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ON THE STAR NUMBER OF A SET-LATTICE

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1. Introduction

Let $\Lambda \subset R^n$ be the integer combination of n linearly independent vectors $b_1, b_2, \ldots, b_n \in R^n$ (the point lattice of full dimension). Let us call the collection $\mathscr{S}$ of the sets $\{S+u\}$, $u \in \Lambda$, the set-lattice, where $S \subset R^n$ is a bounded set ("Figurengitter" by Hadwiger [1]).

Denote by $T(\mathscr{S})$ the number of points $u \in \Lambda$ (the zero point θ included) such that $S \cap (S+u) \neq \emptyset$. This number is called the star number of $\mathscr{S}$ ("Treffenzahl" by Hadwiger [2]). The number $T(\mathscr{S})$ has been introduced and studied when $\mathscr{S}$ is either a packing or covering of R^n (see, e.g. [3], [4] for more details), but it is obviously meaningful in the above most general situation as well.

The set S is called symmetric with centre $x \in R^n$ if $(S-x) = -(S-x)$.

If the centre of symmetric is the point θ then S is called symmetric.

Erdős and Rogers [5] proved that if $S \subset R^n$ is a symmetric convex body (i.e. $S=-S$) such that $\mathcal{S}$ is a covering of R^n (i.e. $\bigcup_{u\in\Lambda}(S+u)=R^n$), then

$$(1.1) \qquad T(\mathcal{S}) \geqslant 2^{n+1}-1.$$

Groemer [6] extended (1.1) to any bounded $S\subset R^n$, i.e. he proved that (1.1) holds for any bounded symmetric S such that $\mathcal{S}$ covers the R^n. Groemer derived his result from a more general inequality. He first introduced the so called reduced star number $t(\mathcal{S})$ ("reduzierte Treffenzahl") as follows.

Given a $u\in\Lambda$ such that $S\cap(S+u)\neq\emptyset$, we take all $v\in\Lambda$ for which there is $w\in\Lambda$ such that $v+w=u$ and $(S+v)\cap(S+w)\neq\emptyset$. Denote by $d(u)$ the number of such v-s (this has been called by Groemer the degree of u).

If we collect all $u\in\Lambda$ such that $d(u)=k$, $k\geqslant 1$, then we get a decomposition of the set $\{u\in\Lambda: \ S\cap(S+u)\neq\emptyset\}$. Hence, denoting by $N_k(\mathcal{S})$ the number of u-s with $d(u)=k$, we get

$$(1.2) \qquad T(\mathcal{S}) = \sum_{k\geqslant 1} N_k(\mathcal{S}).$$

Then, by definition, the number

$$(1.3) \qquad t(\mathcal{S}) := \sum_{k\geqslant 1} k^{-1}\, N_k(\mathcal{S})$$

is called the reduced star number, [6].
It is clear that $d(u)\geqslant 2$ if $u\neq\Theta$, $N_1(\mathcal{S})=0$ if $d(\Theta)\geqslant 2$ and $N_1(\mathcal{S})=1$ if $d(\Theta)=1$, consenquently

(1.4) $N_1(\mathcal{S}) \leq 1.$

Now, clearly ([6], p.23)

(1.5) $t(\mathcal{S}) = \frac{1}{2}N_1(\mathcal{S}) + \frac{1}{2}(N_1(\mathcal{S}) + N_2(\mathcal{S}) + \frac{2}{3}N_3(\mathcal{S}) + \cdots) \leq \frac{1}{2}N_1(\mathcal{S}) + \frac{1}{2}T(\mathcal{S})$,

hence (1.4) implies

(1.6) $T(\mathcal{S}) \geq 2t(\mathcal{S}) - 1$

The relations (1.5) and (1.4) show that equality is in (1.6) if and only if

(1.7) $N_1(\mathcal{S}) = 1$ and $N_k(\mathcal{S}) = 0$ for all $k > 2$.

The $\mathcal{S}$ fulfilling the conditions (1.7) has been called normal set-lattice, [6].

Groemer proved, using an identity for $t(\mathcal{S})$ ([6], Theorem 1) that if

(1.8) $S = -S$ and $\mathcal{S}$ covers the R^n,

then $t(\mathcal{S}) = 2^n$, yielding and extension of (1.1) to non-convex S.

The aim of this paper is to give refinements of the inequality (1.6) in the sense that, say,

(1.9) $T(\mathcal{S}) \geq M_1(\mathcal{S}) \geq M_2(\mathcal{S}) \geq 2t(\mathcal{S}) - 1$,

where $M_1(\mathcal{S})$, $M_2(\mathcal{S})$ are other well defined characteristics of $\mathcal{S}$.

Using (1.9) we get easily extensions and refinements of (1.1). Our inequalities can be used successfully also for giving new characterizations of normal set-lattices (as equality cases of (1.6)).

The clues to our results are two new type identities for $T(\mathcal{S})$. The first of them can be found also in [7], where it served as a tool for sharpening some upper estimations for $T(\mathcal{S})$ (see Section 6 for more details). We shall give also an identity for $N_k(\mathcal{S})$ that gives a new interesting insight into the Groemer's decomposition (1.2) and the quantity $t(\mathcal{S})$. Finally, our representation for $T(\mathcal{S})$ shows an interesting connection between lower estimations for $T(\mathcal{S})$ and a new sharper form of the classical Minkowski-Blichfeldt theorem proved in [8] (see Section 6).

