Discrete majorization type inequalities, comparison of classic results with a recent result

László Horváth
Department of Mathematics, University of Pannonia
Egyetem u. 10.
8200 Veszprém, Hungary
e-mail: horvath.laszlo@mik.uni-pannon.hu

Abstract

In this paper, we examine the relationship between a recent new discrete majorization type inequality and classical majorization type inequalities. The multiplicative analogue of the studied new inequality is also given, which is a wide generalization of Weyl's inequality.As an application, we give a parametric refinement of Popoviciu's version of the Petrović inequality.

Keywords: Majorization type inequalities, Convex functions, Fuchs inequality, Hardy-Littlewood-Pólya inequality, Petrović inequality
MSC2010 Classification: 26A51, 26D15

1 Introduction

By $\mathbb{N}_+$ we denote the set of positive integers.

Let $C \subset \mathbb{R}$ be an interval with nonempty interior (the interior of C is denoted by C°). Let denote F_C the set of all convex functions on C. Let denote F_C^i the set of all increasing and convex functions on C.

The following result is a majorization type inequality which is contained in Theorem 9 of [4].

Theorem 1 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$, let $Y := \{1, \ldots, n\}$ for some $n \in \mathbb{N}_+$, and let $C \subset \mathbb{R}$ be an interval with nonempty interior. Assume $(p_i)_{i=1}^m$ and $(q_j)_{j=1}^n$ are real sequences, and $\mathbf{s} := (s_1, \ldots, s_m) \in C^m$ and $\mathbf{t} := (t_1, \ldots, t_n) \in C^n$. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and $\mathbf{t}$ in decreasing order $(1 \le o \le m + n)$. Then*

(a) For every $f \in F_C^i$ inequality

$$\sum_{i=1}^{m} p_i f(s_i) \le \sum_{j=1}^{n} q_j f(t_j) \tag{1}$$

holds if and only if

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \tag{2}$$

and

$$\sum_{\{i\in X | s_i \geq u_l\}} p_i s_i - \sum_{\{j\in Y | t_j \geq u_l\}} q_j t_j$$

$$\leq u_l \left(\sum_{\{i\in X | s_i \geq u_l\}} p_i - \sum_{\{j\in Y | t_j \geq u_l\}} q_j \right), \quad l = 1, \ldots, o \tag{3}$$

are satisfied.

(b) For every $f \in F_C$ inequality (1) holds if and only if (2) and

$$\sum_{i=1}^{m} p_i s_i = \sum_{j=1}^{n} q_j t_j \tag{4}$$

and (3) are satisfied.

Remark 2 *Note that (3) is always true for $l = 1$ in both cases. If (2) and (4) hold, then (3) is also true for $l = o$.*

Paper [4] was mainly devoted to majorization type inequalities for integrals with signed measures, in this paper we analyze the discrete version (1). It is obvious that Theorem 1 contains Fuchs inequality, the Hardy-Littlewood-Pólya or majorization inequality and the weighted version of it, but the precise relationship between Theorem 1 and these classical results are far from trivial, and this topic was not thoroughly covered in paper [4]. It is worth investigating how the aforementioned classical inequalities can be derived directly from Theorem 1, and this is the basic purpose of this paper. Furthermore, we obtain a precise formulation of the classical results mentioned above for both convex and increasing convex functions. Theorem 1 also provides an opportunity for a new characterisation of both weak majorization and majorization. The multiplicative analogue of Theorem 1 is also given, which is a wide generalization of Weyl's inequality. As an application, we give a parametric refinement of Popoviciu's version of the Petrović inequality.

2 Preliminary results

The following result, interesting in itself, is needed to prove one of the main results.

