unit vector with $w \notin \mathcal{U}(P) \cup \mathcal{U}(-Q)$, then

$$2 \cdot A(P,Q) = \sum_{\substack{u \in \mathcal{U}(P), v \in \mathcal{U}(Q) \\ w \in \text{pos}\{u,-v\}}} |\det(u,v)| S_P(u) S_Q(v). \tag{2.3}$$

In particular, (2.3) yields that if $w \notin \mathcal{U}(P)$ for a (fixed) unit vector w, then

$$A(P, -P) = \sum_{\substack{\{u, v\} \in \mathcal{U}(P) \\ w \in \text{pos}\{u, v\}}} |\det(u, v)| S_P(u) S_P(v), \tag{2.4}$$

where the summation extends over all subsets of $\mathcal{U}(P)$ of cardinality two for which $w \in \text{pos}\{u,v\}$ holds (the factor 2 from the preceding formula cancels, since we do not consider ordered pairs). It is formula (2.4) that was used in a clever way by Betke and Weil [5] to prove the Betke-Weil inequality stated in Theorem 1.2.

Minkowski proved that the notion of mixed area can be extended to any pair K, M of compact convex sets in \mathbb{R}^2 (see [16, 12]), and for compact convex sets $K_1, K_2, M \subset \mathbb{R}^2$ and $\alpha_1, \alpha_2 \geq 0$, it is known that

$$A(K,M) = A(M,K),$$

$$A(K+z_1,M+z_2) = A(K,M) \text{ for } z_1,z_2 \in \mathbb{R}^2,$$

$$A(\Phi K,\Phi M) = |\det \Phi| \cdot A(K,M) \text{ for } \Phi \in \mathrm{GL}(2,\mathbb{R}),$$

$$A(K,K) = A(K), \text{ which is the area of } K,$$

$$A(\alpha_1 K_1 + \alpha_2 K_2,M) = \alpha_1 A(K_1,M) + \alpha_2 A(K_2,M),$$

$$A(K_1,M) \leq A(K_2,M) \text{ if } K_1 \subset K_2.$$

We note that it is a subtle issue to decide under which conditions on convex polygons $P \subset Q$ the inequality $A(P,-P) \leq A(Q,-Q)$ is strict. For example, let P be a triangle with its centroid at the origin o. Let v_1, v_2, v_3 denote the vertices of P, and let Q be the hexagon with vertices $v_1, v_2, v_3, -v_1, -v_2, -v_3$. Then $P \subset Q$ and $P \neq Q$, actually A(Q) = 2A(P), and still we have A(P,-P) = 2A(P) = A(Q) = A(Q,-Q).

Finally, we recall that $A(\cdot, \cdot)$ is additive (a valuation) in both arguments. By this we mean that if K, M, L are compact convex sets in the plane and $K \cup M$ is also convex, then

$$A(K \cup M, L) + A(K \cap M, L) = A(K, L) + A(M, L).$$

By symmetry the same property holds for the second argument.

3 An auxiliary result for associated hexagons

Let $T \subset K$ denote a triangle of maximal area contained in K. Let v_1, v_2, v_3 be the vertices of T, let a_i be the side opposite to v_i , whose length is also denoted by a_i for i = 1, 2, 3, and let h_i be the height of T corresponding to a_i . Then we have

$$2A(T) = a_1h_1 = a_2h_2 = a_3h_3.$$

3We observe that the line passing through v_i and parallel to the side a_i is a supporting line to K, by the maximality of the area of T. The width of K orthogonal to a_i can be expressed in the form $(1+t_i)h_i$ for some $t_i \in [0,1]$, where $t_i \leq 1$ follows since T has maximal area among all triangles in K. (The width of K orthogonal to a_i equals the length of the projection of K to a line orthogonal to a_i .) It follows that K is contained in a circumscribed hexagon H_2 such that, for i=1,2,3, H_2 has two sides parallel to a_i , one of which contains v_i and has length

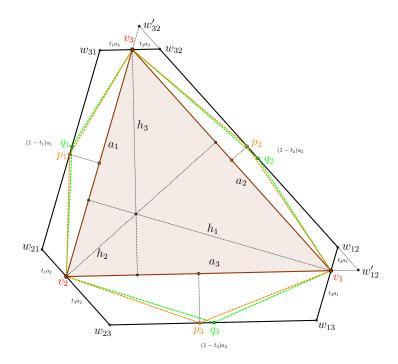


Figure 3.1: Illustration of the geometric construction with the triangle $T = [v_1, v_2, v_3]$, the outer hexagon $H_2 = [w_{32}, w_{31}, w_{21}, w_{23}, w_{13}, w_{12}]$, the inscribed hexagon $H_1 = [v_1, q_2, v_3, q_1, v_2, q_3]$ and the hexagon $H_0 = [v_1, p_2, v_3, p_1, v_2, p_3]$. In particular, we have $||w_{ij} - v_i|| = t_j a_i$ and $||w_{ij} - w_{kj}|| = (1 - t_j)a_j$ for $\{i, j, k\} = \{1, 2, 3\}$.

 $(t_j + t_k)a_i$, $\{i, j, k\} = \{1, 2, 3\}$, and the opposite side has length $(1 - t_i)a_i$ (see Figure 3.1). These assertions follow by elementary geometry from the similarity of corresponding triangles. In fact, with the notation from Figure 3.1 we have

$$\|w_{32}' - w_{32}\| = \frac{a_2}{a_3} \cdot \|w_{32} - v_3\| \quad \text{and} \quad \|w_{32}' - w_{32}\| = \frac{a_2}{h_2} \cdot t_2 h_2,$$

hence

$$||w'_{32} - w_{32}|| = t_2 a_2, \quad ||w_{32} - v_3|| = t_2 a_3,$$

and similarly for permutations of the indices. Moreover,

$$\frac{\|w_{32}' - w_{12}'\|}{a_2} = \frac{h_2(1+t_2)}{h_2},$$

so that $\|w_{32}' - w_{12}'\| = (1 + t_2)a_2$, and hence

$$||w_{32} - w_{12}|| = (1 + t_2)a_2 - ||w'_{32} - w_{32}|| - ||w'_{12} - w_{12}|| = (1 - t_2)a_2.$$

Next, for i=1,2,3 and $\{i,j,k\}=\{1,2,3\}$ we choose a point $q_i \in [w_{ji},w_{ki}] \cap K$. Then we define the (possibly degenerate) hexagon $H_1=[v_1,q_2,v_3,q_1,v_2,q_3] \subset K$. In addition, let p_i be the point on the line determined by $[w_{ji},w_{ki}]$ (and parallel to a_i) which lies on the perpendicular bisector of the side a_i of T, and let H_0 be the hexagon with vertices v_1,p_2,v_3,p_1,v_2,p_3 (see Figure 3.1). Note that in general H_0 may not be convex, but the restricted choice of parameters encountered in the following will always ensure that H_0 is convex.

As $H_1 \subset K \subset H_2$ and $L(H_0) \leq L(H_1)$, we have

$$L(K)^{2} - 6\sqrt{3} A(K, -K)$$

$$\geq L(H_{1})^{2} - 6\sqrt{3} A(H_{2}, -H_{2}) \geq L(H_{0})^{2} - 6\sqrt{3} A(H_{2}, -H_{2}).$$
(3.1)

Clearly, T is also a triangle of maximal area contained in H_1 . As among convex domains of given area, the maximal area of an inscribed triangle is the smallest for ellipses (see Blaschke [3], Sas [15] and Schneider [16, Theorem 10.3.3]), we have

$$A(T) \ge \frac{3\sqrt{3}}{4\pi} \cdot A(H_1) > 0.4 \cdot A(H_1) = 0.4(1 + t_1 + t_2 + t_3)A(T),$$

and hence

$$t_1 + t_2 + t_3 < 1.5. (3.2)$$

The following lemma is the basis for obtaining better bounds on t_1, t_2, t_3 if we know that $L(K)^2 - 6\sqrt{3} A(K, -K)$ is small.

Lemma 3.1. If $a_1 = 2$, $a_2, a_3 \in [2, 2 + \frac{1}{6}]$ and $t_1, t_2, t_3 \in [0, \frac{1}{6}]$, then the hexagons H_0 and H_2 constructed as above satisfy

$$L(H_0)^2 - 6\sqrt{3} A(H_2, -H_2) \ge (a_2 - 2)^2 + (a_3 - 2)^2 + (t_1 - t_0)^2 + (t_2 - t_0)^2 + (t_3 - t_0)^2$$
for $t_0 = (t_1 + t_2 + t_3)/3$.