2. The basic identities

Let $\mathcal{S} = \{(S+u) : u \in \Lambda\}$ be a set lattice in R^n where $S \subset R^n$ is a bounded set and $\Lambda \subset R^n$ a point-lattice generated by the basis $b_1, \ldots, b_n \in R^n$. Denote $P := \{x \in R^n : x = \sum_{i=1}^{n} \lambda_i b_i, 0 \le \lambda_i < 1, i=1,\ldots,n\}$ (a unit cell of Λ). Denote by Λ' the lattice $\frac{1}{2}\Lambda$ i.e. Λ' is generated by the basis $b'_1, b'_2, \ldots, b'_n \in R^n$, where $b'_i = 1/2\, b_i$, $i=1,2,\ldots,n$. It is clear that $\Lambda \subset \Lambda'$.

One can see easily that the set $P' := P \cap \Lambda'$ is in a one-to-one correspondence with the quotient space Λ'/Λ (the set of different cosets $(\Lambda + x)$, $x \in \Lambda'$), i.e.

$$(2.1) \qquad \Lambda' = P' + \Lambda = \bigcup_{x \in P'} (\Lambda + x),$$

where the cosets $(\Lambda+x)$ are mutually disjoint for $x\in P'$. The canonical map $\psi : \Lambda' \to P' \sim \Lambda'/\Lambda$ is defined as

(2.2) $\qquad \psi(u) = x,$ where $u \in \Lambda + x$.

For any set $A \subset \Lambda'$ by definition

(2.3) $$\psi(A) := \bigcup_{a\in A} \psi(a).$$

The $\psi(A)$ is the canonical projection of A into $P' \sim \Lambda'/\Lambda$. One can see easily that

(2.4) $$\psi(A) = \{x\in P' : A \cap (\Lambda + x) \neq \emptyset\} = \bigcup_{u\in\Lambda} (A+u) \cap P'.$$

For any two sets $A,B \subseteq R^n$, A+B means the algebraic (Minkowski) sum of the sets, i.e. the collection of points a+b, $a \in A$, $b \in B$. In particular $A-B := A+(-B)$. Our first identity is a straight consequence of the simple fact

(2.5) $$\{u\in\Lambda : S \cap (S+u) \neq \emptyset\} = (S-S) \cap \Lambda .$$

Hence

(2.6) $$T(\varphi) = |(S-S) \cap \Lambda|,$$

where $|A|$ denotes the cardinality of the finite set A.

Surprisingly enough, to our best knowledge, this almost trivial identity has not been used yet for the calculations concerning $T(\varphi)$ (this identity has been successfully used also in [7]).

The second identity is formulated in the following

Theorem 2.1. For any bounded $S \subset R^n$ and any point lattice $\Lambda \subset R^n$,

(2.7) $$T(\mathcal{S}) = 1 + \sum_{i \geq 1}^{q(\mathcal{S})} \sum_{x \in \psi_i(\mathcal{S})} |2^{-i}(S-S) \cap (\Lambda + x)|,$$

where

(2.8) $$\psi_i(\mathcal{S}) := \psi\left(2^{-i}(S-S) \cap \Lambda'\right) \setminus \{\Theta\}, \quad i = 1, 2, \ldots,$$

(2.9) $$q(\mathcal{S}) := \min\{i : i \geq 0,\ 2^{-i}(S-S) \cap \Lambda = \{\Theta\}\}.$$

If $q(\mathcal{S}) = 0$ or $\psi_i(\mathcal{S}) = \emptyset$ then the respective sums are considered by definition as zeros. □

Proof: Use (2.6). The condition $q(\mathcal{S}) = 0$ means $(S-S) \cap \Lambda = \{\Theta\}$, hence (2.7) is true. Let $q(\mathcal{S}) > 0$. It is clear that

(2.10) $$|(S-S) \cap \Lambda| = |2^{-1}(S-S) \cap \Lambda'|.$$

Using (2.1) we get

(2.11) $$2^{-1}(S-S) \cap \Lambda' = 2^{-1}(S-S) \cap \Lambda \cup \bigcup_{\substack{x \in P' \\ x \neq \Theta}} (2^{-1}(S-S) \cap (\Lambda + x)),$$

where the non-empty sets in the union are mutually disjoint. By the relation (2.4)

(2.12) $$\psi(2^{-1}(S-S) \cap \Lambda') = \{x \in P' : 2^{-1}(S-S) \cap \Lambda' \cap (\Lambda + x) \neq \emptyset\}.$$

But $\Lambda + x \subset \Lambda'$ for $x \in P'$, hence

$$(2.13) \qquad \psi(2^{-1}(S-S) \cap \Lambda') = \{x \in P' : 2^{-1}(S-S) \cap (\Lambda + x) \neq \emptyset\}.$$

The above identities yield

$$(2.14) \qquad T(\mathcal{S}) = \begin{cases} T(\mathcal{S}^{(1)}) + \sum_{x \in \psi_1(\mathcal{S})} |2^{-1}(S-S) \cap (\Lambda + x)| & \text{if } \psi_1(\mathcal{S}) \neq \emptyset, \\ T(\mathcal{S}^{(1)}) & \text{if } \psi_1(\mathcal{S}) = \emptyset, \end{cases}$$

where $\mathcal{S}^{(1)}$ is the set-lattice $\{(2^{-1}S+u) : u \in \Lambda'\}$.
Denote by $\mathcal{S}^{(i)}$ the set-lattice $\{(2^{-i}S+u) : u \in \Lambda\}$.
Puting into (2.14) $\mathcal{S}^{(i)}$ instead of $\mathcal{S}$ and $\mathcal{S}^{(i+1)}$ instead of $S^{(1)}$ we get for all $i \geq 0$

$$(2.15) \qquad T(\mathcal{S}^{(i)}) = \begin{cases} T(\mathcal{S}^{(i+1)}) + \sum_{x \in \psi_{i+1}(\mathcal{S})} |2^{-(i+1)}(S-S) \cap (\Lambda + x)| & \text{if } \psi_{i+1}(\mathcal{S}) \neq \emptyset, \\ T(\mathcal{S}^{(i+1)}) & \text{if } \psi_{i+1}(\mathcal{S}) = \emptyset. \end{cases}$$

