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A DEDUCTIVE REASONING SYSTEM ON THE BASIS OF A NONMONOTONIC LOGIC

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Abstract. This paper presents a deductive reasoning system vs. a set of default theories. Syntactical and semantical aspects of a nonmonotonic logic is considered that provide the background for the deductive reasoning system.

1. Introduction. Nonmonotonicity is the main feature in commonsense reasoning. The statement "Birds fly" is usually given to explain the nonmonotonicity. McDermott and Doyle /1980/ outlines an approach to modeling nonmonotonic reasoning system, McDermott/1982/, Reiter/1980/, Reiter and Crisuolo/1981/, Moore/1983/, Lukaszewicz/1983/ are of much interests in that direction. Various interpretations were made, each gave a specific semantics for a deductive reasoning system. Therefore, it turns out that nonmonotonic logic should be context-sensitive - the set of beliefs of a theory deperds on the determination of a set of axioms for this theory. This paper presents a compromised approach which simultaneously aims to investigate proof-theoretic and model-theoretic aspects of a nonmonotonic logic - modal operators M, L are combined in a single framework of S5-nonmonotonic logic together with a set of default theories. The main intuition is the restriction on the set of needed assumptions when specifying nonmonotonic theorems for a theory. The Computational basis for this deductive system is fixed point properties of an algebraic operator that defines a default theory.

2. Syntactical considerations

Default theories are treated within the framework of propositional language for simplicity sake, after introducing a set of logical axioms and two monotonic inference rules, the nonmonotonic theorems are recognised by terms of modal operators.

2.1. Concepts, definitions and notations.

Definition 2.1.1. Given a classical propositional language Lang /Mendelson - 1965/ which contains:

- a set of proposition letters,
- the set of connectives: $\wedge$ (and), $\vee$ (or), $\sim$ (not), $\Leftrightarrow$ if and only if, ( ) brackets, $\supset$ implication.

to Lang, we attach a modal M "it is consistent", Lang now is usual modal propositional language.

Definition 2.1.2. A term is a constant symbol, a predicate symbol, or an expression $f(t_1,\ldots,t_n)$, where f is a function symbol and $t_1,\ldots,t_n$ are terms.

An atomic formula is an expression $p(t_1,\ldots t_n)$ where p is a predicate symbol and $t_1,\ldots,t_n$ are terms.

A formula is either:

- a proposition letter,
- an expression $\sim p$, where p is an atomic formula,
- $p \supset q$, where p, q are formulas.

Definition 2.1.3. A formula of the form

$$p \wedge Mq_1 \wedge \ldots \wedge Mq_n \supset r$$

or simply

$$Mq_1 \wedge \ldots \wedge Mq_n \supset r$$

where $p, ql, \ldots, q_n, r$ belong to the classical propositional calculus is named a default.

Definition 2.1.4. A default theory A is a set of formulas together with a set of non-logical axioms of that theory. Each non-logical axiom either belongs to propositional calculus or is a default.

We attach the second modal operator L to Lang, and in the following, L is interpreted as " It is believed ".

Definition 2.1.5. Let p, q, r be formulas in a default theory A. Logical axioms schemata is defined as follows:

/la1/ $Lp \supset p$

/la2/ $Mp \supset LMp$

/la3/ $L(p \supset q) \supset (Lp \supset Lq)$

/la4/ $(p \supset (q \supset p)$

/la5/ $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$

/la6/ $(\sim q \supset \sim p) \supset ((q \supset p) \supset q)$

Monotonic inference rules:

/mr1/ $p, p \supset q \vdash q$ / modus ponens /

/mr2/ $p \vdash Lp$ / necessitation /

where " $\vdash$ " is understood in an ordinary monotonic sense as provability: let S be a set of formulas of a default theory, if p S is provable from S and instances of /la1/ - /la6/ and by application of mr1 and mr2, we denote $S \vdash p$. If not, $S \nvdash p$.

From McDermott and Doyle /1980/, we have

$$Th(S) = \{p: S \vdash p\}$$

It is easy to see that Th has the monotonicity:

/i/ $A \subseteq Th(A)$

/ii/ Let A, B be two default theories, from $A \subseteq B$ we have $Th(A) \subseteq Th(B)$.

/iii/ $Th(Th(A)) = Th(A)$ /idempotence/

The last property of Th can also be viewed as fixed point equation, stating that the set of theorems monotonically derivable from a default theory is a fixed point of the operator which computes the closure of a set of formulas under the monotonic inference rules.