Remark In the lemma, we do not need K, only the triangle T and $t_1, t_2, t_3 \ge 0$ are required to define H_0 and H_2 . Moreover, although H_0 will be convex in the situation of the lemma, this will not be needed in the argument.

Proof. By the translation invariance of the mixed area, we can assume that v_2 is the origin. Then we obtain from (2.2) that

$$A(H_2, -H_2) = \frac{1}{2} \left\{ (1 - t_2)a_2 \cdot 0 + (t_1 + t_2)a_3t_3h_3 + (1 - t_1)a_1h_1 + (t_1 + t_3)a_2h_2(1 + t_2) + (1 - t_3)a_3h_3 + (t_2 + t_3)a_1h_1t_1 \right\}$$
$$= 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1).$$

By Heron's formula,

$$2A(T) = a_1 h_1 = a_2 h_2 = a_3 h_3$$

$$= \frac{1}{2} \sqrt{(a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3)},$$

and in addition we have

$$L(H_0) = \sum_{i=1}^{3} \sqrt{a_i^2 + 4t_i^2 h_i^2}.$$

Setting $b_i = 2t_i h_i = \frac{4A(T)t_i}{a_i}$ for i = 1, 2, 3, it follows from the Minkowski inequality (or equivalently, the triangle inequality for (a_i, b_i) , i = 1, 2, 3) that

$$L(H_0)^2 = \left(\sum_{i=1}^3 \sqrt{a_i^2 + b_i^2}\right)^2$$

$$\geq (a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2$$

$$= (a_1 + a_2 + a_3)^2 + 16 A(T)^2 (t_1/a_1 + t_2/a_2 + t_3/a_3)^2$$

$$=: f_1(a_2, a_3, t_1, t_2, t_3).$$

For the subsequent analysis, we set $f_2(a_2, a_3, t_1, t_2, t_3) := 16A(T)^2$, hence

$$f_2(a_2, a_3, t_1, t_2, t_3) = (a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3)$$

and

$$f_1(a_2, a_3, t_1, t_2, t_3) = (a_1 + a_2 + a_3)^2 + f_2(a_2, a_3, t_1, t_2, t_3)(t_1/a_1 + t_2/a_2 + t_3/a_3)^2$$

and finally we set

$$f(a_2, a_3, t_1, t_2, t_3) := f_1(a_2, a_3, t_1, t_2, t_3) - 3\sqrt{3}\sqrt{f_2(a_2, a_3, t_1, t_2, t_3)} (1 + t_1t_2 + t_2t_3 + t_3t_1).$$

Thus we obtain

$$L(H_0)^2 - 6\sqrt{3} \cdot A(H_2, -H_2) \ge f(a_2, a_3, t_1, t_2, t_3),$$
 (3.3)

In the following, we consider

$$W := \left\{ (a_2, a_3, t_1, t_2, t_3)^\top \in \mathbb{R}^5 : a_2, a_3 \in [2, 2 + \frac{1}{6}] \text{ and } t_1, t_2, t_3 \in [0, \frac{1}{6}] \right\},$$

$$z_t := (2, 2, t, t, t)^\top, \quad t \in [0, \frac{1}{6}],$$

and the orthonormal basis

$$e_1 = (1, 0, 0, 0, 0)^{\top}, \quad e_2 = (0, 1, 0, 0, 0)^{\top}, \quad e_3 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)^{\top},$$

$$e_4 = \left(0, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)^{\top}, \quad e_5 = \left(0, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^{\top}.$$

We write Df for the derivative and D^2f to denote the Hessian of f. Using a computer algebra system (for convenience) or direct calculations, we obtain that if $t \in [0, \frac{1}{6}]$, then

$$f(z_t) = 0, \quad Df(z_t) = 0.$$
 (3.4)

By the Taylor formula and (3.4) there is a $\xi \in (0,1)$ such that

$$f(x) = \frac{1}{2} \langle x - z_t, D^2 f(z_t + \xi(x - z_t))(x - z_t) \rangle, \quad x \in W.$$

By relation (3.3), Lemma 3.1 follows once we have shown that

$$f(x) \ge \|x - z_t\|^2 \tag{3.5}$$

for $x = (a_2, a_3, t_1, t_2, t_3)^{\top} \in W$ and $t = (t_1 + t_2 + t_3)/3$.

Since for $x=(a_2,a_3,t_1,t_2,t_3)^{\top}\in W$, we have $x-z_t\in e_5^{\perp}$ (we write e_5^{\perp} for the orthogonal complement of e_5) with $t=(t_1+t_2+t_3)/3$ and $z_t+\xi(x-z_t)\in W$, the proof will be finished if we can verify that

$$\langle v, D^2 f(\bar{x})v \rangle \ge 2||v||^2, \quad \bar{x} \in W, v \in e_5^{\perp}.$$
 (3.6)

Using a computer algebra system (such as SageMath or Maple) or tedious calculations, we obtain that

$$D^{2}f(2,2,0,0,0) = \begin{pmatrix} 12 & -6 & 0 & 0 & 0\\ -6 & 12 & 0 & 0 & 0\\ 0 & 0 & 24 & -12 & -12\\ 0 & 0 & -12 & 24 & -12\\ 0 & 0 & -12 & -12 & 24 \end{pmatrix}$$

has the eigenvalues 6, 18, 36, 36, 0, and as associated pairwise orthogonal eigenvectors, one can choose $(1, 1, 0, 0, 0)^{\mathsf{T}}$ to correspond to the eigenvalue $6, (1, -1, 0, 0, 0)^{\mathsf{T}}$ to correspond to $18, e_3$ and e_4 to correspond to 36, and finally e_5 to correspond to 0.

Define the orthogonal matrix $S := (e_1 \dots e_5) \in O(5)$ and write $y = (y_1, \dots, y_5)^{\top} \in \mathbb{R}^5$. Further, we define

$$\tilde{f}(y_1, \dots, y_5) := f\left(\sum_{i=1}^5 y_i e_i\right) = f(Sy),$$

hence $f(x) = \tilde{f}(S^{\top}x)$, $x \in \mathbb{R}^5$. By the chain rule, for $x \in W$ and $v \in \mathbb{R}^5$ we have

$$\langle v, D^2 f(x) v \rangle = \langle S^\top v, D^2 \tilde{f}(S^\top x) S^\top v \rangle.$$

In addition, note that $||S^{\top}v||^2 = ||v||^2$ and

$$S^{\top}(W) \subset \left[2, 2 + \frac{1}{6}\right]^2 \times \left[-\frac{1}{6}\sqrt{\frac{2}{3}}, \frac{1}{6}\sqrt{\frac{2}{3}}\right]^2 \times \left[0, \frac{\sqrt{3}}{6}\right]$$
$$\subset \widetilde{W} := \left[2, 2 + \frac{1}{6}\right]^2 \times \left[-0.14, 0.14\right]^2 \times \left[0, 0.3\right].$$

Here we use that for $y = S^{\top}(a_2, a_3, t_1, t_2, t_3)^{\top}$ we have

$$y_3^2 + y_4^2 = \frac{1}{2}(t_1 - t_2)^2 + \frac{1}{6}(t_1 + t_2 - 2t_3)^2 = \frac{2}{3}\frac{1}{2}\left[(t_3 - t_2)^2 + (t_3 - t_1)^2 + (t_2 - t_1)^2\right]$$

$$\leq \frac{2}{3}\max\{t_1, t_2, t_3\}^2 = \frac{2}{3}\frac{1}{6^2}$$

and

$$0 \le y_5 = \frac{1}{\sqrt{3}}(t_1 + t_2 + t_3) \le \sqrt{3} \cdot \frac{1}{6}.$$

Moreover, $v \in e_5^{\perp}$ if and only if $\langle S^{\top}v, e_5^{\circ} \rangle = 0$, where $e_5^{\circ} = (0, 0, 0, 0, 1)^{\top}$. Hence, (3.6) follows if we can verify that

$$\langle \widetilde{v}, D^2 \widetilde{f}(y) \widetilde{v} \rangle \ge 2 \|\widetilde{v}\|^2, \quad y \in \widetilde{W}, \ \widetilde{v} \in (e_5^{\circ})^{\perp}.$$

