

ON MOVING AVERAGE COMPOSITION AND PREDICTIVE DECONVOLUTION  
OF LONG-RUN STATIONARY TIME SERIES

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ABSTRACT

The notion of the time mean or time average of time series is widely used in the literature on time series analysis. Analogously, the notion of the time autocorrelation function of the time series  $x_n$  can be defined as the limit of the sum

$$\frac{1}{N} \sum_{n=0}^{N-1} x_{n+s} x_n, \quad s = 0, 1, \dots \text{ when } N \text{ tends to } \infty$$

However, in standard literature on time series analysis only the case when the time mean and the time autocorrelation function coincide with the mean  $Ex_n$  and the autocorrelation function

$\varphi_s = Ex_{n+s} x_n$  is investigated. In [7] we have shown that the seismic signals are not always stationary in usual sense, but they are stationary in slight different sense, i.e. they are so-called long-run stationary autoregressive time series. We have also shown in [7] that for long-run stationary autoregressive time series, the time autocorrelation function is not the same as the autocorrelation function. However, in the predictive deconvolution problem of this time series, instead of the autocorrelation function, the time correlation function plays an important role.

By this reason, in this paper we propose to consider the time mean and the time correlation function as independent notions. / They coincide with the mean and the autocorrelation function when the process  $x_n$  is ergodic and stationary, only/.

We also introduce the notion of a long-run stationary process, and investigate the moving average composition and predictive deconvolution of this process.

§.1. DEFINITIONS AND NOTIONS

Definition 1: The time series  $x_n$  is said to have the time mean  $M\{x_n\}$  in the sense of convergence almost everywhere / in probability/if the sum

$$\frac{1}{N} \sum_{n=m+1}^{m+N} x_n$$

converges to  $M\{x_n\}$  almost everywhere / in probability /

Definition 2: The time series  $x_n$  is said to have the time autocorrelation function  $g_x(s)$  in the sense of convergence almost everywhere / in probability/if for  $s = \dots -1, 0, 1, 2, \dots$  the sums

$$\frac{1}{N} \sum_{n=m+1}^{m+N} x_{n+s} x_n$$

converge to  $g_x(s)$  almost everywhere / in probability/

Remark: The following example shows that the time mean of some time series may be the same as the mean, while the time autocorrelation function may be different from the autocorrelation function.

Example 1: Let  $x_n = \cos nW$   $n=1, 2, \dots$  where  $W$  is a uniformly distributed random variable on  $[0, 2\pi]$ . Then we can see

$$M\{x_n\} = Ex_n = 0$$

$$g_x(1) = \frac{1}{2} \cos W \neq 0 = Ex_{n+1} x_n \quad \text{a.s.}$$

Definition 3: Let  $x_n$  and  $y_n$  be two arbitrary time series. If for  $s = \dots -1, 0, 1, 2, \dots$  the sums

$$\frac{1}{N} \sum_{n=m+1}^{m+N} x_{n+s} y_n$$

converge in a given sense to  $g_{xy}(s)$  then  $g_{xy}(s)$  is called the time crosscorrelation function between  $x_n$  and  $y_n$  in the given sense of convergence .

Definition 4: If the time series  $x_n$  and  $y_n$  have the time cross-correlation function and  $g_{xy}(s) = 0, s = 0, 1, \dots$  then we call they are time-uncorrelated.

Definition 5: If for the time series  $u_n, n = 0, 1, \dots$

$$g_u(0) = \sigma^2 > 0, \quad g_u(s) = 0, \quad s = 1, 2, 3, \dots$$

where  $\sigma^2$  may be random variable, then  $u_n$  is said to be time-uncorrelated / in the given sense of convergence /.

Definition 6: If the time series  $u_n$  is time-uncorrelated and has the time mean 0, then we call  $u_n$  a long-run white noise / in the given sense of convergence /

Definition 7: The time series  $x_n$  is called a long-run stationary time series / in the given sense of convergence / if it has the time mean and the time correlation function / in the given sense of convergence /

We would like to notice that the time mean and the time autocorrelation function may be random variables.