The later identity shows that

$$(2.16) \qquad 2^{-i}(S-S) \cap \Lambda = \{\theta\} \Rightarrow 2^{-j}(S-S) \cap \Lambda = \{\theta\} \quad \text{for all } j > i$$

and

$$(2.17) \qquad 2^{-i}(S-S) \cap \Lambda = \{\theta\} \Rightarrow \psi_j(\mathcal{S}) = \emptyset \quad \text{for all } j > i.$$

These implications and the definition of $q(\mathcal{S})$ show that $T(\mathcal{S}^{(q(\mathcal{S}))}) = 1$ and $\psi_j(\mathcal{S}) = \emptyset$ for all $j > q(\mathcal{S})$.
Applying (2.15) successively we get (2.7). ∎

In spite of its simplicity, the identity (2.7) proved to be quite a powerfull tool in investigating the magnitude of $T(\mathcal{S})$.

3. Refinements of lower estimations

The lower estimations below are almost trivial consequences of the identity (2.7).

Theorem 3.1. For any bounded set $S \subset R^n$ and any point-lattice $\Lambda \subset R^n$ we have

$$(3.1) \qquad T(\mathcal{S}) \geq 1+2 \sum_{i=1}^{q(\mathcal{S})} |\psi_i(\mathcal{S})|,$$

where $q(\mathcal{S})$ and $\psi_i(\mathcal{S})$ are defined by (2.9) and (2.8). □

Proof: For $\theta \neq x \in \Lambda' = \frac{1}{2}\Lambda$, the set $\Lambda+x$ is symmetric, i.e. $\Lambda+x=-(\Lambda+x)$ and does not contain the Θ. Hence the set $2^{-i}(S-S) \cap (\Lambda+x)$ if non-empty is symmetric and does not contain Θ, so it contains at least 2 elements. ■

Remark 3.2. Any finite symmetric set not containing its centre of symmetry contains even number of elements, hence $|2^{-i}(S-S) \cap (\Lambda+x)| = 2r$ for some $r \geq 0$. This simple observation shall have an interesting consequence regarding the number $N_k(\mathcal{S})$, see Lemma 3.4. below. □

Corollary 3.3. For any bounded set $S \subset R^n$ and any point-lattice $\Lambda \subset R^n$ we have

(3.2) $\quad T(\mathcal{S}) \geq 1+2 \sum_{i=1}^{q(\mathcal{S})} |\psi_i(\mathcal{S})| \geq 1+2 \max_{1\leq i\leq q(\mathcal{S})} |\psi_i(\mathcal{S})| \geq 2t(\mathcal{S})-1.$ □

Proof: The second inequality in (3.2) is trivial. To prove the third one it is enough to prove

(3.3) $\quad t(\mathcal{S}) = |\psi(2^{-1}(S-S) \cap \Lambda')|.$

We note that the case $q(\mathcal{S})=0$ means $(S-S)\cap\Lambda = 2^{-1}(S-S)\cap\Lambda' = \{\theta\}$, hence after proving (3.3) we see that $t(\mathcal{S})=1$, so taking by convention zeros instead of the "$\sum$" and the "max" in (3.2) we get that all terms in (3.2) are equal to 1. The identity (3.3) will be a consequence of a following lemma. ■

Lemma 3.4. Let $S\subset R^n$ be bounded and $\Lambda\subset R^n$ a point-lattice. Let $N_k(\mathcal{S})$ be the numbers occuring in (1.2), $k\geq 1$. Then

(3.4) $\quad N_k(\mathcal{S}) = k\cdot|\{x\in P' : |2^{-1}(S-S) \cap (\Lambda+x)| = k\}|, \quad k\geq 1.$ □

Proof: Let $u\in\Lambda$ be such that $S\cap(S+u)\neq\emptyset$ and denote

(3.5) $\quad S(u) := 2^{-1}(S-S+u)\cap\Lambda.$

We claim that

(3.6) $\quad d(u) = |S(u)|$

where $d(u)$ is the degree of u defined in Section 1. Indeed, $d(u)$ is the cardinality of the set

(3.7) $\quad \{v\in\Lambda : \exists w\in\Lambda,\ v+w=u,\ (S+v)\cap(S+w)\neq\emptyset\}.$

But this set is equal to the set $\{ v\in\Lambda : (S+v)\cap(S+u-v)\neq\emptyset \}$ that is in turn equal to $\{ v\in\Lambda: S\cap(S+u-2v)\neq\emptyset \}$ which finally coincides with the $S(u)$.

The condition $S\cap(S+u)\neq\emptyset$ is equivalent to $\Theta\in S(u)$, hence the $N_k(\mathcal{S})$ is the number of points $u\in\Lambda$ such that

(3.8) $\qquad \Theta\in S(u)$ and $|S(u)| = k$.

Let $u,\bar{u}\in\Lambda$ be two points such that $\Theta\in S(u)$, $\Theta\in S(\bar{u})$. Denote $u\sim\bar{u}$ if

(3.9) $$\frac{u-\bar{u}}{2}\in\Lambda .$$

This "$\sim$" is clearly an equivalence relation among the set of points $u\in\Lambda$ such that $\Theta\in S(u)$.