Writing $H(y):=\left(D^2\tilde{f}(y)_{ij}\right)_{i,j=1}^{n-1}$ for the $(n-1)\times(n-1)$ matrix (principal minor) obtained from the $n\times n$ Hessian matrix representing $D^2\tilde{f}(y)$ with respect to the standard basis $e_1^\circ,\ldots,e_5^\circ$ of \mathbb{R}^5 , we want to verify that

$$\langle \bar{v}, H(y)\bar{v}\rangle \ge 2\|\bar{v}\|^2, \quad y \in \widetilde{W}, \ \bar{v} \in \mathbb{R}^4,$$
 (3.7)

that is to say, all eigenvalues of H(y) are at least 2. Let

$$\Xi := \{(\bar{v}_1, \dots, \bar{v}_4)^\top \in [-1, 1]^4 : \bar{v}_i = 1 \text{ for some } i \in \{1, \dots, 4\} \}.$$

By scaling invariance of (3.7) with respect to $\bar{v} \in \mathbb{R}^4$, (3.7) is equivalent to

$$\langle \bar{v}, H(y)\bar{v}\rangle \ge 2\|\bar{v}\|^2, \quad y \in \widetilde{W}, \ \bar{v} \in \Xi,$$
 (3.8)

Since all eigenvalues of $H((2,2,0,0,0)^{\top})$ are positive, this holds if and only if all eigenvalues of $H(y)^2$ are at least 4, for $y \in \widetilde{W}$. The latter means that we have to show that $\langle \bar{v}, H(y)^2 \bar{v} \rangle \geq 4 \|\bar{v}\|^2$ for $y \in \widetilde{W}$ and $\bar{v} \in \mathbb{R}^4$ or, equivalently,

$$||H(y)\bar{v}||^2 \ge 4||\bar{v}||^2, \quad y \in \widetilde{W}, \ \bar{v} \in \mathbb{R}^4.$$
 (3.9)

Again by scaling invariance of (3.9) with respect to \tilde{v} , (3.9) is in turn equivalent to

$$||H(y)\bar{v}||^2 \ge 4||\bar{v}||^2, \quad y \in \widetilde{W}, \, \bar{v} \in \Xi.$$
 (3.10)

Direct rigorous numerical analysis of the eigendecomposition of the hessian $D^2 \tilde{f}$ may be challenging due to requiring too many subdivisions of \widetilde{W} in order to achieve the required precision [14, 11, 8, 10]. As both, \widetilde{W} and Ξ are compact, finite dimensional and as the desired inequalities, either (3.8) or (3.10), are expected to be strict, they are well suited for being studied by rigorous numerics [13, 1, 17].

Namely, for small $\widetilde{W}' \subset \widetilde{W}$, and $\Xi' \subset \Xi$, we preform the following procedure with all computations being carried out rigorously using interval arithmetic and automatic differentiation [13, 1, 17, 9]. First, we bound the jet of \widetilde{f} up to Taylor-coefficients of degree 6 over \widetilde{W}' . In order to increase precision and eliminate some of the dependency issues, for a given \widetilde{W}' , the degree 6 jet of f is bound both over \widetilde{W}' and over the midpoint of \widetilde{W}' . Hence, using the multivariate Taylor-expansions with the appropriate remainder term, we obtain enhanced bounds on the Taylor-coefficients of \widetilde{f} over \widetilde{W}' and, in turn, a better enclosure of the hessian matrix $H = D^2 \widetilde{f}$. Second, we test if we can guarantee that the inequality (3.8) or (3.10) for all $v \in \Xi'$ holds. If that is not the case, then an adaptive bisection scheme of $\widetilde{W}' \times \Xi'$ is utilized and the arising subsets of $\widetilde{W}' \times \Xi'$ are processed separately.

We have implemented our software using the package CAPD [7] and verified both inequalities independently and successfully. The required number of subsets (of $\widetilde{W} \times \Xi$) and the associated computational times (without parallelization on an i7-9750) were

- (3.8): 25880 subsets, 8m14s;
- (3.10): 2440 subsets, 46s.

We note that the increased complexity of (3.8) is most likely just an artefact of the naive computation of the inner product and could be decreased (to that of (3.10)) by choosing a more efficient evaluation scheme. The source code and output logs are available at [2].

In particular, both
$$(3.8)$$
, (3.10) , and, in turn, Lemma 3.1 have been verified.

The following two claims will be used in the proof of Proposition 4.1.

Claim 3.1. If the regular triangle T_0 of side length b contains a triangle T which has a side of length at most a, where $\frac{b}{2} \le a \le b$, then $A(T) \le \frac{a}{b} A(T_0)$.

Proof. Let ℓ be the line containing a side of T of length at most a. We may assume that the vertex v of T opposite to $\ell \cap T$ is also a vertex of T_0 .

If the distance of ℓ from v (that is, the height of T) is at most $b\sqrt{3}/2$, then we are done. Therefore we may now assume that the distance of ℓ from v is larger than $b\sqrt{3}/2$. Let D_v be the circular disc with center v and radius $b\sqrt{3}/2$. Since the side $\ell \cap T$ is disjoint from D_v , it lies in one of the two connected components of $T_0 \setminus D_v$. It follows that T is contained in one of the two triangles which are obtained by cutting T_0 into two sub-triangles by the height emanating from v, thus $A(T) \leq \frac{1}{2} A(T_0)$.

Claim 3.2. If $\varrho_1, \varrho_2, \varrho_3$ are the side lengths of a triangle P and $A(P) \ge \xi \varrho_1$, for some $\xi \in [0, \varrho_1]$, then

$$\varrho_2 + \varrho_3 \ge \varrho_1 + \frac{\xi^2}{\varrho_1}.$$

Proof. The height h of P corresponding to ϱ_1 is at least 2ξ , and $\varrho_2 + \varrho_3$ is minimized under this condition if $\varrho_2 = \varrho_3$, thus

$$\varrho_2 + \varrho_3 \ge 2\sqrt{\left(\frac{\varrho_1}{2}\right)^2 + h^2} \ge \sqrt{\varrho_1^2 + 16\xi^2} = \varrho_1\sqrt{1 + \frac{16\xi^2}{\varrho_1^2}} \ge \varrho_1 + \frac{\xi^2}{\varrho_1},$$

which proves the claim.

4 Local stability

For any convex domain K, let $d_{\rm tr}(K)$ be the minimal $\rho \geq 0$ such that there exists a regular triangle T with centroid z satisfying

$$T-z \subset K-z \subset (1+\rho)(T-z)$$
.

In particular, $d_{tr}(K)$ measures how close K is to a suitable regular triangle.

Proposition 4.1. Suppose that K is a convex domain with $d_{tr}(K) \leq 6^{-2}$ and

$$L(K)^{2} \le (1+\varepsilon)6\sqrt{3}A(K, -K) \tag{4.1}$$

for some $\varepsilon \in [0, (6 \cdot 180)^{-2}]$. Then $d_{tr}(K) \le 400\sqrt{\varepsilon}$.

Proof. Let $d_{\rm tr}(K)=\eta \le 6^{-2}$ and ε be as in the statement of the proposition. There exists a regular triangle T_0 of side length b containing K such that a translate of $\frac{1}{1+\eta}\,T_0$ is contained in K. For a triangle $T\subset K$ of maximal area contained in K, we have

$$A(T) \ge \frac{A(T_0)}{(1+\eta)^2} > \frac{A(T_0)}{1+3\eta}.$$
(4.2)

From now on, we use the notions and auxiliary constructions introduced for K and T at the beginning of Section 3, including the hexagons H_0, H_1, H_2 , the parameters $t_1, t_2, t_3 \ge 0$, etc.

The main part of the proof is divided into several steps. In Step 1, we prove that T is $\sqrt{\varepsilon}$ close to a regular triangle, Step 2 shows that $A(H_2, -H_2) - A(K, -K)$ is ε small. Based on these findings, Step 3 verifies that if H_2 is close to T in the sense that $\max\{t_1, t_2, t_3\} \leq 100 \sqrt{\varepsilon}$ (see (4.21)), then Proposition 4.1 holds.

The rest of the argument is indirect. Starting from Step 4, we assume that the assumption $\max\{t_1,t_2,t_3\}>100\,\sqrt{\varepsilon}$ (see (4.24)) is satisfied and derive a contradiction. Under this assumption, we prove in Step 4 that H_1 is reasonably close to H_0 in the sense that $\|p_i-q_i\|$ is reasonably small for i=1,2,3 (see (4.26)). Then Step 5 verifies that $K\subset D=\frac{1}{2}H_1+\frac{1}{2}H_2$ and clearly $D\subset H_2$. Finally, in Step 6, we prove that the gap between $A(H_2,-H_2)$ and A(D,-D) (and hence the gap between $A(H_2,-H_2)$ and A(K,-K) by Step 5) is too large, which yields the desired contradiction.