From now on, if we do not say else, we always mean that the above definitions are given in the sense of convergence almost everywhere .

## §.2. PROPERTIES OF THE TIME MEAN AND THE TIME -AUTOCORRELATION FUCTION

1. If  $x_n$  and  $y_n$  are long-run stationary time series and there exists the time crosscorrelation function between them, then

$z_n = \alpha x_n + \beta y_n$  has the same property and

$$M\{z_n\} = \alpha M\{x_n\} + \beta M\{y_n\} \quad \text{a.s} \quad (1)$$

$$g_z(s) = \alpha^2 g_x(s) + \alpha\beta g_{xy}(s) + \alpha\beta g_{yx}(s) + \beta^2 g_y(s) \quad \text{a.s} \quad (2)$$

Thus if  $x_n$  and  $y_n$  are time-uncorrelated then

$$g_z(s) = \alpha^2 g_x(s) + \beta^2 g_y(s) \quad (3)$$

The proof is straightforward.

$$2. \quad \varepsilon_{xy}(s) = \varepsilon_{yx}(-s) \quad \text{a.s.} \quad (4)$$

$$3. \quad |\varepsilon_x(s)| \leq \varepsilon_x(0) \quad \text{a.s.} \quad (5)$$

Proof: It is obvious that

$$0 \leq M \left\{ (x_{n+s} \pm x_n)^2 \right\} = M \left\{ x_{n+s}^2 \pm 2x_{n+s}x_n + x_n^2 \right\} = 2q_x(0) \pm 2q_x(s)$$

which proves (5) .

$$4. \quad q_{xy}^2(s) \leq q_x(0)q_y(0) \quad (6)$$

Proof: We can see

$$\begin{aligned} 0 \leq M \left\{ (ax_{n+s} + by_n)^2 \right\} &= M \left\{ a^2x_{n+s}^2 + 2abx_{n+s}y_n + b^2y_n^2 \right\} = \\ &= a^2q_x(0) + 2abq_{xy}(s) + b^2q_y(0) \end{aligned}$$

The above quadratic form is nonnegative for any a, b therefore its discriminant is nonnegative. Thus (6) is proved.

$$5. \quad \varepsilon_x(s) \text{ is nonnegative definite a.s.} \quad (7)$$

Proof: Let  $a_1, a_2, \dots, a_M$  be arbitrary values, then we have

$$\begin{aligned} \sum_{s, \ell=1}^M q_x(s-\ell) a_s a_\ell &= \sum_{s, \ell=1}^M \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=m+1}^{m+N} x_{n+s} x_{n+\ell} a_s a_\ell = \\ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=m+1}^{m+N} \sum_{s, \ell=1}^M x_{n+s} x_{n+\ell} a_s a_\ell &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=m+1}^{m+N} \left( \sum_{s=1}^M x_{n+s} a_s \right)^2 \geq 0 \end{aligned}$$

Thus we have (7) .

6. There exists a random function  $F(\lambda)$  , which is a spectral distribution function a.s. and

$$q_x(s) = \int_{-0.5}^{0.5} e^{2\pi i \lambda s} dF(\lambda) \quad \text{a.s.} \quad (8)$$

We can see that (8) is an immediate consequence of (7) .

We call  $F(\lambda)$  the time-spectral distribution function of the time series  $x_n$  .

If  $F(\lambda)$  is absolutely continuous a.s. then  $f(\lambda) = F'(\lambda)$  is called the time-spectral density function of  $x_n$  .

$$7. \int_{\mu_1}^{\mu_2} \frac{1}{N} \left| \sum_{k=0}^{N-1} x_k e^{-i2\pi\lambda k} \right|^2 d\lambda = F(\mu_2) - F(\mu_1) \quad \text{a.s.} \quad (9)$$

where the random variables  $\mu_1, \mu_2$  are the continuity points of  $F(\lambda)$ .