Definition 2.1.6. Let S be a set of formulas. S is consistent if and only if $S \nvdash p$ for only some $p \in S$. A default theory is consistent if and only if its non-logical axioms are consistent.

The above monotonic structure is identical to S5 modal propositional logic /see Hughes and Cresswell, 1972/. In the logical axiom schemata, /la1/ means that everything believable is true, /la2/ shows that p is unprovable only if it provable only if it is provably unprovable, this assertion is useful in nonmonotonic system, /la3/ describes behaviour of modus ponens: it allows to infer q from p   q and p, where modus ponens is activated. The instances of the last three axioms /la4/-/la6/ form the axiomatisation for the sentential calculus.

In the following, we settle up the nonmonotonic structure of our default theories, a set of assumptions is added to a default theory by the usual way

Definition 2.1.7. Let d be a default, a formula of the form

$Mq_1 \wedge \dots \wedge Mq_n$ or simply Mq

is called an assumption of d, and is denoted M d ..

Definition 2.1.8. Let d be a default. Condition for d, denoted by cond d , is defined as follows

$$\mathrm{cond}(d) = \begin{cases} p & \text{if } d = p \wedge Mq_1 \wedge \dots \wedge Mq_n \supset r \\ p \vee \sim p & \text{if } d = Mq_1 \wedge \dots \wedge Mq_n \supset r \end{cases}$$

Comment. We give here the similar definition with the ones in Moore /1983/ about objective /resp. subjective/ inference in which we mixture objective and subjective inferences, but define for mixed inferences a condition /in definition 2.1.8/, this serves for convenience of some forms of proof later.

Definition 2.1.9. Let S be a set of formulas, the set of assumptions for S, denoted as As S,d, is defined as

$$As(S,d) = \begin{cases} \{M(d)\} & \text{if } \mathrm{cond}(d) \in S \text{ and } S \cup \{M(d)\} \text{ is consistent} \\ \emptyset & \text{if otherwise} \end{cases}$$

<u>Definition 2.1.10.</u> The set of assumptions for a default theory A, denoted by As_A S , is defined as:

$$As_A(S) = \bigcup_{d \in \text{def } A} As(S, d)$$

where def A denotes the set of all default of A.

<u>Defimition 2.1.11.</u> Let A be a default theory and S be any set of formulas. We define operator NM_A as follows

$$NM_A(S) = Th(A \cup As_A(S))$$

Before giving a definition of the special extension, we consider an example belows to clarify some intuitive idea supporting that definition

<u>Example 2.1.12.</u> Consider the theory

$$A = \{ p \wedge Mq \supset q, (p \supset q) \wedge Mp \supset p \}$$

There are two fixed points with respect to NM_A : Th(A) and $Th(A \cup \{Mp, Mq\})$. There exists only one extension for A, which is Th(A) , because we have no reason to believe p or $p \supset q$, so it results in the fact that none of the default of A can be activated. The available way to avoid such situations is that by analogy with the monotonic case, we should treate extensions for a default theory A as minimal fixed points of NM_A. We come to the following definition

<u>Definition 2.1.13.</u> Let A be a default theory. A set S of formulas is called a minimal extension for A if and only if S is a minimal fixed point with respect to NM_A, i.e., S is minimal set of formulas such that

$$S = NM_A(S) = Th(A \cup As_A(S))$$

The above definition naturally leads to the following definition of beliefs.

Definition 2.1.14. Let A be a default theory. The intersection of all minimal extension for A is called the set of beliefs derivable from A and is denoted by TH(A).

We have the following theorem.

Theorem 2.1.15. There exists a minimal extension for every default theory A.

Proof. **In the case the default theory A is inconsistent** it is clear that the set of all formulas becomes the only minimal extension for A. With this, now on we may suppose that A is consistent.Our treatment now is to build up a minimal extension for A.