Step 1 T is $\sqrt{\varepsilon}$ close to a regular triangle.

By scaling invariance of the assertion (and symmetry), we may assume that the side lengths of T satisfy

$$a_1 \le a_2 \le a_3$$
 and $a_1 = 2$.

Assuming $a_1 < \frac{b}{2}$, we can apply Claim 3.1 with a = b/2 and get

$$A(T) \le \frac{1}{2}A(T_0) \le \frac{1}{2}(1+\eta)^2 A(T)$$

by (4.2), which is a contradiction, since $\eta \le 6^{-2}$. Hence, we have $\frac{b}{2} \le a_1 \le b$, and another application of Claim 3.1 now yields that

$$A(T) \le \frac{a_1}{b} A(T_0) \le \frac{a_1}{b} (1 + 3\eta) A(T),$$

so that

$$\frac{b}{1+3\eta} \le a_1 = 2 \le a_3 \le b,$$

and hence

$$a_3 \le b \le 2(1+3\eta) = 2+6\eta \le 2+\frac{1}{6}.$$
 (4.3)

It also follows from (4.2) that

$$(1+t_1+t_2+t_3)A(T) = A(H_1) \le A(K) \le A(T_0) < (1+3\eta)A(T),$$

therefore

$$t_i < \frac{1}{12} < \frac{1}{6}$$
 for $i = 1, 2, 3$. (4.4)

Thus, the condition $d_{tr}(K) \le 6^{-2}$ ensures that Lemma 3.1 can be applied.

Hence, by a combination of (3.1) with Lemma 3.1 and the assumption (4.1) of the proposition, we see that

$$(a_i - 2)^2 \le 6\sqrt{3} A(K, -K)\varepsilon < 11 A(K, -K)\varepsilon \text{ for } i = 2, 3,$$
 (4.5)

$$(t_i - t)^2 \le 6\sqrt{3} A(K, -K)\varepsilon < 11 A(K, -K)\varepsilon \text{ for } t = \frac{1}{3}(t_1 + t_2 + t_3), i = 1, 2, 3.$$
 (4.6)

To estimate A(K, -K), we observe that the height h_1 of T corresponding to the side $a_1 = 2$ satisfies $h_1 \le \sqrt{a_3^2 - 1} < 2$, thus $A(T) \le \sqrt{a_3^2 - 1} < 2$ by (4.3), and hence

$$A(K, -K) \le A(T_0, -T_0) = 2A(T_0) \le 2(1+3\eta)A(T) < \frac{39}{9} < 5.$$
 (4.7)

Now (4.5) and (4.6) imply that

$$2 \le a_i \le 2 + 7\sqrt{\varepsilon} \quad \text{for } i = 2, 3, \tag{4.8}$$

$$|t_i - t| \le 7\sqrt{\varepsilon} \quad \text{for } i = 1, 2, 3.$$
 (4.9)

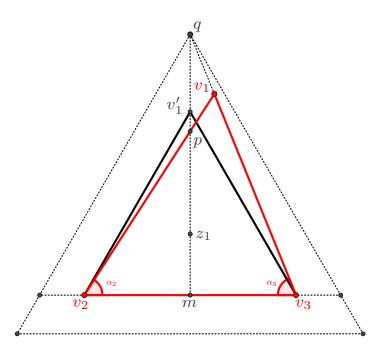


Figure 4.1: Illustration of the triangles $T = [v_1, v_2, v_3], T_1' = [v_1', v_2, v_3]$ together with the auxiliary points m, p, q, z_1 in the case where $p \in [v_1', m]$. In the case $p \notin [v_1', m]$ (not shown in the figure) we have $T_1' \subset T$ and $p = q = v_1$.

We observe that $\frac{a_1^2 + a_2^2 - a_3^2}{a_2}$ is an increasing function of $a_2 \ge 2$ as $a_3 \ge a_1$. Writing α_i to denote the angle of T opposite to the side a_i and using that $7^2\sqrt{\varepsilon} \leq 1$, we have

$$\cos \alpha_3 = \frac{a_1^2 + a_2^2 - a_3^2}{2a_1a_2} \ge \frac{8 - (2 + 7\sqrt{\varepsilon})^2}{8} \ge \frac{4 - 29\sqrt{\varepsilon}}{8} \ge \frac{1}{2} - 4\sqrt{\varepsilon} > 0.$$

In particular, we have $\alpha_1 \le \alpha_2 \le \alpha_3 < \pi/2$, $\alpha_3 \ge \pi/3$ and $\alpha_1 \le \pi/3$. Since $\cos'(s) = -\sin s \le -\sqrt{3}/2$ if $s \in [\frac{\pi}{3}, \frac{\pi}{2}]$, the mean value theorem then implies that

$$\alpha_3 \le \frac{\pi}{3} + \frac{2}{\sqrt{3}} 4\sqrt{\varepsilon} \le \frac{\pi}{3} + 5\sqrt{\varepsilon}.$$

Since $\pi \le \alpha_1 + 2\alpha_3 \le \alpha_1 + 2(\frac{\pi}{3} + 5\sqrt{\varepsilon})$, we conclude that

$$\frac{\pi}{3} - 10\sqrt{\varepsilon} \le \alpha_1 \le \alpha_2 \le \alpha_3 \le \frac{\pi}{3} + 5\sqrt{\varepsilon}. \tag{4.10}$$

As a first estimate, we deduce $\tan \alpha_3 \le 2$, thus $\tan'(s) = 1 + (\tan s)^2 \le 5$ if $s \in [\frac{\pi}{3}, \alpha_3]$. In particular, (4.10) yields

$$\sqrt{3} - 50\sqrt{\varepsilon} \le \tan \alpha_1 \le \tan \alpha_2 \le \tan \alpha_3 \le \sqrt{3} + 25\sqrt{\varepsilon}. \tag{4.11}$$

Now let T'_1 be the regular triangle of edge length 2 positioned in such a way that the side a_1 is common with T and int $T \cap \operatorname{int} T_1' \neq \emptyset$. Recalling that v_2 and v_3 are the endpoints of a_1 , we write v_1' to denote the third vertex of T_1' , z_1 to denote the centroid of T_1' and $m=\frac{1}{2}(v_2+v_3)$ to denote the midpoint of a_1 . As $a_3 \geq a_2$ and $\frac{\pi}{2} > \alpha_3 \geq \frac{\pi}{3}$, there exists a point q such that $v_1' \in [m,q]$ and $v_1 \in [v_3, q]$. In addition, we consider the intersection point $\{p\} = [q, m] \cap [v_1, v_2] \neq \emptyset$ (see Figure 4.1).

We deduce from (4.11) that

$$||q - v_1'|| = \tan(\alpha_3) - \sqrt{3} \le 25\sqrt{\varepsilon}$$

and

$$||p - m|| = \tan(\alpha_2) \ge \tan(\alpha_1) \ge \sqrt{3} - 50\sqrt{\varepsilon}$$

and if $p \in [v'_1, m]$, then also

$$||p - v_1'|| = ||v_1' - m|| - ||p - m|| = \sqrt{3} - \tan(\alpha_2) \le 50\sqrt{\varepsilon}.$$

Therefore, $||v_1' - z_1|| = \frac{2}{\sqrt{3}}$ implies that in any case

$$\left(1 - 25\sqrt{3}\sqrt{\varepsilon}\right)\left(T_1' - z_1\right) \subset T - z_1 \subset \left(1 + \frac{25\sqrt{3}}{2}\sqrt{\varepsilon}\right)\left(T_1' - z_1\right). \tag{4.12}$$

For the regular triangle

$$T_1 = z_1 + (1 - 25\sqrt{3}\sqrt{\varepsilon})(T_1' - z_1),$$

with centroid at z_1 , we have

$$T_1 - z_1 \subset T - z_1 \subset \left(1 + 70\sqrt{\varepsilon}\right) (T_1 - z_1). \tag{4.13}$$

In addition, let z be the centroid of T. The argument above also shows that

$$v_1 \in v_1' + \frac{25\sqrt{3}}{2}\sqrt{\varepsilon} (T_1' - z_1).$$

Since $z = \frac{1}{3}(v_1 + v_2 + v_3)$ and $z_1 = \frac{1}{3}(v_1' + v_2 + v_3)$, we get

$$z - z_1 \in \frac{25\sqrt{3}}{2\cdot 3} \sqrt{\varepsilon} \cdot (T_1' - z_1) \subset 10\sqrt{\varepsilon} \cdot (T_1 - z_1),$$

$$z_1 - z \in 20\sqrt{\varepsilon} \cdot (T_1 - z_1),$$

$$(4.14)$$

where for the second containment we used that $-(T_1 - z_1) \subset 2(T_1 - z_1)$. In summary, (4.13) and (4.14) show that the regular triangle T_1 is a very good approximation of T.