8. Linear operator on long-run stationary time series.

Suppose the sequence  $h_z, z = \dots, -1, 0, 1, \dots$  satisfies the following conditions:

$$\sum_{z=-\infty}^{\infty} |h_z| < \infty, \quad \sum_{z=-\infty}^{\infty} h_z^2 < \infty \quad (10)$$

and  $x_n$  is arbitrary long-run stationary time series then

$$y_n = \sum_{z=-\infty}^{\infty} h_z x_{n-z} \quad (11)$$

is also long-run stationary time series with

$$M\{y_n\} = M\{x_n\} \sum_{z=-\infty}^{\infty} h_z \quad (12)$$

$$g_y(s) = \sum_{z=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_z h_k g_x(s-z+k) \quad (13)$$

Proof: The proof of (12) is straightforward. To prove (13) let us consider

$$\frac{1}{N} \sum_{n=m+1}^{m+N} y_{n+s} y_n = \sum_{z=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_z h_k \frac{1}{N} \sum_{n=m+1}^{m+N} x_{n+s-z} x_{n-k}$$

from which

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=m+1}^{m+N} y_{n+s} y_n = \sum_{z=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_z h_k g_x(s-z+k)$$

We notice that by (5) the series on the right hand side of (13) converges.

### §.3. MOVING AVERAGE COMPOSITION OF LONG-RUN STATIONARY TIME SERIES

Theorem 1: a. If the time series  $x_n$  has the form

$$x_n = \sum_{s=0}^{\infty} b_s u_{n-s} \quad (14)$$

where  $b_0, b_1, b_2, \dots$  are random variables for which

$$\sum_{s=0}^{\infty} b_s^2 < \infty \quad (15)$$

and  $u_n$  is time-uncorrelated then  $x_n$  has the time spectral density function  $f(\lambda)$  and

$$\int_{-0.5}^{0.5} \ln f(\lambda) d\lambda > -\infty \quad \text{a.s.} \quad (16)$$

b. Conversely, if the time series  $x_n$  has the time spectral density function  $f(\lambda)$ , for which (16) holds then  $x_n$  can be written in the form (14), where  $u_n$  is time-uncorrelated. If  $u_n$  is a long-run white noise, then  $x_n$  is a long-run stationary process with  $M\{x_n\} = 0$ .

Proof: a. By (13) we have

$$g_x(s) = \sum_{\tau=0}^{\infty} \sum_{k=0}^{\infty} b_{\tau} b_k g_u(s - \tau + k) = \sigma^2 \sum_{n=0}^{\infty} b_{n+s} b_n$$

Thus

$$f(\lambda) = \sigma^2 \left| \sum_{n=0}^{\infty} b_n e^{i2\pi n\lambda} \right|^2$$

i.e. the problem is reduced to the theory of analytic functions /  $H^2$  classes /

b. For  $f(\lambda)$  satisfying (16) we can write

$$f(\lambda) = \sigma^2 \left| \sum_{n=0}^{\infty} b_n e^{i2\pi n\lambda} \right|^2 \quad \text{a.s.}$$

where  $b_0 = 1$  and  $\sum_{n=0}^{\infty} b_n z^n \neq 0$  for  $|z| \leq 1$ .

By the definition of the time spectral density we have

$$g_x(s) = \sigma^2 \sum_{n=0}^{\infty} b_{n+s} b_n$$

Set

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots = \frac{1}{\sum_{n=0}^{\infty} b_n z^n}$$

and

$$u_n = \sum_{s=0}^n a_s x_{n-s}, \quad n = 0, 1, 2, \dots$$

Then

$$x_n = \sum_{s=0}^n b_s u_{n-s}$$

Now we prove that  $u_n$  is time-uncorrelated.