Consider an arbitrary sequence of defaults of A: (d_j) .From this sequence we define a sequence(S_i) by the following manner

Put

$$S_1 = Th(A)$$

From a given S_i we define

$$S_i^1 = S_i$$

$$S_i^{J+1} = S_i^j \cup As(S_i^j, d_j)$$

Put

$$S = \bigcup_{i=1}^{\infty} S_i$$

It is easy to see that $S_i^1 \subseteq S_i^2 \subseteq \ldots$

We prove that S is a minimal extension for A, i.e., S is minimal set of formulas and

$$S = Th\left(A \cup As_A(S)\right)$$

S is consistent by induction on i, and also by induction on i, we have

$$S_i \subseteq Th(A \cup As_A(S))$$

which immediately leads to

$$S \subseteq Th(A \cup As_A(S)) \qquad (1)$$

Let $p \in A \cup As_A(S)$. With some $d_k \in def(A)$, we have $p \in As(S, d_k)$. By definition 2.1.9 we have $cond(d_k) \in S$ and $S \cup \{p\}$ is consistent. It implies that for some natural m, $cond(d_m) \in S_m$, furthermore, we have $cond(d_k) \in S_m^k$ because $S_m \subseteq S_m^k$. By the construction of S, we have $S_m^k \subseteq S$. Hence $S_m^k \cup \{p\}$ is consistent. From here we have

$$p \in S_m^{k+1} \subseteq S_{m+1} \subseteq S$$

It implies that

$$A \cup As_A(S) \subseteq S \qquad (2)$$

By definition of Th, we have $S \subseteq Th(S)$.
Let $p \in S$, with $S = \bigcup_{i=1}^{\infty} S_i$, thus

$$\bigcup_{i=1}^{\infty} S \vdash p$$

because $S_o \subseteq S_1 \subseteq \ldots$ and for some natural m, we get $S_m \vdash p$. It implies that $p \in Th(S_m) \subseteq S_{m+1}$.

Altogether we get $p \in S$. So

$$S = Th(S) \qquad (3)$$

From (1), (2), (3) we obtain

$$S = Th(A \cup As_A(S)) \quad (4)$$

In the rest, we show that S is minimal fixed point.

Suppose that there is a fixed point S_x such that $S_x \subseteq S$. We have $S_i \subseteq S_x$ by the result of induction on i, it implies that $S \subseteq S_x$. Thus $S = S_x$.

This completes the proof of the theorem 2.1.15.

Example 2.1.16./Reiter, Criscuolo, 1981/.

Consider theory $C = \{p \wedge Mr \supset r,\ q \wedge M(\sim p \wedge \sim r) \supset r\}$. For this theory, we have three possibilities: if p is given, then it is consistent to infer r; if q is given, then it is consistent to infer $\sim r$; if p, q are given simultaneously, it is consistent to infer r. Suppose q is is given, we then have theory $C_q = \{A \cup \{q\}\}$ and its extension $Th(C_q \cup \{M(\sim p \wedge \sim r)\})$. If we add the assumption Mr to A, we then get two extensions for C, which are: $Th(C_q \cup \{M(\sim p \wedge \sim r)\})$ and $Th(C_q \cup \{Mr\})$. The second one contains the formula of the form $p \supset r$ that contradicts to the given conditions, thus we can not accept it in reality.

This example put forwards the fact that when considering a default theory, it is strictly necessary to give attention to those assumptions which are needed for drawing available conclusions.

Notation 2.1.17. The set of needed assumptions for a default theory A is denoted as NA A and

$$NA(A) = \bigcup_{d \in def A} \{M(d)\}$$

3. Semantical considerations.

3.1. definitions

<u>Definition 3.1.1</u>. Model is a tuple $M = \langle W, f \rangle$ where W is a nonempty set of possible worlds and f is a function from the set of all proposition letters of Lang to 2^W.

<u>Definition 3.1.2</u>. Let $p \in M$. The truth value for p with respect to $w \in W$, denoted by $u(w,p)$,is defined by mapping v:

$$v: \; W \times S \longrightarrow \{0,1\}$$

so that

$$/t1/ \; v(w,a) = \begin{cases} 1 & \text{iff } w \in f(a) \\ 0 & \text{otherwise} \end{cases}$$

where a is an arbitrary proposition letter in Lang.

$$/t2/ \; v(w, \sim p) = 1 - v(w, p)$$

$$/t3/ \; v(w, p \supset q) = \begin{cases} 1 & \text{iff } v(w,p) = 0 \text{ or } v(w,q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$/t4/ \; v(w, Mq) = \begin{cases} 1 & \text{iff } v(w_x,p) = 1 \text{ for some } w_x \in W \\ 0 & \text{otherwise} \end{cases}$$

<u>Definition 3.1.3</u>. Let S be a set of formulas, p is true in M, denoted by $M \models p$, if and only if $v(w,p) = 1$ for every $w \in W$.