Note that p_i, q_i lie on the same side of H_2 (above a_i) for i=1,2,3. This follows from $\alpha_i \leq \frac{\pi}{2}, \ a_i \geq 2$, (4.3) and (4.4), which imply that $t_i a_j \leq \frac{1}{12}(2+\frac{1}{6}) < 0.2 < 1 \leq \frac{a_k}{2}$ for $i,j,k \in \{1,2,3\}$. Moreover, for i=1,2,3 we have

$$1.77 > (1 + 4\sqrt{\varepsilon})(\sqrt{3} + 25\sqrt{\varepsilon}) \ge \frac{a_i}{2}\tan(\alpha_3) \ge h_i \ge \frac{a_i}{2}\tan(\alpha_1) \ge \sqrt{3} - 50\sqrt{\varepsilon} > 1.68, (4.15)$$

and hence in particular

$$2 > 1.77 > A(T) = h_1 > 1.68 > 1.5.$$
 (4.16)

Step 2 A(K, -K) and $A(H_2, -H_2)$ are ε close, and $L(H_0)$, $L(H_1)$, L(K) are ε close.

We deduce from (4.7) that $6\sqrt{3}\,A(K,-K) \le 26\sqrt{3} < 50$. Therefore (4.1), Lemma 3.1, $H_1 \subset K \subset H_2$ and $L(H_0) \le L(H_1)$ imply that

$$50\varepsilon \ge L(K)^2 - 6\sqrt{3} A(K, -K) \ge L(K)^2 - 6\sqrt{3} A(H_2, -H_2)$$

$$\ge L(H_1)^2 - 6\sqrt{3} A(H_2, -H_2) \ge L(H_0)^2 - 6\sqrt{3} A(H_2, -H_2) \ge 0. \tag{4.17}$$

In particular,

$$L(K)^2 - 6\sqrt{3} A(K, -K) \le 50\varepsilon,$$

 $6\sqrt{3} A(H_2, -H_2) - L(K)^2 \le 0,$

which yields

$$A(H_2, -H_2) \le A(K, -K) + 5\varepsilon. \tag{4.18}$$

Using $L(K) \ge L(H_1) \ge L(H_0) \ge L(T) \ge 6$ and

$$L(K)^2 - 6\sqrt{3} A(H_2, -H_2) \le 50\varepsilon,$$

 $6\sqrt{3} A(H_2, -H_2) - L(H_1)^2 \le 0.$

we get

$$50\varepsilon \ge L(K)^2 - L(H_1)^2 = (L(K) - L(H_1))(L(K) + L(H_1)) \ge 12(L(K) - L(H_1)).$$

We deduce that

$$L(K) \le L(H_1) + 5\varepsilon. \tag{4.19}$$

A similar argument shows that

$$L(H_1) \le L(H_0) + 5\varepsilon. \tag{4.20}$$

This completes Step 2.

For the remaining part of the proof, we set $\gamma = 10^2$. In the following, we distinguish whether

$$\max\{t_1, t_2, t_3\} \le \gamma \sqrt{\varepsilon} \tag{4.21}$$

is satisfied or not. If (4.21) holds, then H_2 is $\sqrt{\varepsilon}$ close to K (see the argument below).

Step 3 If (4.21) holds, then $d_{tr}(K) \leq 4\gamma\sqrt{\varepsilon}$.

It follows from (4.13) and $T \subset K$ that

$$T_1 - z_1 \subset K - z_1.$$
 (4.22)

Using (4.21) and recalling that z is the centroid of T, we get

$$K - z \subset H_2 - z \subset (1 + 3\max\{t_1, t_2, t_3\}) (T - z) \subset (1 + 3\gamma\sqrt{\varepsilon})(T - z).$$

Therefore (4.13) and (4.14) imply that if (4.21) holds, then

$$K - z_{1} = K - z + (z - z_{1}) \subset (1 + 3\gamma\sqrt{\varepsilon})(T - z_{1}) + 3\gamma\sqrt{\varepsilon}(z_{1} - z)$$

$$\subset [1 + 70\sqrt{\varepsilon} + 3\gamma\sqrt{\varepsilon} + 210\gamma\varepsilon + 60\gamma\varepsilon](T_{1} - z_{1})$$

$$\subset (1 + 4\gamma\sqrt{\varepsilon})(T_{1} - z_{1}). \tag{4.23}$$

In view of (4.22), we conclude Proposition 4.1 if (4.21) holds.

It remains to consider the case where

$$\max\{t_1, t_2, t_3\} > \gamma \sqrt{\varepsilon}. \tag{4.24}$$

This case will finally lead to a contradiction.

It follows from (4.9) and (4.24) that

$$\min\{t_1, t_2, t_3\} \ge 86\sqrt{\varepsilon}.\tag{4.25}$$

Step 4 Assuming (4.24), H_1 is reasonably close to H_0 .

More precisely, we claim that

$$||p_i - q_i|| < 0.06 < 0.1 \text{ for } i = 1, 2, 3,$$
 (4.26)

which is what we mean by saying that H_1 is "reasonably close" to H_0 .

Let $\{i, j, k\} = \{1, 2, 3\}$, and assume that $q_i \neq p_i$ (otherwise (4.26) readily holds). For the line ℓ_i through p_i and q_i and parallel to a_i , let \tilde{v}_j be the reflection of v_j through ℓ_i , and hence p_i is the midpoint of $[\tilde{v}_j, v_k]$. For the triangle $\tilde{T}_i = [q_i, \tilde{v}_j, v_k]$, we have

$$A(\widetilde{T}_i) = 2A([q_i, p_i, v_k]) = ||p_i - q_i||t_i h_i.$$

Recall from (4.15) that $1.68 < h_i < 1.77$. Using (4.8) and $t_i \le \frac{1}{12}$ by (4.4), we get

$$\|\tilde{v}_j - v_k\| = 2\|p_i - v_k\| \le 2\left(\frac{a_i}{2} + \frac{1}{12}h_i\right) \le a_i + \frac{1}{6}h_i \le 2 + \frac{7}{6 \cdot 180} + \frac{1}{6} \cdot 1.77 \le 2.31.$$

Therefore

$$A(\widetilde{T}_i) \ge \|p_i - q_i\|t_i \cdot 1.68 \ge \frac{\|p_i - q_i\|t_i}{2.31} \cdot 1.68 \cdot \|\widetilde{v}_j - v_k\|.$$

Since

$$\frac{\|p_i - q_i\|t_i}{2.31} \cdot 1.68 \le \frac{a_i}{12 \cdot 2.31} \cdot 1.68 < a_i \le \|\tilde{v}_j - v_k\|,$$

Claim 3.2 can be applied. In Combination with $t_i \ge 86\sqrt{\varepsilon}$ (by (4.25)), this leads to

$$||q_{i} - v_{j}|| + ||q_{i} - v_{k}|| = ||q_{i} - \tilde{v}_{j}|| + ||q_{i} - v_{k}||$$

$$\geq ||\tilde{v}_{j} - v_{k}|| + \frac{1}{2.31} \left(\frac{||p_{i} - q_{i}||t_{i} \cdot 1.68}{2.31} \right)^{2}$$

$$= ||p_{i} - v_{j}|| + ||p_{i} - v_{k}|| + ||p_{i} - q_{i}||^{2} t_{i}^{2} \frac{1.68^{2}}{2.31^{3}}$$

$$\geq ||p_{i} - v_{j}|| + ||p_{i} - v_{k}|| + ||p_{i} - q_{i}||^{2} \frac{86^{2} \cdot 1.68^{2}}{2.31^{3}} \varepsilon.$$

We deduce from (4.20) that

$$5 \varepsilon \ge ||p_i - q_i||^2 \frac{86^2 \cdot 1.68^2}{2.31^3} \varepsilon,$$

and hence (4.26) follows.

Step 5 Assuming (4.24), we have $K \subset D := \frac{1}{2} H_1 + \frac{1}{2} H_2$.

The polygon $D \subset H_2$ has twelve sides (see Figure 4.2).

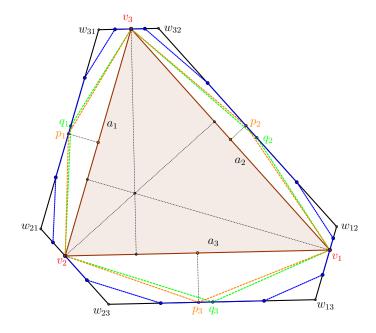


Figure 4.2: Illustration for $D = \frac{1}{2}H_1 + \frac{1}{2}H_2$.