Lemma 3 *Assume*

$$w_1 \geq w_2 \geq \ldots \geq w_{k-1} \geq w_k \geq 0 > z_1 \geq z_2 \geq \ldots \geq z_{k-1} \geq z_k \tag{5}$$

for some $k \in \mathbb{N}_+$. Then there are no real numbers $r_1, \ldots, r_k$ for which

$$\left.\begin{array}{l} r_1 w_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + r_k w_k < 0 \\ r_1 z_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + r_k w_k < 0 \\ \vdots \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + r_k w_k < 0 \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + r_k z_k < 0 \end{array}\right\}. \qquad (6)$$

Proof. We argue by induction on k, the case $k = 1$ is obvious. Let $k \in \mathbb{N}_+$ such that the result holds for every pair of k-tuples $(w_1, \ldots, w_k)$ and $(z_1, \ldots, z_k)$ satisfying (5). Assume

$$w_1 \geq w_2 \geq \ldots \geq w_{k-1} \geq w_k \geq w_{k+1} \geq 0 > z_1 \geq z_2 \geq \ldots \geq z_{k-1} \geq z_k \geq z_{k+1},$$

and suppose that there exist real numbers $r_1, \ldots, r_{k+1}$ for which

$$\left.\begin{array}{l} r_1 w_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + r_k w_k + r_{k+1} w_{k+1} < 0 \\ r_1 z_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + r_k w_k + r_{k+1} w_{k+1} < 0 \\ \vdots \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + r_k w_k + r_{k+1} w_{k+1} < 0 \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + r_k z_k + r_{k+1} w_{k+1} < 0 \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + r_k z_k + r_{k+1} z_{k+1} < 0 \end{array}\right\}.$$

If $r_{k+1} \geq 0$, then $r_{k+1} w_{k+1} \geq 0$, and hence $(r_1, \ldots, r_k)$ is a solution of (6), giving a contradiction.

Assume $r_{k+1} < 0$. Then $w_k \geq w_{k+1}$ and $z_k \geq z_{k+1}$ imply that

$$r_k w_k + r_{k+1} w_{k+1} \geq (r_k + r_{k+1}) w_k, \quad r_k z_k + r_{k+1} z_{k+1} \geq (r_k + r_{k+1}) z_k,$$

and therefore $(r_1, \ldots, r_{k-1}, r_k + r_{k+1})$ is a solution of

$$\left.\begin{array}{l} r_1 w_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + (r_k + r_{k+1}) w_k < 0 \\ r_1 z_1 + r_2 w_2 + \ldots + r_{k-1} w_{k-1} + (r_k + r_{k+1}) w_k < 0 \\ \vdots \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + (r_k + r_{k+1}) w_k < 0 \\ r_1 z_1 + r_2 z_2 + \ldots + r_{k-1} z_{k-1} + (r_k + r_{k+1}) z_k < 0 \end{array}\right\},$$

which is also a contradiction.

The proof is complete. ■

3 Majorization type inequalities

We start by comparing the classical Fuchs inequality (see below) and inequality (1).

Theorem 4 *(Fuchs inequality, see [1]) Let $C \subset \mathbb{R}$ be an interval with nonempty interior. If $(s_1, \ldots, s_m) \in C^m$, $(t_1, \ldots, t_m) \in C^m$ and $p_1, \ldots, p_m$ are real numbers such that*

(a) $s_1 \geq \ldots \geq s_m$ and $t_1 \geq \ldots \geq t_m$,

(b) $\sum\limits_{i=1}^{k} p_i s_i \leq \sum\limits_{i=1}^{k} p_i t_i$ $(k = 1, \ldots, m-1)$,

(c) $\sum\limits_{i=1}^{m} p_i s_i = \sum\limits_{i=1}^{m} p_i t_i$,

then for every $f \in F_C$ inequality

$$\sum_{i=1}^{m} p_i f(s_i) \leq \sum_{i=1}^{m} p_i f(t_i) \tag{7}$$

holds.

Clearly, Theorem 1 contains Fuchs inequality, but as the result below shows, proving it directly is not trivial.