To do this let us consider

$$\begin{aligned}
 g_u(s) &= \sum_{z=0}^{\infty} \sum_{k=0}^{\infty} a_z a_k g_x(s-z+k) = \\
 &= \sigma^2 \sum_{z=0}^{\infty} \sum_{k=0}^{\infty} a_z a_k \sum_{n=0}^{\infty} b_{n+s-z} b_{n-k} = \\
 &= \sigma^2 \sum_{n=0}^{\infty} \sum_{z=0}^{\infty} a_z b_{n+s-z} \sum_{k=0}^{\infty} a_k b_{n-k} = \\
 &= \sigma^2 \sum_{n=0}^{\infty} d_{n+s} d_n = \begin{cases} \sigma^2 & s=0 \\ 0 & s \neq 0 \end{cases}
 \end{aligned}$$

where  $D(z) = 1 + d_1 z + d_2 z^2 + \dots = A(z)B(z) = 1$   
 Thus the theorem is proved.

#### §.4. PREDICTIVE DECONVOLUTION OF AUTOREGRESSIVE TIME SERIES WITH RANDOM COEFFICIENTS

In [7] we have investigated the predictive deconvolution problem of long-run stationary autoregressive time series  $x_n$  which satisfies the following equation:

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_p x_{n-p} = u_n, \quad n=0,1,2,\dots \quad (17)$$

where the  $u_n$ 's are independent random variables with mean 0 and variance  $E u_n^2 = \sigma_n^2$  and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} E u_n^2 = \sigma^2 > 0$$

However, it is reasonable to assume that among the  $u_n$ 's there may be dependent random variables and variables having non-zero mean.

Similarly, the coefficients  $a_1, a_2, \dots, a_p$  may depend on experiment, i.e. they may be random variables. In this section we investigate this case.

Theorem 2: Let  $x_n$  be a process satisfying the equation

$$x_n + \alpha_1 x_{n-1} + \dots + \alpha_p x_{n-p} = u_n \quad n=0,1,2,\dots \quad (18)$$

where  $\alpha_1, \alpha_2, \dots, \alpha_p$  are random variables and the following three conditions:

a. Let  $\mathcal{N}$  denote the set of non-negative integers, and let  $H$  be a subset of  $\mathcal{N}$  such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \in \mathcal{N} \\ n+s \in H}} u_{n+s} u_n = 0 \quad \text{a.s.} \quad (19)$$

b. The  $u_n$ 's,  $n \in \mathcal{N} \setminus H$  are independent variables with mean 0 and

$$E |u_n|^{2+\varepsilon_0} < K < \infty, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \in \mathcal{N} \\ n \notin H}} E u_n^2 = \sigma^2 > 0 \quad (20)$$

$$c. \quad 1 + \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_p z^p \neq 0 \quad \text{for } z \leq 1 \quad (21)$$

Then  $x_n$  is a long-run stationary process with  $M\{x_n\} = 0$ .  
The matrix

$$\begin{pmatrix} g_x(0) & g_x(1) & \dots & g_x(p-1) \\ g_x(1) & g_x(0) & \dots & g_x(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ g_x(p-1) & g_x(p-2) & \dots & g_x(0) \end{pmatrix} \quad (22)$$

is positive definite a.s. and we have

$$\begin{pmatrix} g_x(0) & g_x(1) & \dots & g_x(p-1) \\ g_x(1) & g_x(0) & \dots & g_x(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ g_x(p-1) & g_x(p-2) & \dots & g_x(0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix} = - \begin{pmatrix} g_x(1) \\ g_x(2) \\ \vdots \\ g_x(p) \end{pmatrix} \quad (23)$$