Six of these sides are subsets of the six sides of H_2 as $h_{H_1}(u) = h_{H_2}(u)$ holds for any $u \in \mathcal{U}(H_2)$, and the other six sides of D are parallel to the sides of H_1 . We prove

$$K \subset D$$
 (4.27)

indirectly, so we assume that there exists $x \in K \setminus D$, and seek a contradiction. Then there exists $u \in \mathcal{U}(D)$ such that $h_K(u) \geq \langle x, u \rangle > h_D(u)$. Since $x \in K \subset H_2$, u is normal to a side of D parallel to a side of H_1 . Let u be normal to $[v_i, q_j]$, $i \neq j$, so that w_{ij} is the vertex of H_2 where u is a normal to H_2 . For

$$\Delta = h_{H_2}(u) - h_{H_1}(u) = \langle u, w_{ij} - q_j \rangle,$$

we have

$$\Delta \cdot ||q_j - v_i|| = 2A([q_j, v_i, w_{ij}]) = t_j h_j ||w_{ij} - q_j||.$$

We have $h_j \ge 1.68$ by (4.15), and since $t_i \le \frac{1}{12}$ by (4.4) and $h_i < 2$, we obtain using (4.26) that

$$||q_j - v_i|| \le ||p_i - v_i|| + 0.1 \le \frac{a_i}{2} + \frac{1}{12}h_i + 0.1 \le \frac{3}{2} + \frac{1}{6} + 0.1 < 2,$$

and

$$||w_{ij} - q_j|| \ge ||w_{ij} - p_j|| - 0.1 \ge \frac{a_j}{2} - 0.1 \ge 1 - 0.1 = 0.9.$$

Thus we get

$$\Delta \ge \frac{1}{2} t_j \cdot 1.68 \cdot 0.9 \ge 60 \sqrt{\varepsilon}.$$

We deduce that

$$\langle u, x - q_j \rangle > h_D(u) - h_{H_1}(u) = \frac{1}{2} (h_{H_2}(u) - h_{H_1}(u)) = \frac{\Delta}{2} \ge 30\sqrt{\varepsilon},$$

therefore

$$A([q_j, v_i, x]) = \frac{\langle u, x - q_j \rangle \cdot \|q_j - v_i\|}{2} \ge 15\sqrt{\varepsilon} \cdot \|q_j - v_i\|.$$

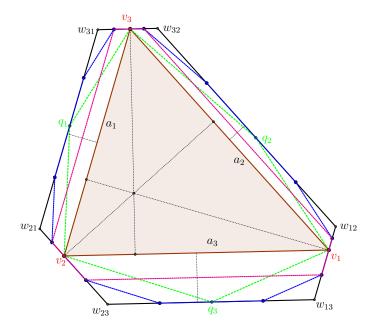


Figure 4.3: Illustration for $D = \frac{1}{2}H_1 + \frac{1}{2}H_2$ and $D_0 = \frac{1}{2}T + \frac{1}{2}H_2$.

Note that $15\sqrt{\varepsilon} \le 0.02$ and $||q_j - v_i|| \ge ||p_j - v_i|| - 0.1 \ge 0.9$. It follows from Claim 3.2 and $||q_j - v_i|| \le 2$ that

$$||x - v_i|| + ||x - q_j|| \ge ||q_j - v_i|| + \frac{1}{||q_j - v_i||} \cdot (15\sqrt{\varepsilon})^2 \ge ||q_j - v_i|| + 15\varepsilon.$$

Writing \widetilde{H} to denote the polygon that is the convex hull of x and H_1 , we have that $[v_i, x]$ and $[x, q_i]$ are sides of \widetilde{H} , and hence

$$L(K) \ge L(\widetilde{H}) \ge L(H_1) + 15\varepsilon,$$

contradicting (4.19). In turn, we conclude (4.27).

Step 6 $A(H_2, -H_2) - A(K, -K)$ is too large, assuming (4.24).

In order to calculate A(D,-D) for $D=\frac{1}{2}H_1+\frac{1}{2}H_2$, we claim that $D_0=\frac{1}{2}T+\frac{1}{2}H_2$ satisfies

$$A(D, -D) = A(D_0, -D_0) + A(T)(t_1t_2 + t_2t_3 + t_1t_3).$$
(4.28)

We prove (4.28) by applying Betke's formula (2.4) three times. Let b_i be the side of D containing q_i (and hence b_i is contained in the "long" side of H_2 parallel to a_i), and let $\tilde{u}_i \in \mathcal{U}(D)$ be the normal to D at the vertex v_i of T, i=1,2,3, and hence $-\tilde{u}_i$ is the exterior unit normal to b_i . In addition, let d_i be the diagonal of D that cuts off b_i and the two sides neighboring b_i from D, and hence d_i is parallel to b_i , i=1,2,3. We write v_{ij} to denote the exterior unit normal to the side of D cut off by and parallel to $[q_i, v_j]$, $i \neq j$, and hence v_{ij} and v_{ik} are the normals to the two sides neighboring b_i , $\{i,j,k\} = \{1,2,3\}$.

Now d_1 dissects D into a trapezoid and a polygon D_1 with 10 sides, and on the way to verify (4.28), we first claim that

$$A(D, -D) = A(D_1, -D_1) + S_D(\tilde{u}_1)S_D(\nu_{1,2})|\det(\tilde{u}_1, \nu_{1,2})|.$$
(4.29)

To prove (4.29), we choose a unit vector $u_1 \neq \tilde{u}_1$ very close to \tilde{u}_1 and such that $\langle u_1, v_2 - v_3 \rangle > 0$. When we apply Betke's formula (2.4) to calculate the difference $A(D, -D) - A(D_1, -D_1)$ using u_1 as the reference vector, we deduce after cancellation of summands common to A(D, -D) and $A(D_1, -D_1)$, that the exterior unit normal $-\tilde{u}_1$ to the sides b_1 of D and d_1 of D_1 does not occur in either term and precisely one of the two exterior unit normals of the two sides of D neighboring b_1 , in this case v_{12} , occurs. To see this, we observe that if $\{i, j, k\} = \{1, 2, 3\}$, then

$$\angle(-\tilde{u}_i, \nu_{ij}) < 0.16 \tag{4.30}$$

because $\angle(-\tilde{u}_i,\nu_{ij})=\angle(q_i-v_j,v_k-v_j)$ satisfies $\tan\angle(-\tilde{u}_i,\nu_{ij})\leq (t_ih_i)/(1-0.06)<0.16$ using the estimates (4.26), $h_i<1.77$ and $t_i\leq\frac{1}{12}$ (cf. (4.4)). On the one hand, if $\{i,j\}=\{2,3\}$, then (4.10), (4.30) and $\sqrt{\varepsilon}<(6\cdot180)^{-1}$ yield

$$\angle(\tilde{u}_1, \nu_{i1}) = \angle(\tilde{u}_1, -\tilde{u}_i) - \angle(-\tilde{u}_i, \nu_{i1}) = \alpha_j - \angle(-\tilde{u}_i, \nu_{i1}) \ge \frac{\pi}{3} - 10\sqrt{\varepsilon} - 0.16 > 0.87;$$

therefore, the angle of \tilde{u}_1 with any other exterior unit normal to D or D_1 is at least 0.87. On the other hand, $\angle(-\tilde{u}_1, \nu_{1j}) < 0.16$ for j = 2, 3 by (4.30), concluding the proof of (4.29).

Since

$$S_D(\nu_{1,2}) = \frac{1}{2} ||q_1 - v_2||, \quad S_D(\tilde{u}_1) = \frac{1}{2} (t_2 + t_3) a_1, \quad |\det(\tilde{u}_1, \nu_{12})| = \frac{t_1 h_1}{||q_1 - v_2||},$$

we deduce from (4.29) that

$$A(D, -D) = A(D_1, -D_1) + \frac{1}{2}t_1(t_2 + t_3)A(T).$$
(4.31)

Next we observe that d_2 dissects D_1 into a trapezoid and a polygon D_2 with 8 sides. Choosing a unit vector $u_2 \neq \tilde{u}_2$ very close to \tilde{u}_2 , and applying Betke's formula (2.4) to calculate $A(D_1, -D_1) - A(D_2, -D_2)$ with u_2 as reference vector, we conclude as before that

$$A(D_1, -D_1) = A(D_2, -D_2) + \frac{1}{2}t_2(t_1 + t_3)A(T).$$

Finally, the analogous argument for a a unit vector $u_3 \neq \tilde{u}_3$ very close to \tilde{u}_3 implies that

$$A(D_2, -D_2) = A(D_0, -D_0) + \frac{1}{2}t_3(t_1 + t_2)A(T).$$

Thus we arrive at

$$A(D, -D) = A(D_0, -D_0) + \frac{1}{2}[t_1(t_2 + t_3) + t_2(t_1 + t_3) + t_3(t_1 + t_2)]A(T),$$

which completes the proof of (4.28).