<u>Definition 3.1.4</u>. Let S be a set of formula. S is true in M if and only if $M \models p$ for every $p \in S$. In this case we call that M is a model for S.

Definition 3.1.5. Let A be default theory. A set $X \subseteq NA(A)$ is called an activation set of a set $Y \subseteq def\ A$ if and only if the following conditions are satisfied:

/act1/ $A \cup X$ is consistent.
/act2/ $Y = \{d \in def(A): cond(d) \wedge M(d) \in Th(A \cup X)\}$.
/act3/ if $p \in X$, then $p = M(d)$ for some $d \in Y$.
/act4/ for every $d_1 \in def(A) - Y$
$cond(d_1) \in Th(A \cup X)$ or
$Th(A \cup X) \cup \{M(d_1)\}$ is inconsistent.

Definition 3.1.6. A set $X \subseteq NA\ A$ is called a minimal activation set of a set $Y \subseteq def\ A$ if and only if the following conditions are fulfiled:

/ma1/ X is an activation set of Y by definition 3.1.5
/ma2/ There is no activation set of any $Y_1 \subsetneq X$

In the case $Y \subseteq def\ A$ satisfies /ma1/ and /ma2/, we call Y minimally activable.

Definition 3.1.7. Let M be a model for a default theory A. M is called a minimal model for A if and only if M is a model for a minimal activation set $X \subseteq NA\ A$ of a set $Y \subseteq def\ A$.

3.2. Some results.

Theorem 3.2.1. Let A be default theory and suppose that $X \subseteq NA(A)$ is a minimal activation set of a set $Y \subseteq def(A)$. Then $Th(A \cup X)$ is a minimal fixed point with respect to operator NM_A.

Proof. By definition 2.1.13, we have to prove that

$$Th(A \cup X) = Th(A \cup As_A(A \cup X)) \qquad (5)$$

Firstly we prove

$$As_A(Th(A \cup X)) \subseteq Th(A \cup X)$$

Let $p \in As_A(Th(A \cup X))$. There is $d \in def\ A$ such that $p = M(d)$ $cond(d) \in Th(A \cup X)$ and $Th(A \cup X) \cup \{M(d)\}$ is consistent. By the theorem's hypothesis X is an activation set of Y, hence by /act4/ we get $p \in Y$. By /act2/ we have moreover $M(d) \in Th(A \cup X)$, because $p = M(d)$, so $p \in Th(A \cup X)$.

We prove now that $X \subseteq As_A(Th(A \cup X))$.

Let $p \in X$. Because X is an activation set of Y and by /act3/, we have $p = M(d)$ for some $d \in Y \subseteq def(A)$. By /act2/, $cond(d) \wedge M(d) \in Th(A \cup X)$. Therefore, $Th(A \cup X) \cup \{M(d)\}$ is consistent by /act1/ and /act2/. Thus $M(d) \in As_A(Th(A \cup X))$. As $p = M(d)$, so $p \in As_A(Th(A \cup X))$ which completes the proof of (5).

Let $Z \subseteq NA(A)$ be a fixed point of NM_A and suppose that Z is consistent. Consider $As_A(Z)$, we have

$$As_A(Z) = \bigcup_{d \in def(A)} As_A(Z, d)$$

by verifying through /act1/ - /act4/ we conclude that $As_A(Z)$ is an activation set of the set

$$\{d \in def(A) : cond(d) \wedge M(d) \in Th(A \cup As_A(Z))\}$$

Suppose that $Z \subseteq Th(A \cup X)$. We have

$$Th(A \cup As_A(Z)) \subseteq Th(A \cup X) \qquad (6)$$

Denote

$$Y_1 = \{d \in def(A) : cond(d) \wedge M(d) \in Th(A \cup As_A(Z))\}$$

$$Y = \{d \in def(A) : cond(d) \wedge M(d) \in Th(A \cup X)\}$$

From (6) we have $Y_1 \subseteq Y$. But Y is also a minimal activation set, so by (3) / in Thorem 2.1.15 / we get $Y = Y_1$.

It is clear that $Z \supseteq Th(A \cup X)$ because from $Y = Y_1$ we can naturally take $Y \subseteq Y_1$.

