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STRONGLY DOMINATING SETS OF VARIABLES

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In this paper we introduce and investigate strongly dominating and regular dominating sets of variables for the functions.

DEFINITION 1.[2]. A function $f(x_1,\dots,x_n)$ is said to depend on the variable x_i, $1 \le i \le n$ if there exist $n-1$ constants $c_1,\dots,c_{i-1},c_{i+1},\dots,c_n$ for the variables $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ such that the function $f(x_1=c_1,\dots,x_{i-1}=c_{i-1},x_{i+1}=c_{i+1},\dots,x_n=c_n)$ assumes at least two different values. R_f denotes the set of all the variables on which f depends.

DEFINITION 2.[2]. A set M, $M \subseteq R_f$ is called separable for f with respect to the set $N = \{x_1,\dots,x_l\} \subseteq R_f$ if there exist l constants $c_1,\dots,c_l$ such that $M \subseteq R_{f(x_1=c_1,\dots,x_l=c_l)}$. $S_{f,N}$ denotes the set of all the separable sets for f with respect to the set N.

When M is separable for f with respect to $R_f \setminus M$ then M is called separable for f. S_f denotes the set of all the separable sets for f.

DEFINITION 3.[1]. A set M, $M \subseteq R_f$ is called c-separable for f with respect to the set $N = \{x_1,\dots,x_l\} \subseteq R_f$ iff for every l constants $c_1,\dots,c_l$, $M \subseteq R_{f(x_1=c_1,\dots,x_l=c_l)}$ holds true. $S^*_{f,N}$ denotes the set of all the c-separable sets for f with respect to the set N.

When M is c-separable for f with respect to $R_f \setminus M$ then M is called c-separable for f.

DEFINITION 4.[1]. A set $M = \{x_1,\dots,x_m\} \subseteq R_f$ is called dominating of N, $N \subseteq R_f$, ($M \xrightarrow{d} N$) if there exist m constants $c_1,\dots,c_m$ such that (*) $N \cap R_{f(x_1=c_1,\dots,x_m=c_m)} = \emptyset$ and M is minimal with respect to this property. When M satisfies (*) then it is called α-dominating of N, ($M \xrightarrow{\alpha d} N$).

$M \xrightarrow{\bar{d}} N$ denotes that M is not dominating of N for f and

$M \xrightarrow{\overline{\alpha d}} N$ denotes that M is not α-dominating of N for f.

DEFINITION 5.[3]. A set $P = \{x_1,\dots,x_p\} \subseteq R_f$ is called strongly dominating of Q , $Q \subseteq R_f$, ($P \xrightarrow{sd} Q$) if for f there exists an p-tuple $P^* = (c_1^*,\dots,c_p^*)$ such that P is dominating of Q with P^* and

$$\bigcup_{i=1}^{p} C_{x_i,Q} = Q \text{ where } C_{x_i,Q} = Q \setminus R_{f(x_i=c_i^*)} .$$

The set $C_{x_i,Q}$ is called active zone of x_i in Q. $P \xrightarrow{\overline{sd}} Q$ denotes that P is not strongly dominating of Q for f.

When $P \xrightarrow{sd} Q$ and for any i,j , $1 \le i,j \le p$

$$C_{x_i,Q} \cap C_{x_j,Q} = \emptyset$$

the set P is called regular dominating of Q for f ($P \xrightarrow{rd} Q$).

$P \xrightarrow{\overline{rd}} Q$ denotes that P is not regular dominating of Q for f.

We now present an example to illustrate these definitions.

EXAMPLE 1. Let $f = x_1x_5 + x_2x_5 + x_3x_5 + x_3x_5\bar{x}_6 + x_4x_6 \pmod 2$, $P = \{x_5,x_6\}$ and $Q = \{x_1,x_2,x_3,x_4\}$. It is obvious that $P \xrightarrow{sd} Q$ with $P^* = (0,0)$ and $C_{x_5,Q} = \{x_1,x_2,x_3\}$, $C_{x_6,Q} = \{x_4\}$. Since $C_{x_5,Q} \cap C_{x_6,Q} = \emptyset$ it follows that $P \xrightarrow{rd} Q$.

THEOREM 1. If $P \xrightarrow{sd} Q$ then for every $x_i \in P$, $C_{x_i,Q} \not\subseteq M$, where $M = \bigcup_{\substack{x_j \in P \\ x_j \ne x_i}} C_{x_j,Q}$.