We recall that A(T, -T) = 2A(T), $A(H_2, -H_2) = 2A(T)(1+t_1t_2+t_2t_3+t_3t_1)$, and observe that $A(H_2, -T) = A(T, -H_2) = A(T, -T)$ by the symmetry and rigid motion invariance of the mixed area and Minkowski's formula (2.1). Thus (4.28) and the linearity of the mixed area imply

$$A(D, -D) = A(D_0, -D_0) + A(T)(t_1t_2 + t_2t_3 + t_1t_3)$$

$$= A\left(\frac{1}{2}T + \frac{1}{2}H_2, -\frac{1}{2}T - \frac{1}{2}H_2\right) + A(T)(t_1t_2 + t_2t_3 + t_1t_3)$$

$$= \frac{1}{4} \cdot 2A(T) + \frac{2}{4} \cdot 2A(T) + \frac{1}{4} \cdot 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1)$$

$$+ A(T)(t_1t_2 + t_2t_3 + t_1t_3)$$

$$= 2A(T) + \frac{3}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1).$$

We deduce from (4.18), $K \subset D$ (cf. (4.27)), (4.25) and $A(T) \ge 3/2$ that

$$5\varepsilon \ge A(H_2, -H_2) - A(K, -K) \ge A(H_2, -H_2) - A(D, -D)$$

$$= 2A(T)(1 + t_1t_2 + t_2t_3 + t_3t_1) - 2A(T) - \frac{3}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1)$$

$$= \frac{1}{2}A(T)(t_1t_2 + t_2t_3 + t_3t_1) \ge \frac{1}{2} \cdot \frac{3}{2} \cdot 2 \cdot (86 \cdot \gamma)\sqrt{\varepsilon^2} \ge \gamma^2 \varepsilon,$$

which is a contradiction, proving that (4.24) does not hold. Therefore the argument in Step 3 proves Proposition 4.1.

5 Proof of Theorem 1.3

Before starting the actual proof of Theorem 1.3, we recall Proposition 5.1 and Lemma 5.2 proved in essence by Betke and Weil [5] as Lemma 1 and Lemma 2 in [5]. We slightly modified the argument from of (2.4) and added an observation which turns out to be useful.

Proposition 5.1 (Betke, Weil (1991)). If P is a polygon with $k \ge 3$ sides, and P is not a regular polygon with an odd number of sides, then there exists a polygon P' with k sides and arbitrarily close to P such that

$$\frac{L(P')^2}{A(P',-P')} < \frac{L(P)^2}{A(P,-P)}.$$

Proof. For a vertex v of the polygon P, we write $N_P(v)$ for the normal cone of P at v; namely, if $u_1, u_2 \in \mathcal{U}(P)$ are the exterior unit normals of the two sides meeting at v, then $N_P(v) = \text{pos}\{u_1, u_2\}$.

Case 1 There exist vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$.

In this case, P is not a triangle. Let v_3, v_4 be the neighbors of v_1 . For a $v_1' \in [v_1, v_4] \setminus \{v_1, v_4\}$, let P' be obtained from P by replacing the vertex v_1 by v_1' . In particular, there exists a unit vector $w \in -\text{int } N_P(v_1)$ orthogonal to $[v_1', v_3]$. Using this vector w in (2.4) and the property $-N_P(v_1) \subset N_P(v_2)$, we get A(P', -P') = A(P, -P), while obviously L(P') < L(P) by strict containment.

Case 2 There exist no vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$ as in Case 1, but there exist a vertex v and a side e with exterior normal $u_0 \in \mathcal{U}(P)$ such that $-u_0 \in N_P(v)$ and $-u_0$ does not halve the angle of $N_P(v)$.

In this case, P does not have any parallel sides, since we are not in Case 1. The line ℓ through v parallel to e is a support line of P. Let v_4 denote the point preceding v and v_3 the point following v on ∂P in the clockwise order. Let ν_0 denote the unit vector orthogonal to u_0 such that $(v_4-v)/\|v_4-v\|, (v_3-v)/\|v_3-v\|, \nu_0$ are in counter-clockwise order on the unit circle. Let α_4 denote the angle enclosed by v_4-v and $-\nu_0$ and α_3 denote the angle enclosed by v_3-v and ν_0 . Let u_i denote the exterior unit normal of $[v_i,v]$ for i=3,4. Since $-u_0$ does not halve the angle enclosed by u_3,u_4 , we have $\alpha_3\neq\alpha_4$. We may assume that $\alpha_4>\alpha_3$. By Fermat's principle, moving v along ℓ an arbitrarily small amount in the direction of ν_0 to v' and denoting by P' the polygon obtained from P by replacing v by v', we get L(P')< L(P). Clearly, we thus still get a convex k-gon if v' is sufficiently close to v.

To prove A(P, -P) = A(P', -P'), we denote by Q the (nonempty) convex hull of the (common) vertices of P and P' (thus v, v' are removed). Note that in the case where P (and hence also P') is a triangle, Q is a segment. We consider the triangles $\triangle = [v_3, v_4, v]$ and

 $\triangle' = [v_3, v_4, v']$, hence we have $P = Q \cup \triangle$ and $P' = Q \cup \triangle'$. For the segment $I = [v_3, v_4]$ we have $A(\triangle, -I) = A(\triangle, I) = A(\triangle)$. Using this and the additivity of the mixed area in both arguments, we obtain

$$A(P, -P) = A(Q, -Q) + 2A(Q, -\Delta) - 2A(Q, -I).$$

A similar expression is obtained for P' with \triangle replaced by \triangle' .

We choose v_5, v_6 such that $e = [v_5, v_6]$ and $v_6 - v_5$ is a positive multiple of v' - v. If v' is sufficiently close to v, then the first assumption in Claim 2 ensures that for the exterior unit normals u_i of the sides of Q between v_4 and v_5 (in counter-clockwise order), we have

$$h_{\triangle}(u_i) = \langle v_5, u_i \rangle = h_{\triangle'}(u_i)$$

and that for the exterior unit normals u_j of the sides of Q between v_6 and v_3 (in counter-clockwise order), we have

$$h_{\triangle}(u_j) = \langle v_6, u_j \rangle = h_{\triangle'}(u_j).$$

But then (writing e also for the length of the edge e)

$$A(Q, -\Delta) = A(Q, -\Delta') = e\langle u_0, -v \rangle - e\langle u_0, -v' \rangle = e\langle u_0, v' - v \rangle = 0,$$

which proves the assertion.

Case 3 There exist no vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$, and for any vertex v and side e with exterior normal $u_0 \in \mathcal{U}(P)$ such that $-u_0 \in N_P(v)$ the vector u_0 halves the angle of $N_P(v)$, but P is not a regular polygon with an odd number of sides.

Let u_1, \ldots, u_k be the exterior unit normals to the sides of P in clockwise order. We observe that no closed half plane having the origin on its boundary contains u_1, \ldots, u_k . We set $u_{i+pk} = u_i$ for any $i = 1, \ldots, k$ and $p \in \mathbb{Z}$.

Since there exist no vertices v_1 and v_2 of P such that $-N_P(v_1) \subset N_P(v_2)$, we have $u_i \neq -u_j$ for any i, j, and there exists an $m \in \{2, \ldots, k-1\}$ such that $-u_i \in \text{int pos}\{u_{i+m-1}, u_{i+m}\}$ for any i. The conditions in Case 3 imply that $-u_i$ actually halves the angle $\angle(u_{i+m-1}, u_{i+m})$ between u_{i+m-1} and u_{i+m} .

Let us assume that $\angle(u_m, u_{m+1})$ is minimal among the angles of the from $\angle(u_j, u_{j+1})$. We deduce that no u_j lies in int $pos\{u_1, -u_m\} \subset pos\{-u_m, -u_{m+1}\}$; therefore, $u_1 = u_{2m}$. We conclude that k = 2m - 1, and hence k is odd.