Proof: Let

$$v_n = \begin{cases} 0 & n \in H \\ u_n & n \notin H \end{cases}$$

then the sequence  $v_n, n=0,1,2,\dots$  satisfies the conditions of the theorem in [7], hence we have

$$g_u(s) = g_v(s) = \begin{cases} \sigma^2 & s=0 \\ 0 & s \neq 0 \end{cases}$$

Now we can write

$$x_n = \sum_{s=0}^n \beta_s u_{n-s} \quad (24)$$

where

$$1 + \beta_1 z + \beta_2 z^2 + \dots = \frac{1}{1 + \alpha_1 z + \dots + \alpha_p z^p} \quad (24')$$

By (18) and (24)

$$\sum_{\tau=0}^p \alpha_{\tau} x_{n-\tau} x_{n-k} = \sum_{s=0}^{n-k} \beta_s u_n u_{n-k-s}$$

from which we have

$$\sum_{\tau=0}^p \alpha_{\tau} g_x(k-\tau) = \sum_{s=0}^{\infty} \beta_s g_u(s+k)$$

For  $k=1, 2, \dots, p$  we get (24).

Notice that

$$g_x(s) = \sum_{n=0}^{\infty} \beta_{n+s} \beta_n$$

then it is well known that (21) and (24) provide the positivity of the matrix (23). The proof is complete.

Remark: 1. If  $H$  is finite, then (19) is satisfied.

2. If  $|u_n| < K, n \in H$  and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_H(n) = 0 \quad \text{where} \quad \chi_H(n) = \begin{cases} 1 & n \in H \\ 0 & n \notin H \end{cases}$$

then (19) is also satisfied.

3. If we know  $g_x(\tau), \tau=0, \dots, p$  then  $\alpha_{\tau}$ 's are determined by (23). In practice we can estimate  $g_x(\tau)$  by

$$r_{\tau} = \frac{1}{N} \sum_{n=0}^{N-1} x_{n+\tau} x_n$$

and then  $\alpha_1, \alpha_2, \dots, \alpha_p$  are estimated by

$$\begin{pmatrix} r_0 & r_1 & \dots & r_{p-1} \\ r_1 & r_0 & \dots & r_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p-1} & r_{p-2} & \dots & r_0 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_p \end{pmatrix} = - \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{pmatrix} \quad (25)$$

Therefore we can estimate  $u_n$  by  $\hat{u}_n = x_n + \hat{\alpha}_1 x_{n-1} + \dots + \hat{\alpha}_p x_{n-p}$  and we have

$$\lim_{N \rightarrow \infty} \hat{u}_n = u_n \quad \text{a.s.}$$

Numerical example:

Example 2: Let  $x_n$  be a long-run stationary process defined by

$$x_n + \alpha x_{n-1} = u_n \quad \alpha = 0.5, \quad n = 0, 1, \dots, 499 \quad (26)$$

$$u_n = \begin{cases} W_n & v_n \leq 0.5 \\ 0 & v_n > 0.5 \end{cases} \quad n \neq 4, 5, 6, 9 \quad (27)$$

$$u_4 = u_5 = u_6 = u_9 = v$$

where  $W_n$ 's are independently, normally r.v.'s with mean 0 and variance 1,  $v_n$ 's and  $v$  are independently, uniformly r.v.'s on  $[0,1]$  and independent of  $W_n$ 's.

Thus

$$Eu_4 = Eu_5 = Eu_6 = Eu_9 = 0.5 \neq 0, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} Eu_n^2 = 0.5$$

By (25) we get the estimate  $\hat{\alpha} = 0.4941$ . In table 1 the first 30 values of  $u_n$  and its estimate are printed.