The Theorem 3.2.1. is proved.

Theorem 3.2.2 /Completeness theorem/ Let A be a default theory and p be any formula. Then $p \in TH(S)$ if and only if p is true in every minimal model for A.

Proof. The belows lemmae immediatey lead to the completeness theorem.

Lemma 3.2.2.1. /McDermott - 1982, pp.39-40/ Let S be a set of formulas and p be an arbitrary formula in S. Then $p \in TH(S)$ if and only if $M \models S$ for every model for S.

Lemma 3.2.2.2. Let A be a default theory ans Z be a fixed point with respect to the operator NM_A. Then every model M for Z is a minimal model for A.

Proof. Let M be a model for Z. It is clear that every model for Z is also a model for A. Because $Z = TH(A \cup As_A(Z))$ M is a model for Z, so M is model for $As_A(Z)$. By (3) /in Theorem 2.1.15 / $As_A(Z)$ is an activation set for

$$Y_1 = \{ d \in def(A) : cond(d) \wedge M(d) \in Th(A \cup As_A(Z)) \}$$

In the rest, it suffices to prove that M is a model for a set $X \subseteq NA(A)$ which is an minimal activation set of Y.

Suppose that some set $X \subseteq NA(A)$ is an minimal activation set of a set $Y_1 \subseteq Y$.

Aimimg to prove that X is minimal activation set for Y, we show that

$$Th(A \cup X) \subseteq Th(A \cup As_A(Z)) \qquad (7)$$

To prove (7) it is equivalent to prove

$$X \subseteq Th \left(A \cup As_A (Z) \right)$$

Assume that $p \in X$. For some $d \in Y_1$, we have $p = M(d)$ /by /act3/ /, as $Y_1 \subseteq Y$ we have $p \in Y$. Moreover we have $p \in Th \left(A \cup As_A(Z)\right)$, this completes the proff of (7).

From 4 , we have

$$Th \left(A \cup X\right) = Th \left(A \cup As_A (Z)\right) \qquad (8)$$

(8) together with $Y_1 \subseteq Y$ implies that $Y_1 = Y$. This finishes the proof of Lemma 3.2.2.2.

Lemma 3.2.2.3. Let A be a default theory and p be any formula. Then p is true in each minimal model for A if and only p belongs to each minimal fixed point with respect to NM_A.

Proof. /If/ By applying Theorem 2.1.15 we immediately fulfil the "If" part.
/Only if/ This part is direct result of application of two Lemmae 3.2.2.1 and 3.2.2.2.

From lemma 3.2.2.3 we have directly the Completeness Theorem.

4. Conclusion. This paper shows a compromised approach to nonmonotonic reasoning system in comparison with those of McDermott, Doyle, Moore, Reiter and Lukaszewicz : we treate simultaneously two modal operators M and L which allows to consider not only in the light of proof-theoretic but also of model-theoretic aspects, furthermore default theories are manipulated here with the intuitive idea that

every time when a theory is activated, the set of assumptions is carefully considered in order to provide plausible conclusions. We shows the context-sensitivity of our system. It should be noted that our nonmonotonic reasoning system is not semi-decisive, so some intuititions and heuristics are used in building this system - definition 3.5.1 and Theorem 3.5.15 are instances. Moreover, well-defined nonmonotonic theorems are derived from each default theory. Our approach, instead of competing the previous ones, is above all the completion of them.

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Summary

The paper presents a deductive reasoning system, where both syntactical and semantical aspects of a non-monotonic logic are considered.
Non-monotonicity is the main feature in commonsense reasoning. Many approaches to modelling non-monotonicity are known. The author presents a compromised approach which simultaneously aims to investigate proof-theoretic and model-theoretic aspects of non-monotonic logic.

EGY NEM-MONOTON LOGIKÁN ALAPULÓ DEDUKTIV KÖVETKEZTETÉSI RENDSZER

Ha Hoang Hop

Összefoglaló

A cikk egy deduktiv következtetési rendszert mutat be, amely a nem-monoton logikák mind szintaktikai, mind szemantikai aspektusain alapszik. A szerző a nem-monotonitásnak /amely a "józan következtetésnek" fő tulajdonsága/ egy kompromisszumos modelljét mutatja meg, amely a nem-monoton logikák mindkét tárgyalásának /modell-elméleti illetve bizonyitás-elméleti/ aspektusait felhasználja.