Let $u_1,u_2,\dots,u_j\in\Lambda$, $\Theta\in S(u_i)$, $i=1,\dots,j$, be mutually equivalent points. Then there are $s_i,\tilde{s}_i\in S$ such that $s_i-\tilde{s}_i+u_i=\Theta$, $i=1,2,\dots,j$, hence using the equivalence of u_i-s, there are $z_{i\nu}\in\Lambda$ for which

(3.10) $$s_i-\tilde{s}_i+u_\nu = u_\nu-u_i = 2z_{i\nu} \qquad 1\le i,\nu\le j.$$

This implies

(3.11) $$|S(u_\nu)| \geqslant j, \qquad \nu=1,2,\dots,j.$$

On the other hand, if $\Theta\in S(u)$ and $|S(u)| = k$, i.e. $S(u)=\{v_1,v_2,\dots,v_k\}$, then for the points

(3.12) $$u-2v_i,\ i=1,2,\dots,k,$$

we have $\theta \in S(u-2v_i)$, they are mutually equivalent (one of them is equal to u) and there are $s_\nu, \bar{s}_\nu \in S$, $\nu=1,2,\ldots,k$ such that

(3.13) $$s_\nu - \bar{s}_\nu + u - 2v_i = 2(v_\nu - v_i), \quad 1 \le i,\nu \le k.$$

If we had $s_{k+1}, \bar{s}_{k+1} \in S$ such that $s_{k+1} - \bar{s}_{k+1} + u - 2v_i = 2z$ for some $z \neq v_\nu - v_i$, $l=1,\ldots,k$, (i.e. if $|S(u-2v_i)| > k$) then $z+v_i \neq v_\nu$, $\nu=1,2,\ldots,k$, and $s_{k+1} - \bar{s}_{k+1} + u = 2(z+v_i)$, implying $|S(u)| > k$. Hence

(3.14) $$|S(u-2v_i)| = |S(u)| = k, \quad i=1,2,\ldots,k.$$

If there were a point u_{k+1} such that $\theta \in S(u_{k+1})$ and $u_{k+1} \sim u-2v_1$, then using (3.11) we would have $|S(u-2v_1)| \geqslant k+1$, that contradicts to (3.14).

The above show that the set of elements $u \in \Lambda$ satisfying (3.8) is equal to the union of k-tuples

$\{u_1^{(1)}, u_2^{(1)}, \ldots, u_k^{(1)}\}, \{u_1^{(2)}, u_2^{(2)}, \ldots, u_k^{(2)}\}, \ldots, \{u_1^{(r)}, u_2^{(r)}, \ldots, u_k^{(r)}\}$, such that

(3.15) $$u_j^{(i)} \sim u_\nu^{(i)} \quad \text{for } 1 \le i \le r, \quad 1 \le j,\nu \le k,$$

and

(3.16) $$u_j^{(i)} \not\sim u_j^{(\nu)} \quad \text{for } 1 \le j \le k, \ 1 \le i,\nu \le r.$$

To finish the proof it is enough to show that there is a one-to-one correspondence between the sets

(3.17) $$A := \{ u_1^{(1)}, u_1^{(2)}, \ldots, u_1^{(r)} \}$$

and

(3.18) $$B := \{x \in P' : |2^{-1}(S-S) \cap (\Lambda + x)| = k\}.$$

The canonical projection ψ, (2.4), of Λ' onto P' will provide this correspodence.
We first show that

(3.19) $$\psi(-\tfrac{1}{2}A) = B$$

By the definition of ψ'

(3.20) $$\Lambda + x = \Lambda + \psi(x) \quad \text{for all } x \in \Lambda'.$$

It is clear that

(3.21) $$|2^{-1}(S-S) \cap (\Lambda - \frac{u_1^{(i)}}{2})| = k, \; i=1,2,\dots,r.$$

Hence, taking into account (3.20) we get

(3.22) $$\psi(-\tfrac{1}{2}A) \subseteq B.$$

Let $x = \frac{v}{2} \in B$. Then there are exactly k points $v_i \in \Lambda$, $i=1,2,\dots,k$, such that

(3.23) $$s_i - \tilde{s}_i = 2v_i + v, \; i=1,\dots,k,$$

for some $s_i, \tilde{s}_i \in S$.

One can see easily that this implies

(3.24) $$\theta \in S(v+2v_1) \quad \text{and} \quad |S(v+2v_1)| = k,$$

hence $v+2v_1$ should be equal to some $u_j^{(i)}$, say $u_p^{(s)}$. The definition of ψ implies that for $x,y \in \Lambda'$,

(3.25) $\psi(x)=\psi(y)$ if and only if $x-y \in \Lambda$,

consequently

(3.26) $\psi(\frac{u}{2})=\psi(\frac{\bar{u}}{2})$ if and only if $u \sim \bar{u}$.

These imply that

(3.27) $x=\psi(-\frac{v+2v_1}{2})=\psi(-\frac{u_p^{(s)}}{2})=\psi(-\frac{u_p^{(i)}}{2})$

so (3.19) is true.

As the elements of A are not mutually equivalent, by (3.26) we have $\psi(-\frac{1}{2}A)=|A|$ hence $|B|=r$ and by this the lemma is proved. ■

<u>Remark 3.5.</u> The identity (3.4) and the Remark 3.2 imply that $N_k(\mathcal{P})=0$ for such odd numbers $k \geq 3$ that are not equal to $|2^{-1}(S-S)\cap\Lambda|$. This fact is not explicitly mentioned in [6], although it is not difficult to derive it from the definition of $N_k(\mathcal{P})$. □

<u>Corollary 3.6.</u> The identity (3.3) is true. □

<u>Proof</u>: By (2.4) we have

(3.28) $\psi(2^{-1}(S-S)\cap\Lambda')=\{x\in P': 2^{-1}(S-S)\cap\Lambda'\cap(\Lambda+x)\neq\phi\}$.