Table 1: Long-run white noise defined by (27) and its estimate

	long-run	estimated		long-run	estimated
n	white noise	white noise	n	white noise	white noise
0	0.1346	0.1346	15	0.0	-0.0061
1	0.2081	0.2073	16	0.0	0.0031
2	-0.1960	-0.1968	17	0.0	-0.0015
3	0.0	0.0016	18	0.0	0.0008
4	0.1527	0.1519	19	0.7724	0.7720
5	0.1527	0.1522	20	0.6930	0.6886
6	0.1527	0.1521	21	0.6723	0.6704
7	0.0	-0.0006	22	0.0541	0.0510
8	0.0	0.0003	23	0.0	0.0012
9	0.1527	0.1526	24	0.0	-0.0006
10	0.0	-0.0008	25	0.1007	0.1010
11	0.5493	0.5497	26	0.0	-0.0007
12	0.0	-0.0035	27	0.0	0.0004
13	0.0	0.0017	28	0.3580	0.3579
14	1.1048	1.1040	29	-0.0502	-0.0522

We can see that  $u_4, u_5, u_6$  and  $u_9$  are well estimated inspite of their dependence and the fact that they have non-zero mean.

§.5. PREDICTIVE DECONVOLUTION OF ARI-TIME SERIES

It is reasonable to think that in some applications

$$A(z) = 1 + \alpha_1 z + \dots + \alpha_p z^p \neq 0 \quad \text{for } |z| \leq 1$$

there may be some roots which are close to the unit circle. In this section we investigate the predictive deconvolution problem of the limiting case, i.e., the case in which there are some roots on the unit circle.

It is easy to prove the following theorem:

Theorem 3: Suppose the process  $x_n$  satisfies the conditions (18), (19), (20) and the Z-transform can be written in the form

$$A(z) = (1-z)^d (1 + \gamma_1 z + \dots + \gamma_q z^q) \tag{28}$$

where  $q + d = p$

$$1 + \gamma_1 z + \dots + \gamma_q z^q \neq 0 \quad \text{for } |z| \leq 1 \tag{29}$$

Let  $y_n = \Delta^d x_n$  where  $\Delta x_n = x_{n+1} - x_n$   
 Then  $y_n$  is a long-run stationary process with  $M\{y_n\} = 0$ .  
 The matrix

$$\begin{pmatrix} g_y(0) & g_y(1) & \dots & g_y(q-1) \\ \vdots & & & \\ g_y(q-1) & g_y(q-2) & \dots & g_y(0) \end{pmatrix}$$

is positive definite and we have

$$\begin{pmatrix} g_y(0) & g_y(1) & \dots & g_y(q-1) \\ g_y(1) & g_y(0) & \dots & g_y(q-2) \\ \vdots & & & \\ g_y(q-1) & g_y(q-2) & \dots & g_y(0) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_q \end{pmatrix} = - \begin{pmatrix} g_y(1) \\ g_y(2) \\ \vdots \\ g_y(q) \end{pmatrix} \tag{30}$$

In practice the predictive deconvolution of ARI - time series can be done as follows :

1. Compute  $y_n = \Delta^d x_n$
2. Estimate  $g_y(z)$  by  $r_y(z)$
3. Solve the normal equations to get the estimates  $\hat{\delta}_1, \hat{\delta}_2, \dots, \hat{\delta}_q$
4. Estimate  $\alpha_1, \alpha_2, \dots, \alpha_p$  by

$$1 + \hat{\alpha}_1 z + \dots + \hat{\alpha}_p z^p = (1-z)^d (1 + \hat{\delta}_1 z + \dots + \hat{\delta}_q z^q)$$

5. Estimate  $u_n$ 's by

$$\hat{u}_n = x_n + \hat{\alpha}_1 x_{n-1} + \dots + \hat{\alpha}_p x_{n-p}$$

Then we have

$$\lim_{N \rightarrow \infty} \hat{u}_n = u_n \quad \text{a.s.}$$

Numerical example:

Example 3: Let  $x_n$  be ARI-time series defined by

$$x_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2} = u_n, \quad \alpha_1 = -0.5, \alpha_2 = -0.5, n = 0, \dots, 499 \quad (31)$$

where  $u_n$  is defined by (27).

Then we get the estimates  $\hat{\alpha}_1 = -0.5058$ ,  $\hat{\alpha}_2 = -0.4942$ . In table 2 the first 20 values of  $u_n$  and its estimate are printed.