As $-u_i$ halves $\angle(u_{i+m-1}, u_{i+m})$ for any i, and $-u_{i+m-1}$ halves $\angle(u_{i-1}, u_i)$, we deduce that $\angle(u_{i+m-1}, u_{i+m}) = \angle(u_{i-1}, u_i)$ for any i. In turn, we conclude that all exterior angles of P are $2\pi/k$.

For any fixed $i \in \mathbb{Z}$, we write e_i to denote both the side corresponding to u_i of P and its length, and ℓ_i to denote the line containing e_i . Since P is not regular, the side lengths of P are not the same; therefore, there exists $1 \le p < q \le k$ such that

$$e_{p-1} + e_p \neq e_{q-1} + e_q. (5.1)$$

Fixing $i \in \{1, ..., k\}$, if |t| is small, then let $P_{i,t}$ be the k-gon bounded by the lines ℓ_j , $j \in \{1, ..., k\} \setminus \{i\}$ and $\ell_i + tu_i$. It follows that

$$L(P_{i,t}) = L(P) + 2t \cdot \left(\frac{1}{\sin\frac{2\pi}{k}} - \frac{1}{\tan\frac{2\pi}{k}}\right) = L(P) + \kappa \cdot t$$

and, choosing $w = -u_i$ in (2.4), we see that

$$A(P_{i,t}, -P_{i,t}) = A(P, -P) + (e_{i+m} + e_{i+m-1}) \cdot \frac{t}{\sin \frac{2\pi}{k}} \cdot \sin \frac{\pi}{k}$$
$$= A(P, -P) + (e_{i+m} + e_{i+m-1}) \cdot \varrho \cdot t.$$

where $\kappa=2\frac{1-\cos\frac{2\pi}{k}}{\sin\frac{2\pi}{k}}$ and $\varrho=\frac{\sin\frac{\pi}{k}}{\sin\frac{2\pi}{k}}.$ We deduce that

$$\frac{d}{dt} \frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} \bigg|_{t=0} = \frac{(e_{i+m} + e_{i+m-1}) \cdot \varrho \cdot L(P) - A(P, -P) \cdot 2\gamma}{L(P)^3}.$$
 (5.2)

We conclude from (5.1) and (5.2) the existence of some $i \in \{1, ..., k\}$ such that

$$\frac{d}{dt} \frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} \bigg|_{t=0} \neq 0.$$

In particular, we can choose a $t \neq 0$ with arbitrarily small absolute value such that

$$\frac{A(P_{i,t}, -P_{i,t})}{L(P_{i,t})^2} > \frac{A(P, -P)}{L(P)^2},$$

and hence we can choose $P' = P_{i,t}$, completing the proof of Proposition 5.1.

For regular polygons with an odd number of sides, we have the following estimates.

Lemma 5.2. If P is a regular polygon with an odd number $k \geq 5$ of sides, then

$$d_{\rm tr}(P) > 0.25$$
 (5.3)

$$\frac{L(P)^2}{A(P, -P)} \ge 20\sin\frac{\pi}{5} > 1.1 \cdot 6\sqrt{3}.$$
 (5.4)

Proof. We may assume that L(P)=1, and hence $A(P,-P)=(4k\sin\frac{\pi}{k})^{-1}$ according to (2.2), proving (5.4). For (5.3), assuming that the origin is the centroid of P, we have $-P\subset \left(\cos\frac{\pi}{k}\right)^{-1}P$, and hence $A(P,-P)\leq \left(\cos\frac{\pi}{k}\right)^{-1}A(P)$. On the other hand, if $T_0\subset P$ is a regular triangle with centroid z_0 and $P-z_0\subset (1+d)(T_0-z_0)$, then

$$A(P, -P) \ge A(T_0, -T_0) = 2A(T_0) \ge 2(1+d)^{-2}A(P),$$

and hence $d_{\rm tr}(P) \ge \sqrt{2\cos\frac{\pi}{5}} - 1 > 0.25$.

Proof of Theorem 1.3. Let $\varepsilon \in [0, 2^{-28}]$ and let $K \subset \mathbb{R}^2$ be a convex domain with $L(K)^2 \le (1+\varepsilon) \, 6\sqrt{3} \, A(K,-K)$. Then $0 \le \varepsilon < (6\cdot 2400)^{-2} < (6\cdot 180)^{-2}$. We distinguish two cases.

Case 1: $d_{\rm tr}(K) \le 6^{-2}$. Then Proposition 4.1 implies that $d_{\rm tr}(K) \le 400\sqrt{\varepsilon}$, and the proof is finished.

Case 2: $d_{\rm tr}(K) > 6^{-2}$. We will show that in fact this case does not occur. We fix a number ε' with $0 \le \varepsilon < \varepsilon' < (6 \cdot 2400)^{-2}$. Since $d_{\rm tr}(\cdot), A(\cdot, \cdot), L(\cdot)$ are continuous, there is a polygon $P \subset \mathbb{R}^2$ with $d_{\rm tr}(P) > 6^{-2}$ and $L(P)^2 \le (1 + \varepsilon') 6\sqrt{3} A(P, -P)$. Since $d_{\rm tr}(\cdot), A(\cdot, \cdot), L(\cdot)$ are translation invariant, $d_{\rm tr}(\cdot)$ is scaling invariant and $K \mapsto L^2(K)/A(K, -K)$ (for convex domains $K \subset \mathbb{R}^2$) is also scaling invariant, we can assume that $o \in P$ and A(P, -P) = 1. Let

 $k \geq 3$ be the number of vertices of P. We write \mathcal{P}_k for the set of all polygons $G \subset \mathbb{R}^2$ with at most k vertices, $d_{\mathrm{tr}}(G) \geq 6^{-2}$, $o \in G$, A(G, -G) = 1 and $L(G)^2 \leq (1 + \varepsilon') 6\sqrt{3}$. Then in particular we have $P \in \mathcal{P}_k \neq \emptyset$.

We claim that there is some $P_0 \in \mathcal{P}_k$ such that $L(P_0) = \inf\{L(G) : G \in \mathcal{P}_k\}$. For the proof, it is sufficient to consider a minimizing sequence $P_i \in \mathcal{P}_k$ with $L(P_i) \leq L(P)$ for $i \in \mathbb{N}$. Let B^2 denote the unit disc with center at the origin. Then $P_i \subset L(P)B^2$ for $i \in \mathbb{N}$. An application of Blaschke's selection theorem shows that the sequence P_i , $i \in \mathbb{N}$, has a convergent subsequence with limit $P_0 \subset \mathbb{R}^2$. Since all conditions involved in the definition of \mathcal{P}_k are preserved under limits and $L(\cdot)$ is continuous, we conclude that $P_0 \in \mathcal{P}_k$ realizes the infimum.

Since $d_{\rm tr}(P_0) \ge 6^{-2} > 0$, P_0 is not a regular triangle. Assuming (for the moment) that P_0 is a regular r-gon with an odd number r, we have $k \ge r \ge 5$. Then Lemma 5.2 shows that $L(P_0)^2 \ge 1.1 \cdot 6\sqrt{3}$. Since also $L(P_0)^2 \le (1+\varepsilon')6\sqrt{3}$, we get $\varepsilon' \ge 0.1$, a contradiction.

Hence $P_0 \in \mathcal{P}_k$ is a k-gon, but not a regular polygon with an odd number of edges. Assume (for the moment) that $d_{\rm tr}(P_0) > 6^{-2}$. Proposition 5.1 then shows that there is a k-gon P_1 such that $d_{\rm tr}(P_1) > 6^{-2}$ and

$$\frac{L(P_1)^2}{A(P_1, -P_1)} < \frac{L(P_0)^2}{A(P_0, -P_0)} = L(P_0)^2.$$

Again by scaling and translation invariance, we obtain a k-gon P_2 for which $d_{\rm tr}(P_2) > 6^{-2}$, $A(P_2, -P_2) = 1$, $o \in P_2$ and $L(P_2)^2 < L(P_0)^2 \le (1+\varepsilon')\,6\sqrt{3}$, that is, $P_2 \in \mathcal{P}_k$. But this contradicts the minimality of $L(P_0)$. Therefore we conclude that $d_{\rm tr}(P_0) = 6^{-2}$. Since $L(P_0)^2 \le (1+\varepsilon')\,6\sqrt{3}\,A(P_0, -P_0)$, it follows from another application of Proposition 4.1 that

$$6^{-2} = d_{tr}(P_0) \le 400\sqrt{\varepsilon'} < 400 \cdot (6 \cdot 2400)^{-1} = 6^{-2},$$

a contradiction. This finally shows that the present case does not occur, which completes the argument. \Box

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