But $\Lambda + x \subset \Lambda'$, hence

(3.29) $\quad |\gamma_{\cdot}(2^{-1}(S-S)\cap\Lambda')| = \sum_{k>1} |\{x\in P' : |2^{-1}(S-S)\cap(\Lambda+x)| = k\}$

and the definition (1.3) of $t(\mathscr{S})$ and the identity (3.4) imply (3.3). ∎

4. Lower estimations in special cases

Let $\mathscr{S}$ be a set-lattice in R^n and denote

(4.1) $\quad p(\mathscr{S}) := |\{i : i\geq 1,\ \gamma_i(\mathscr{S}) = P' \setminus \{\theta\}\}|$

It is clear that $|P'| = 2^n$ hence we can write for the right hand side of (3.1)

(4.2) $$1+2\sum_{i=1}^{q(\mathscr{S})} |\gamma_i(\mathscr{S})| = $$
$$= 1 + 2p(\mathscr{S})(2^n-1) + 2\sum_{i=1}^{q(\mathscr{S})}{}' |\gamma_i(\mathscr{S})|,$$

where $\sum'$ runs over all i-s such that $|\gamma_i(\mathscr{S})| < |P'|-1$.
One can easily see using (2.4) that

(4.3) $\quad \gamma_i(\mathscr{S}) = P'\setminus\{\theta\} \iff 2^{-i}(S-S)+\Lambda \supseteq \Lambda',\ i=1,2,\dots$

Hence

(4.4) $\quad p(\mathscr{S}) = |\{i : i\geq 1,\ 2^{-i}(S-S)+\Lambda \supseteq \tfrac{1}{2}\Lambda\}|.$

The definition of $q(\mathscr{S})$ shows that $p(\mathscr{S}) \leq q(\mathscr{S})$.

<u>Theorem 4.1.</u> Let $\mathcal{S} = \{(S+u) : u \in \Lambda\}$ be a set-lattice such that the set S is bounded and symmetric with the centre of symmetry $x \in R^n$. Denote

$$(4.5) \qquad b(\mathcal{S}) := |\{ i : i \geq 0,\ 2^{-i}(S-x) + \Lambda \supseteq \tfrac{1}{2}\Lambda \}| .$$

Then

$$(4.6) \qquad b(\mathcal{S}) \leq p(\mathcal{S})$$

and equality hold in (4.6) if S is convex. $\square$

<u>Proof:</u> Let $i \geq 0$ be such that

$$(4.7) \qquad 2^{-i}(S-x) + \Lambda \supseteq \tfrac{1}{2}\Lambda$$

i.e. to any $u \in \Lambda$ there are $s \in S$ and $v \in \Lambda$ such that

$$(4.8) \qquad u = 2^{-i+1}(s-x) + 2v = 2^{-i}(s-x) + 2^{-i}(s-x) + 2v.$$

Using the condition $S-x = -(S-x)$, there is $\bar{s} \in S$ such that

$$(4.9) \qquad u = 2^{-i}(s-x) - 2^{-i}(\bar{s}-x) + 2v = 2^{-i}(s-\bar{s}) + 2v .$$

This show that

$$(4.10) \qquad 2^{-(i+1)}(S-S) + \Lambda \supseteq \tfrac{1}{2}\Lambda$$

and by this (4.6) is proved.

Let S be convex and assume that (4.10) holds for some $i \geq 0$, i.e. for any $u \in \Lambda$ there are $s_1, s_2 \in S$ and $v \in \Lambda$ such that

(4.11) $u=2^{-i}((s_1-x)-(s_2-x))+2v.$

By the symmetry of S-x, there is $s_3 \in S$ such that

(4.12) $u=2^{-i}(s_1+s_3-2x) +2v.$

The convexity of S implies that $s_4 = \frac{s_1+s_3}{2} \in S$, so we get

(4.13) $\frac{u}{2} = 2^{-i}(s_4-x) + v$

that proves (4.7). ■

Remark 4.2. The convexity and symmetry of S-x shows that

(4.14) $2^{-(i+1)}(S-S)=2^{-i}(S-x), \quad i=0,1,\ldots,$

that immediately implies, by the definitions of $p(\mathcal{S})$ and $b(\mathcal{S})$ that $p(\mathcal{S}) = b(\mathcal{S})$.

The proof of the theorem shows that the condition

(4.15) $\frac{(S-x)+(S-x)}{2} \subseteq S-x$

is enough (together with the symmetry of S-x) to ensure $p(\mathcal{S}) = b(\mathcal{S})$.

The condition (4.15) is a little weaker than the condition of convexity of S-x. It is a special case of a condition introduced by R.Rado (so called C-set, see [8] for details). □

<u>Corollary 4.3.</u> Let $\mathcal{S}$ be a set-lattice in R^n such that S is bounded and symmetric with centre $x \in R^n$.
Then

(4.16) $T(\mathcal{S}) \geq 1 + 2\sum_{i=1}^{q(\mathcal{S})} |\psi_i(\mathcal{S})| \geq 1 + 2\,b(\mathcal{S})(2^n-1) + 2\sum_{i=1}^{q(\mathcal{S})}{}' |\psi_i(\mathcal{S})|$. □

This inequality both extends and sharpens the result of Groemer (consenquently the result of Erdős and Rogers) mentioned in Section 1. Namely, if

(4.17) $S=-S$ and $S+\Lambda = R^n$

(these are the assumptions of Groemer), then clearly $d(\mathcal{S}) \geq 1$, hence

(4.18) $T(\mathcal{S}) \geq 2^{n+1} - 1 + 2\sum_{i=1}^{q(\mathcal{S})}{}' |\psi_i(\mathcal{S})|$.

We have to note that, as Groemer observed ([6], p. 26, 6-th row from the top), for the proof of the condition $t(\mathcal{S})=2^n$ (that yield (1.1) via (1.6)), instead of (4.17) it is enough to assume

(4.19) $S=-S$ and $S+\Lambda \supseteq \frac{1}{2}\Lambda$.