Table 2: Long-run white noise defined by (27) and its estimate

	long-run	estimated		long-run	estimated
n	white noise	white noise	n	white noise	white noise
0	0.1346	0.1346	10	0.0	-0.0008
1	0.2081	0.2073	11	0.5493	0.5497
2	-0.1960	-0.1968	12	0.0	-0.0034
3	0.0	0.0016	13	0.0	0.0017
4	0.1527	0.1519	14	1.1048	1.1040
5	0.1527	0.1522	15	0.0	-0.0060
6	0.1527	0.1521	16	0.0	0.0030
7	0.0	-0.0006	17	0.0	-0.0015
8	0.0	0.0003	18	0.0	0.0008
9	0.1527	0.1526	19	0.7724	0.7720

§.6. THE DETERMINATION OF THE ORDER OF ARI-TIME SERIES

As we have seen in the previous section, in the predictive deconvolution of ARI-time series we need to know the parameters  $d$  and  $q$ . However, in the majority of applications the observations  $x_n$ s are available only, thus we have to estimate the parameters  $d$  and  $q$ . In [7] we have shown that the usual methods for testing statistical hypotheses cannot be always applied to the long-run stationary processes. Here we propose to estimate  $d$  and  $q$  as follows:

1. For  $k=0,1,2,\dots$  compute  $y_n = \Delta^k x_n$

If for some  $k_0$  and for fixed  $s = 0,1,2,\dots$  the average

$$\frac{1}{N} \sum_{n=m+1}^{m+N} y_{n+s} y_n$$

stabilizes for large  $N$  then we estimate  $d$  by  $k_0$

/ It means that the process  $y_n = \Delta^{k_0} x_n$  is long-run stationary time series /

2. For  $j = 1,2,\dots$  we solve the equations

$$\begin{pmatrix} r_y(0) & r_y(1) & \dots & r_y(j-1) \\ r_y(1) & r_y(0) & \dots & r_y(j-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_y(j-1) & r_y(j-2) & \dots & r_y(0) \end{pmatrix} \begin{pmatrix} \hat{\delta}_1^j \\ \hat{\delta}_2^j \\ \vdots \\ \hat{\delta}_j^j \end{pmatrix} = - \begin{pmatrix} r_y(1) \\ r_y(2) \\ \vdots \\ r_y(j) \end{pmatrix}$$

Then we compute

$$\hat{u}_n^j = 1 + \hat{\alpha}_1^j x_{n-1} + \dots + \hat{\alpha}_{j+d}^j x_{n-j-d}$$

$$q_{\hat{u}}^j(s) = \frac{1}{N} \sum_{n=0}^{N-1} \hat{u}_{n+s}^j \hat{u}_n^j$$

If for some  $j_0$  we have

$$q_{\hat{u}}^{j_0}(0) = \sigma^2, \quad q_{\hat{u}}^{j_0}(s) \approx 0, \quad s = 1,2,\dots$$

i.e.  $q_{\hat{u}}^{j_0}(s), s = 1,2,\dots$  are negligible compared with  $q_{\hat{u}}^{j_0}(0)$  then we estimate  $q$  by  $j_0$ .

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On moving average composition and predictive deconvolution  
of long-run stationary time series

Phan Dang Cau

Summary

The notions of empirical and long-run stationary processes respectively, are introduced.

It is proposed to consider the time mean and the time correlation functions as independent notions /they coincide with the mean and the autocorrelation function when the process is ergodic and stationary, only/.

The moving average composition and predictive deconvolution of the long-run process are investigated.

Empirikusan stacionárius folyamat mozgóátlag felbontása  
és prediktív dekonvolúciója

Phan Dang Cau

Összefoglaló

A cikkben a szerző bevezeti az empirikusan stacionárius folyamat fogalmát, vizsgálja e folyamat mozgóátlag felbontását, prediktív dekonvolúcióját.