PROOF. Obviously $P \xrightarrow{d} Q$. Without loss of generality assume that $P = \{x_1,\dots,x_p\}$, $Q = \{x_{p+1},\dots,x_q\}$, $p < q \le n$. Now suppose that the theorem is false and let for example

$$C_{x_1,Q} \subseteq C_{x_2,Q} \cup \dots \cup C_{x_s,Q} , \quad s \le p .$$

Consequently there exist s-1 constsnts $c_2^*,\dots,c_s^*$ such that

$$C_{x_1,Q} \cap R_{f(x_2=c_2^*,\dots,x_s=c_s^*)} = \emptyset.$$

This implies

$$Q \cap R_{f(x_2=c_2^*,\dots,x_p=c_p^*)} = \emptyset$$

for every p-s constants $c_{s+1}^*,\dots,c_p^*$ for the variables from the set.

$P \setminus (Q \cup \{x_1\})$. So we have $P \setminus \{x_1\} \xrightarrow{\alpha d} Q$ and $P \xrightarrow{\bar{d}} Q$. A contradiction. The theorem is proved.

COROLLARY 1. If $P \xrightarrow{sd} Q$ then for every $x_j, x_i \in P$, $x_i \neq x_j$

$$C_{x_i,Q} \not\subset C_{x_j,Q}.$$

COROLLARY 2. If $P \xrightarrow{sd} Q$ then for every $x_i \in P$, $C_{x_i,Q} \neq \emptyset$.

THEOREM 2. If $P \xrightarrow{sd} Q$ then $Card(P) \le Card(Q)$.

This theorem follows by Theorem 1.

$P \xleftrightarrow{sd} Q$ denotes that $P \xrightarrow{sd} Q$ and $Q \xrightarrow{sd} P$.

PROPOSITION 3. If $P \xleftrightarrow{sd} Q$ then $Card(P) = Card(Q)$.

THEOREM 4. If $P \xrightarrow{sd} Q$ and $Card(P) = Card(Q)$ then $P \xrightarrow{rd} Q$ and $Q \xrightarrow{d} Q$.

PROOF. By $P \xrightarrow{sd} Q$, $Card(P) = Card(Q)$ and Theorem 1 it follows that there exist $Card(P)$ different active zones in Q which contain at least one variable belonging to only one active zone. Hence for every $x_i \in P$ it is true $Card(C_{x_i,Q}) = 1$. This implies $P \xrightarrow{rd} Q$.

Now suppose that $Q \xrightarrow{\bar{d}} Q$. Obviously $Q \xrightarrow{\alpha d} Q$ and there exists a subset Q_1 of Q for which $Q_1 \xrightarrow{d} Q$. Without loss of generality assume that $P = \{x_1, \dots, x_p\}$, $Q = \{x_{p+1}, \dots, x_{2p}\}$, $2p \le n$ and $C_{x_i,Q} = \{x_{p+i}\}$ for $i=1,2,\dots,p$. Let $x_{p+t} \in Q \setminus Q_1$ for some $t \le p$. Consequently $C_{x_t,Q} \cap Q_1 = \emptyset$. As in Theorem 1 it follows that

$P \setminus \{x_t\} \xrightarrow{\alpha d} Q$ and $P \xrightarrow{\bar{d}} Q$. This contradiction completes the proof of the theorem.

COROLLARY. If $P \xrightarrow{sd} P$ then $P \xrightarrow{rd} P$.

In [1] the s-system of a family Ω of sets is introduced as follows: A set $\Sigma = \{x_1, \dots, x_t\}$ is called s-system of $\Omega = \{P_1, \dots, P_n\}$, $P_i \neq \emptyset$ if $\Sigma \cap P_i \neq \emptyset$, $i=1,2,\dots,n$ and for every $j \le t$ there exists $P_k \in \Omega$ such that $P_k \cap \Sigma = \{x_j\}$.

THEOREM 5. Let $P \xrightarrow{sd} Q$. The following two conditions are equivalent:

(i) $Card(P) = Card(Q)$

(ii) Q is an s-system of the family $\{C_{x_i,Q} | x_i \in P\}$.

PROOF. (ii)==>(i). Sinse Q is an s-system of $\{C_{x_i,Q} | x_i \in P\}$ it follows that for every $x_j \in Q$ there exists $x_i \in P$ such that $C_{x_i,Q} \cap Q = \{x_j\}$. By the definition of $C_{x_i,Q}$ it follows that $C_{x_i,Q} \subseteq Q$. Consequently $C_{x_i,Q} = \{x_j\}$. So every $x_j \in Q$ is an active zone of some $x_i \in P$. By Corollary 1 of Theorem 1 it follows that

$$Card(P) = Card(Q).$$

(i)==>(ii). Sinse $Card(P) = Card(Q)$ then as in the proof of Theorem 4 it follows that $Card(C_{x_i,Q}) = 1$ for every $x_i \in P$. So, any $x_j \in Q$ is an active zone of some $x_i \in P$. Hence Q is an s-system of the family $\{C_{x_i,Q} | x_i \in P\}$.