Lemma 5 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$. Let $\mathbf{s} := (s_1, \ldots, s_m) \in \mathbb{R}^m$, $\mathbf{t} := (t_1, \ldots, t_m) \in \mathbb{R}^m$ such that $s_1 \geq \ldots \geq s_m$ and $t_1 \geq \ldots \geq t_m$. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and $\mathbf{t}$ in decreasing order $(1 \leq o \leq 2m)$. Let $p_1, \ldots, p_m$ be real numbers. If*

$$\sum_{i=1}^{k} p_i s_i \leq \sum_{i=1}^{k} p_i t_i, \quad k = 1, \ldots, m, \tag{8}$$

are satisfied, then

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i s_i - \sum_{\{j \in X \mid t_j \geq u_l\}} p_j t_j$$
$$\leq u_l \left(\sum_{\{i \in X \mid s_i \geq u_l\}} p_i - \sum_{\{j \in X \mid t_j \geq u_l\}} p_j \right), \quad l = 1, \ldots, o. \tag{9}$$

Proof. If $l = 1$, then (9) has equality, and if $l = o$, then (9) follows from the fact that (8) is satisfied for $k = m$ (see Remark 2).

Assume $1 < l < o$, and let

$$\{1, \ldots, n_1\} := \{i \in X \mid s_i \geq u_l\}$$

and

$$\{1, \ldots, n_2\} := \{j \in X \mid t_j \geq u_l\}.$$

For $n_1 = n_2$, (9) can be obtained from the fact that (8) is true for $k = n_1$.

Assume $n_2 > n_1$ (if $\{i \in X \mid s_i \geq u_l\}$ is empty, let $n_1 := 0$, and in this case the empty sum is defined to equal 0). Then (9) can be written in the form

$$\sum_{i=1}^{n_1} p_i s_i - \sum_{j=1}^{n_2} p_j t_j \leq u_l \left(\sum_{i=1}^{n_1} p_i - \sum_{j=1}^{n_2} p_j \right) = -u_l \sum_{j=n_1+1}^{n_2} p_j. \tag{10}$$

By (8),

$$\sum_{i=1}^{n_1} p_i s_i - \sum_{j=1}^{n_2} p_j t_j \leq \begin{cases} -\sum\limits_{j=n_1+1}^{n_2} p_j t_j \\ -\sum\limits_{j=n_1+2}^{n_2} p_j t_j - p_{n_1+1} s_{n_1+1} \\ \vdots \\ -\sum\limits_{j=n_1+1}^{n_2} p_j s_j \end{cases} . \tag{11}$$

It can be seen from (10) and (11) that it is enough to show the next: at least one of the expressions

$$\begin{array}{l} \sum\limits_{j=n_1+1}^{n_2} p_j \left(t_j - u_l\right) \\ \sum\limits_{j=n_1+2}^{n_2} p_j \left(t_j - u_l\right) + p_{n_1+1} \left(s_{n_1+1} - u_l\right) \\ \vdots \\ \sum\limits_{j=n_1+1}^{n_2} p_j \left(s_j - u_l\right) \end{array}$$

is nonnegative. But this follows from Lemma 3, since

$$t_{n_1+1} - u_l \geq \ldots \geq t_{n_2} - u_l \geq 0 > s_{n_1+1} - u_l \geq \ldots \geq s_{n_2} - u_l.$$

We can prove similarly if $n_1 > n_2$.