By the definition of $b(\mathcal{S})$, this assumption also implies $b(\mathcal{S}) \geq 1$.

Let us rewrite (3.1) using (4.2).

(4.20) $T(\mathcal{S}) \geq 1+2p(\mathcal{S})\cdot(2^n-1) + 2\sum_{i=1}^{q(\mathcal{S})}{}' |\psi_i(\mathcal{S})|$

(This inequality holds without any assumptions on S!)

We see that for (4.18) to hold a weaker condition $p(\mathcal{f})\geq 1$ is also enough. To see more clearly what the conditions $b(\mathcal{f})\geq 1$, $p(\mathcal{f})\geq 1$ mean, let us write them more explicitly:

(4.21) There is $i\geq 0$ such that $2^{-i}(S-x)+\Lambda \supseteq \frac{1}{2}\Lambda$

(4.22) There is $i\geq 1$ such that $2^{-i}(S-S)+\Lambda \supseteq \frac{1}{2}\Lambda$.

These conditions are equivalent if S-x is symmetric and convex. As we have seen in the proof of Theorem 4.1, for $i\geq 0$

(4.23) $2^{-i}(S-x)+\Lambda \supseteq \frac{1}{2}\Lambda \Rightarrow 2^{-(i+1)}(S-S)+\Lambda \supseteq \frac{1}{2}\Lambda$

(of course the symmetry of S-x has to be also assumed). We quess that for non-convex S, $p(\mathcal{f})$ can be greater than $b(\mathcal{f})$, more exactly the following problem seems to be not hopeless.

<u>Problem 4.4</u> Find a bounded non-convex set $S\subset R^n$ and a point-lattice $\Lambda\subset R^n$ of full-dimension such that: $S=-S$ and $2^{-i}(S-S)+\Lambda \supseteq \frac{1}{2}\Lambda$ and $2^{-(i-1)}S+\Lambda \not\supseteq \frac{1}{2}\Lambda$ for some $i\geq 1$. □

The conditions (4.21) and (4.22) are especially illustrative i S-x is a star-shaped set (or a ray-set, as these sets are called in [3]), i.e.

(4.24) and $\lambda(S-x)\subseteq S-x$ for all $0\leq\lambda\leq 1$.

These conditions imply that

(4.25) $2^{-i}(S-x) \leq 2^{-(i-1)}(S-x)$, $2^{-i}(S-S) \leq 2^{-(i-1)}(S-S)$, $i \geq 1$.

These imply that $b(\mathcal{S})$ and $p(\mathcal{S})$ are equal to the maximal $i \geq 0$ and $i \geq 1$ such that $2^{-i}(S-x)+\Lambda \supseteq \frac{1}{2}\Lambda$ and $2^{-i}(S-S)+\Lambda \supseteq \frac{1}{2}\Lambda$, respectively.

Summarizing the meaning of (4.20) we can say that the principal term in the right hand side of (1.1) is not 2^{n+1} but $c \cdot 2^{n+1}$, where the constant c can be quite big depending on how much the set lattice $\{((S-S)+u) : u \in \Lambda\}$ covers $\frac{1}{2}\Lambda$ in the sense how many times we can half the set S-S to maintain the covering.
Moreover, an addive term $\sum_i' |\psi_i(\mathcal{S})|$ also occurs on the right hand side of (4.20) and no symmetry assumptions on S are needed.
Expressing this phenomenon, we could call the constants $b(\mathcal{S})$ and $p(\mathcal{S})$ the (2 based) degrees of coverings $\frac{1}{2}\Lambda$ by $\mathcal{S} := \{(S+u) : u \in \Lambda\}$ and $\{((S-S)+u) : u \in \Lambda\}$, respectively.
So, rather the degrees of covering than the coverings itselves decide upon the lower estimations for $T(\mathcal{S})$.

5. Equality criteriae

We have seen that the conditions (1.7) are sufficient and necessary conditions for the equality in (1.6).
The set-lattices fulfilling these conditions are called normal set-lattices. Of course, these conditions can be derived in a more complicated way using our refinements

of (1.6).
Below we shall show this more complicated route.
This is not for the proof of the conditions in themselves, because the proof of (1.7) is almost trivial, see Section 1.
But we gain some more necessary conditions for the normality of a set-lattice and a little deeper insight into the matter.
First, let us write once more the inequalities (3.2).

$$(5.1)\quad T(\mathcal{S}) \overset{①}{\geq} 1 + 2 \sum_{i=1}^{q(\mathcal{S})} |\psi_i(\mathcal{S})| \overset{②}{\geq} 1 + 2 \max_{1 \leq i \leq q(\mathcal{S})} |\psi_i(\mathcal{S})| \overset{③}{\geq} 2t(\mathcal{S}) - 1$$

If $q(\mathcal{S})=0$ then $t(\mathcal{S})=1$ and equalities are in all inequalities of (5.1). The same is true if $\psi_i(\mathcal{S})=\phi$ for all $1 \leq i \leq q(\mathcal{S})$.
So assume that $q(\mathcal{S})>0$ and $\sum |\psi_i(\mathcal{S})| > 0$.
Taking into account identities (2.7) and (3.3) we can state:

- ① is equality if and only if

$$(5.2)\quad |2^{-i}(S-S) \cap (\Lambda+x)| = 2 \text{ for all } x \in \psi_i(\mathcal{S}), \text{ for all } i \text{ s.t. } \psi_i(\mathcal{S}) \neq \phi ;$$

- ② is equality if and only if

(5.3) there is exactly one j, $1 \leq j \leq q(\;)$ s.t. $\psi_j(\mathcal{S}) \neq \emptyset$;

- ③ is equality if and only if

$$(5.4)\quad |\psi_1(\mathcal{S})| = \max_{1 \leq i \leq q(\mathcal{S})} |\psi_i(\mathcal{S})| .$$

These give:
$T(\mathcal{S}) = 2t(\mathcal{S})-1$ if and only if

(5.5) $\quad \psi_i(\varphi) = \emptyset \quad$ for all $i \geq 2$

and

(5.6) $\quad |2^{-1}(S-S) \cap (\Lambda + x)| = 2 \quad$ for all $x \in \psi_1(\varphi)$.