It is easy to prove that Q is unique s-system of the family $\{C_{x_i,Q} | x_i \in P\}$.

COROLLARY. If $P \xrightarrow{sd} Q$ and Q is an s-system of the family $\{C_{x_i,Q} | x_i \in P\}$ then $Q \xrightarrow{d} Q$.

EXAMPLE 2. Let $f = x_1x_4 + x_2x_3\bar{x}_4x_5 \pmod 2$, $P=\{x_4, x_5\}$, $Q=\{x_1, x_2, x_3\}$ and $P^*=(0,0)$. It is easy to see that $P \xrightarrow{sd} Q$, $P \xrightarrow{rd} Q$ but $P \xrightarrow{\bar{d}} P$ and $Q \xrightarrow{\bar{d}} Q$.

THEOREM 6. If $P=\{x_1, \ldots, x_p\} \subseteq R_f$ and $P \xrightarrow{d} P$ then $P \xrightarrow{rd} P$.

PROOF. Let x_i be an arbitrary variable from P. Obviously $\{x_i\}$ is dominating of $\{x_i\}$ for any constant c_i. Hence $x_i \in C_{x_i,P}$, $C_{x_i,P} \neq \emptyset$ and $P \subseteq \bigcup_{x_i \in P} C_{x_i,P}$.

On the other hand $\bigcup_{x_i \in P} C_{x_i,P} \subseteq P$. So, we obtain $P \xrightarrow{sd} P$ with any $P^*=(c_1^*, \ldots, c_p^*)$ and by Corollary of Theorem 4 it follows that $P \xrightarrow{rd} P$.

COROLLARY. If $P \xrightarrow{d} P$ then for each subset P_1 of P it is true $P_1 \xrightarrow{rd} P_1$.

THEOREM 7. If $P \xrightarrow{sd} Q$ and $P \cap Q = \emptyset$ then each non-empty subset P_1

of P is separable for f with respect to $P\setminus P_1$.

PROOF. Let $P = (x_1,\ldots,x_p)$, $P_1 = (x_1,\ldots,x_s)$, $s \le p$ and $P^* = (c^*_1,\ldots,c^*_p)$ be an p-tuple with which P is strongly dominating of Q for f.

Suppose the theorem is false i.e. $P_1 \notin S_{f,P\setminus P_1}$. Then for any p-s tuple $(c_{s+1},\ldots,c_p)$ it is true

$$P_1 \setminus R_{f(x_{s+1}=c_{s+1},\ldots,x_p=c_p)} \ne \emptyset .$$

Hence

$$P_2 = P_1 \setminus R_{f(x_{s+1}=c^*_{s+1},\ldots,x_p=c^*_p)} \ne \emptyset$$

Let $x_i \in P_2$. We have

$$x_i \notin R_{f(x_{s+1}=c^*_{s+1},\ldots,x_p=c^*_p)}$$

and

$$C_{x_i,Q} \cap R_{f(x_{s+1}=c^*_{s+1},\ldots,x_p=c^*_p)} = \emptyset .$$

This implies

$$C_{x_i,Q} \subseteq \bigcup_{k=s+1}^{p} C_{x_k,Q}.$$

This contradicts to Theorem 1.Consequently $P_1 \in S_{f,P\setminus P_1}$.The theorem is proved.

COROLLARY 1. If $P \xrightarrow{sd} Q$ and $P \cap Q = \emptyset$ then for any subset P_1 of P the set $(Q\setminus(\bigcup_{x_k\in P\setminus P_1} C_{x_k,Q})) \cup P_1$ is separable for f with respect to $P\setminus P_1$.

COROLLARY 2. If $P \xrightarrow{sd} Q$, $P_1, P_2 \subseteq P$ and $P_1 \cap P_2 = \emptyset$ then $P_1 \in S_{f,P_2}$.

THEOREM 8. Let $P=(x_1,\ldots,x_p) \subseteq R_f$ and $P^* = (c^*_1,\ldots,c^*_p)$. If $Q = R_f \setminus (T \cup P)$, where $T = R_{f_1}$ and

$$f_1 = f(x_1=c^*_1,\ldots,x_p=c^*_p)$$

then $T \in S^{*}_{f,Q}$.