The proof is complete. ∎

Remark 6 *Theorem 1 is much more general than Fusch inequality. On the one hand,* $m = n$ *and* $p_i = q_i$ $(i = 1, \ldots, m)$ *in the Fusch inequality, on the other hand Fusch inequality gives only sufficient but not necessary conditions for satisfying inequality (7). This last remark can be easily checked with the following simple example: let*

$$s_1 = s_2 := 2, \quad s_3 = s_4 = s_5 := 1, \quad t_1 = t_2 = t_3 = t_4 = t_5 := 3$$

and

$$p_1 := -1,\ p_2 := 2,\ p_3 := -2,\ p_4 := 3,\ p_5 := -\frac{3}{2}.$$

In this case

$$\sum_{i=1}^{5} p_i s_i = \sum_{i=1}^{5} p_i t_i = \frac{3}{2}$$

and

$$p_1 s_1 = -2 > -3 = p_1 t_1 \text{ and } \sum_{i=1}^{3} p_i s_i = 0 > -3 = \sum_{i=1}^{3} p_i t_i,$$

and hence the conditions of Fusch inequality are not satisfied. However,

$$\sum_{i=1}^{5} p_i f(s_i) = f(2) - \frac{1}{2} f(1) \leq \frac{1}{2} f(3) = \sum_{i=1}^{5} p_i f(t_i)$$

for every convex function $f : C \to \mathbb{R}$ for which 1, 2 and 3 belong to the interval C.

Of course, conditions (9) are true.

The previous result and Theorem 1 give Fusch inequality for increasing and convex functions.

Corollary 7 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior. If $(s_1, \ldots, s_m) \in C^m$, $(t_1, \ldots, t_m) \in C^m$ and $p_1, \ldots, p_m$ are real numbers such that*

(a) $s_1 \geq \ldots \geq s_m$ and $t_1 \geq \ldots \geq t_m$,

(b) $\sum_{i=1}^{k} p_i s_i \leq \sum_{i=1}^{k} p_i t_i$ $(k = 1, \ldots, m)$,

then for every $f \in F_C^i$ inequality (7) holds.

Next, we compare Theorem 1 with the majorization inequality.

Majorization is a binary relation (preorder) for finite sequences of real numbers, and the theory of majorization is a significant topic in mathematics (see [5]).

Definition 8 *Let $\mathbf{s} := (s_1, \ldots, s_n) \in \mathbb{R}^n$ and $\mathbf{t} := (t_1, \ldots, t_n) \in \mathbb{R}^n$.*

(a) We say that $\mathbf{s}$ is weakly majorized by $\mathbf{t}$, written $\mathbf{s} \prec_w \mathbf{t}$, if

$$\sum_{i=1}^{k} s_{[i]} \leq \sum_{i=1}^{k} t_{[i]}, \quad k = 1, \ldots, m, \tag{12}$$

where $s_{[1]} \geq s_{[2]} \geq \ldots \geq s_{[m]}$ and $t_{[1]} \geq t_{[2]} \geq \ldots \geq t_{[m]}$ are the entries of $\mathbf{s}$ and $\mathbf{t}$, respectively, in decreasing order.

(b) We say that $\mathbf{s}$ is majorized by $\mathbf{t}$, written $\mathbf{s} \prec \mathbf{t}$, if (12) holds, and in addition

$$\sum_{i=1}^{m} s_{[i]} = \sum_{i=1}^{m} t_{[i]}. \tag{13}$$

The Hardy-Littlewood-Pólya or majorization inequality is the next:

Theorem 9 *(see [3] and [6]) Let $C \subset \mathbb{R}$ be an interval with nonempty interior, and let $\mathbf{s} := (s_1, \ldots, s_m) \in C^m$ and $\mathbf{t} := (t_1, \ldots, t_m) \in C^m$. Then*

(a) Inequality

$$\sum_{i=1}^{m} f(s_i) \leq \sum_{i=1}^{m} f(t_i) \tag{14}$$

holds for every $f \in F_C$ if and only if $\mathbf{s} \prec \mathbf{t}$.

(b) Inequality (14) holds for every $f \in F_C^i$ if $\mathbf{s} \prec_w \mathbf{t}$.

Since the majorization inequality gives a necessary and sufficient condition for satisfying inequality (14), which is a special case of inequality (1), condition $\mathbf{s} \prec \mathbf{t}$ must be equivalent to a suitable condition of Theorem 1. However, the relation of conditions (2), (4) and (3) with condition $\mathbf{s} \prec \mathbf{t}$ is not directly evident, so we analyze this relation in the following statement.