The condition (5.5) is by (2.15) equivalent to

(5.7) $\quad 2^{-1}(S-S) \cap \Lambda = \{\Theta\}$

By (3.4) the conditions (5.7) and (5.6) are equivalent to $N_1(\varphi)=1$ and $N_k(\varphi)=0$ for all $k>2$.

If S is symmetric with the centre of symmetry $x \in R$, then one more inequality holds

(5.8) $\quad 1 + 2\sum_{i=1}^{q(\varphi)} |\psi_i(\varphi)| \overset{④}{\geq} 1 + 2b(\varphi)(2^n-1) + 2\sum_{i=1}^{q(\varphi)}{}' |\psi_i(\varphi)|$.

- (4) is equality if and only if

(5.9) $\quad p(\varphi) = b(\varphi)$.

A sufficient condition for (5.9) is the convexity of S. If either

(5.10) $\quad b(\varphi) \geq 1$

or

(5.11) $\quad b(\varphi) = 0 \quad$ and $\quad |\psi_1(\varphi)| < 2^n - 1$,

then the right hand side of (5.8) is not less then $2t(\varphi) - 1$.

The only exceptional case when $2t(\mathcal{S})-1$ might be greater than the right hand side of (5.8) is

$$(5.12)\qquad 2^{-1}(S-S)+\Lambda \supseteq \tfrac{1}{2}\Lambda\,,\ 2^{-i}(S-x)+\Lambda \not\supseteq \tfrac{1}{2}\Lambda \qquad \text{for all } \ i \geq 0.$$

(Of course, this cannot occur if S is convex.)
We could formulate a problem similar to the Problem 4.4.

<u>Problem 5.1.</u> Find a bounded non-convex set $S\subset R^n$ and a point-lattice $\Lambda \subset R^n$ of full dimension such that: $S=-S$, $2^{-1}(S-S)+\Lambda \supseteq \frac{1}{2}\Lambda$ and $2^{-i}S+\Lambda \not\supseteq \frac{1}{2}\Lambda$ for all $i\geq 0$. □
If we had a solution of this problem, then $t(\mathcal{S})=t^n$ and it has to be still checked whether

$$(5.13)\qquad 2^n-1 \ > \sum_{i=1}^{q(\mathcal{S})}{}' \, |\gamma_i(\mathcal{S})|$$

to quarantee that $2t(\mathcal{S})-1$ is greater than the right hand side of (5.8).

<u>6. Remarks</u>

1.Let $\Lambda \subset R^n$ be a point-lattice of full dimension, P be its unit cell i.e.

$$(6.1)\qquad R^n = P + \Lambda = \bigcup_{x\in P} (\Lambda + x)\,,$$

where the cosets $(\Lambda+x)$ are mutually disjoint for $x\in P$.
Denote by φ the canonical map of R^n onto $P \sim R^n/\Lambda$ i.e.

$$(6.2)\qquad \varphi(y):=x \text{ where } y\in\Lambda+x.$$

For any set $A \subseteq R^n$

(6.3) $$\mathcal{f}(A) := \bigcup_{a \in A} \mathcal{f}(a) = \{ x \in P : A \cap (\Lambda + x) \neq \emptyset \} = \bigcup_{u \in \Lambda} (A + u) \cap P .$$

The volume (Lebesgue-measure) in R^n is denoted by V. We have proved in [8] (see also [7],[9],[10]), that for any bounded Lebesgue-measurable set $S \subset R^n$ we have

(6.4) $$T(\mathcal{S}) = |(S-S) \cap \Lambda| \geq 2 \frac{V(S)}{V(\mathcal{f}(S))} - 1$$

It would be quite interesting to compare the right hand side of (6.4) with the lower estimations for $T(\mathcal{S})$ proved in previous sections.
A refinement of (6.4) where affine dimensions of the sets $(S-x) \cap \Lambda$, $x \in \mathcal{f}(S)$, play a role, has been proved in [9],[10]. The inequality (6.4) holds in much more general structures as well, when instead of R^n, Λ and V we take a locally compact Abelian group G, its "sufficiently large" discrete subgroup Λ and Haar-measure α, respectively, [9]. It would be interesting to see how the methods of this paper work in more general structures (the main obstacle seems to be the interpretation of $\frac{x}{2}$).

2. A result of Hadwiger [11] implies that if $S \subset R^n$ is a compact body and $\mathcal{S}$ is a covering of R^n then

(6.5) $$T(\mathcal{S}) \leq d\Lambda^{-1} (V(S-S+S) - V(S)) + 1$$

where V is the L-measure and $d\Lambda = V(P)$ is the determinant of the lattice.
Using the identity (2.6) we have proved a sharpening of (6.5):

$$T(\mathcal{S}) \leq d\Lambda^{-1}\left(\tilde{V}(S) - V(S)\right) + 1 \tag{6.6}$$

where $\tilde{V}(S) \leq V(S-S+S)$ is a sort of "measure" of S (see [7] for details).
Both papers [5] and [6] contain some upper estimations for $T(\mathcal{S})$ for some special $\mathcal{S}$. Our identity (2.7) seems to be a useful tool for investigating these questions as well.