PROOF. Suppose the theorem is false. Then there exist q constants $c_{i_1},\ldots,c_{i_q}$ for the variables in Q such that $T \not\subset R_{f_2}$, where

$$f_2=f(x_{i_1}=c_{i_1},\dots,x_{i_q}=c_{i_q}).$$

Without loss of generality assume that $x_r \in T\setminus R_{f_2}$. Obviously $Q \cap R_{f_1}=\emptyset$. Consequently for every q constants $d_{i_1},\dots,d_{i_q}$ it is true

$$f_1=f_1(x_{i_1}=d_{i_1},\dots,x_{i_q}=d_{i_q}).$$

This implies that

$$f_1=f_1(x_{i_1}=c_{i_1},\dots,x_{i_q}=c_{i_q}).$$

Now $x_r \notin R_{f_2}$ shows that $x_r \notin R_{f_1}$. Remember that $T = R_{f_1}$. Thus we have $x_r \notin T$. A contradiction. The theorem is proved.

COROLLARY 1. If $P\xrightarrow{sd}Q$ then $Q\setminus C_{x_i,Q} \in S^*_{f,C_{x_i,Q}}$ and $C_{x_i,Q}\in S^*_{f,M}$, where $M = Q\setminus C_{x_i,Q}$ for any $x_i \in P$.

COROLLARY 2. If $P\xrightarrow{rd}Q$ then for every $x_i,x_j \in P$, $i\neq j$ it is true $C_{x_j,Q}\in S^*_{f,C_{x_i,Q}}$.

THEOREM 9. If $P\xrightarrow{sd}Q$ then $P\xrightarrow{sd}Q\cup P$.

PROOF. Obviously $P\xrightarrow{d}Q\cup P$. Then for each variable $x_i\in P$ there exists an active zone of x_i in Q such that $C_{x_i,Q}\subseteq Q$. Thus we have $\{x_i\}\xrightarrow{d}C_{x_i,Q}$. By Corollary 2 of Theorem 7 it follows that $C_{x_i,M}=\{x_i\}\cup C_{x_i,Q}$ and $\bigcup_{x_i\in P} C_{x_i,M} = M$, where $M = P \cup Q$. The theorem is proved.

THEOREM 10. If $P\xrightarrow{sd}Q$ and $P \cap Q =\emptyset$ then for every $x_i\in P$ and for every $x_j\in C_{x_i,Q}$ there exists at least one m-set M, $Card(M)=m$ which is separable for f and $x_i,x_j\in M$.

This theorem follows by Theorem 4.10 [1] and Theorem 6.1 [6].

COROLLARY 1. If $P\xrightarrow{sd}Q$ then each variable $x_i\in P$ forms a separable pair with each variable $x_j\in C_{x_i,Q}$.

Denote by $\delta(x_i,L)$ the number of all the separable pairs which are formed between x_i and the variables from the set $L\subseteq R_f$ and by

$\delta(P,Q)$ the number of all the separable pairs from $P \times Q$.

COROLLARY 2. If $P \xrightarrow{sd} Q$ then for any $x_j \in Q$

$$\delta(x_j, P) \geq Card(\{C_{x_i,Q} | x_j \in C_{x_i,Q} \& x_i \in P\}).$$

COROLLARY 3. If $P \xrightarrow{sd} Q$ then

$$\delta(P,Q) \geq \sum_{x_i \in P} Card(C_{x_i,Q}).$$

COROLLARY 4. If $P \xrightarrow{rd} Q$ then

$$\delta(P,Q) \geq Card(Q).$$

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Summary

The paper studies strongly dominating and regular dominating sets of variables of functions. It is a continuation of the research of the authors and K. Chimev. If is proved that if P strongly dominates Q for f, then all subsets $P_1 \subset P$ are separable for f with respect to $P \setminus P_1$.

A VÁLTOZÓK ERŐSEN DOMINÁLÓ HALMAZAI

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Összefoglaló

A cikk a függvények változóinak un. erősen domináló és regulárisan domináló halmazaival foglalkozik /a pontos definiciók a cikkben megtalálhatók/. Ez folytatása a szerzők illetve K. Csimev korábbi kutatásainak. Be van bizonyitva, hogy ha P erősen dominálja Q-t /egy f függvényre nézve/ akkor minden $P_1 \subset P$ sz f-nek a $P \setminus P_1$-re nézve vett szeparábilis részhalmaza.