3. The basic recursion identity (2.15) fits into a following more general framework. Let $M \subset L \subset R^n$ be two point lattices of full dimension. Let Q be a unit cell of M (with respect to L), i.e.

$$L = Q + M = \bigcup_{v \in Q} (M+v) \tag{6.7}$$

where the cosets $(M+v)$ are mutually disjoint for $v \in Q$.
Let $\omega : L \to Q \sim L/M$ be the canonical map

$$\omega(u) = v \text{ if } u \in M+v. \tag{6.8}$$

For a set $A \subset L$ we can write

$$\omega(A) = \bigcup_{a \in A} \omega(a) = \{v \in Q : A \cap (M+v) \neq \emptyset\} = \bigcup_{y \in M} (A+y) \cap Q. \tag{6.9}$$

One can prove, (analogously to the "continuous" cases studied in [9], [12]), that for any bounded set $A \subset L$

$$(6.10)\qquad |A| = \sum_{k\geq 1} k \cdot |A(k)|,$$

$$(6.11)\qquad |\omega(A)| = \sum_{k\geq 1} |A(k)|,$$

where

$$(6.12)\qquad A(k) := \{ v \in Q : |A \cap (M+v)| = k \}, \quad k = 1, 2, \ldots .$$

Now, as to (2.15), taking into account $|2^{-i}(S-S) \cap \Lambda| = |2^{-(i+1)}(S-S) \cap \Lambda'|$ and dentoing

$$(6.13)\qquad S_{i+1}(k) := \{ x \in P' : |2^{-(i+1)}(S-S) \cap (\Lambda + x)| = k \}, \quad k \geq 1,$$

we get by (6.10)

$$(6.14)\qquad T(\mathcal{S}^{(i)}) = \sum_{k\geq 1} k \cdot |S_{i+1}(k)| .$$

The definition (2.8) of $\psi_{i+1}(\mathcal{S})$ shows that

$$(6.15)\qquad \sum_{k\geq} k\, |S_{i+1}(k)| = T(\mathcal{S}^{(i+1)}) + \sum_{x \in \psi_{i+1}(\mathcal{S})} |2^{-(i+1)}(S-S) \cap (\Lambda + x)| .$$

It is clear that

$$(6.16)\qquad |\psi(2^{-(i+1)}(S-S) \cap \Lambda')| = \sum_{k\geq 1} |S_{i+1}(k)| .$$

Writing (6.14) and (6.16) for $i=0$ and taking into account the identities (3.4) and (3.3) we get the Groemer's formulas (1.2) and (1.3).

The proof of (6.10) and (6.11) together with a detailed study of this more general approach including "discrete" versions of the results in [12], will be a matter of another paper. Let us note that for the results of this paper a special connection of the two lattices, namely $\Lambda' = \frac{1}{2}\Lambda$, was also important.

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ON THE STAR NUMBER OF A SET-LATTICE

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Summary

Let $\Lambda \subset R^n$ be a point lattice (discrete subgroup of full dimension) and $S \subset R^n$ be a bounded set. The family $\mathcal{S} := \{(S+u) : u \in \Lambda\}$ is called the set lattice and the cardinality of the set $\{u \in \Lambda : S \cap S+u \neq \emptyset\}$, denoted by $T(\mathcal{S})$, is the star number of $\mathcal{S}$. Groemer introduced the so called reduced star number $t(\mathcal{S})$ and proved: (i) $T(\mathcal{S}) \geq 2t(\mathcal{S})-1$ and (ii) if $S = -S$ and $\mathcal{S}$ covers the R^n then $t(\mathcal{S}) = 2^n$, i.e. $T(\mathcal{S}) \geq 2^{n+1}-1$. The last inequality has been proved earlier by Erdős and Rogers for convex symmetric sets S such that $\mathcal{S}$ covers the R^n.

In the paper, after proving two new type identities for $T(\mathcal{S})$, both sharpenings and extensions of above results are given. The results also show an interesting connection of estimations for $T(\mathcal{S})$ to recent sharpenings of some classical results in geometry of numbers.

A HALMAZ-RÁCS CSILLAG SZÁMÁRÓL

Uhrin Béla

Összefoglaló

Legyen $\Lambda \subset R^n$ egy pont-rács /teljes dimenziója rész-csoport/ és $S \subset R^n$ egy korlátos halmaz. Az $\mathcal{S} := \{(S+u) : u \in \Lambda\}$ családot halmaz-rácsnak nevezzük és az $\{u \in \Lambda : S \cap (S+u) \neq \emptyset\}$ halmaz számosságát /jelöljük $T(\mathcal{S})$-el/ az $\mathcal{S}$ csillag-számának nevezzük. Groemer definiálta a $t(\mathcal{S})$ u.n. redukált csillag-számot és bebizonyította: (i) $T(\mathcal{S}) \geq 2t(\mathcal{S})-1$ és (ii) ha $S = -S$ és $\mathcal{S}$ lefedi R^n-et, akkor $t(\mathcal{S}) = 2^n$, azaz ebben az esetben $T(\mathcal{S}) \geq 2^{n+1}-1$. Az utóbbi egyenlőtlenséget olyan origóra szimmetrikus S konvex halmazokra, hogy $\mathcal{S}$ lefedi R^n-et, Erdős és Rogers is bebizonyították. A cikkben tetszőleges korlátos S-re, a $T(\mathcal{S})$ számra két uj azonosságot látunk be, amelyekből a fenti eredményeknél mind általánosabb, mind élesebb eredmények nyerhetők. A cikk eredményei egy érdekes összefüggésre is rámutatnak, amely a $T(\mathcal{S})$ illetve a geometriai számelmélet bizonyos klasszikus eredményei között fennáll.