The cardinality of a set A is denoted by $|A|$.

Lemma 10 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$. Let $\mathbf{s} := (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $\mathbf{t} := (t_1, \ldots, t_m) \in \mathbb{R}^m$. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and $\mathbf{t}$ in decreasing order $(1 \leq o \leq 2m)$. Then*

(a) $\mathbf{s} \prec_w \mathbf{t}$ if and only if

$$\sum_{\{i \in X \mid s_i \geq u_l\}} s_i - \sum_{\{j \in X \mid t_j \geq u_l\}} t_j$$

$$\leq u_l \left(|\{i \in X \mid s_i \geq u_l\}| - |\{j \in X \mid t_j \geq u_l\}| \right), \quad l = 1, \ldots, o. \tag{15}$$

(b) $\mathbf{s} \prec \mathbf{t}$ if and only if

$$\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} t_i$$

and (15) are satisfied.

Proof. (a) It follows from Lemma 5 that $\mathbf{s} \prec_w \mathbf{t}$ implies (15).

Conversely, suppose on the contrary that $s_{[1]} > t_{[1]}$, and let l_1 be defined such that $u_{l_1} = t_{[1]}$, and let

$$\{1, \ldots, n_1\} := \left\{ i \in X \mid s_{[i]} \geq u_{l_1} \right\}$$

and

$$\{1, \ldots, n_2\} := \left\{ j \in X \mid t_{[j]} = u_{l_1} \right\}.$$

By (15),

$$\sum_{i=1}^{n_1} s_{[i]} - n_2 t_{[1]} \leq t_{[1]} \left(n_1 - n_2 \right),$$

and therefore

$$\sum_{i=1}^{n_1} s_{[i]} \leq n_1 t_{[1]},$$

which is a contradiction.

Let $k \in \{1, \ldots, m-1\}$ for which

$$\sum_{i=1}^{k-1} s_{[i]} \leq \sum_{i=1}^{k-1} t_{[i]}. \tag{16}$$

If

$$\sum_{i=1}^{k} s_{[i]} > \sum_{i=1}^{k} t_{[i]}$$

would be satisfied, then $s_{[k]} > t_{[k]}$ necessarily.

Let l_2 be defined such that $u_{l_2} := t_{[k]}$, and let

$$\{1, \ldots, n_3\} := \left\{ i \in X \mid s_{[i]} \geq u_{l_2} \right\}$$

and

$$\{1, \ldots, n_4\} := \left\{ j \in X \mid t_{[j]} = u_{l_2} \right\}.$$

Then by (15) (the empty sum is defined to equal 0),

$$\sum_{i=1}^{k-1} s_{[i]} - \sum_{i=1}^{k-1} t_{[i]} - (n_4 - (k-1))\, t_{[k]} \leq t_{[k]} \,(n_3 - n_4),$$

and hence (16) shows that

$$\sum_{i=k}^{n_3} s_{[i]} \leq t_{[k]} \,(n_3 - (k-1)),$$

which is also a contradiction.

(b) It is an immediate consequence of (a).

The proof is complete. ■

Remark 11 *(a) By Theorem 5 (a) and Lemma 10 (a), condition* $\mathbf{s} \prec_w \mathbf{t}$ *in Theorem 9 (b) is not only sufficient but also a necessary condition.*

(b) There are different characterisations of the relation of majorization (for example, by doubly stochastic matrices, see [6]), the previous lemma gives a new characterization of both weak majorization and majorization.

Finally, we turn to the relationship between the weighted version of Hardy-Littlewood-Pólya inequality and Theorem 1. The weighted version of Hardy-Littlewood-Pólya inequality is the next:

Theorem 12 *Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior. If* $(s_1, \ldots, s_m) \in C^m$, $(t_1, \ldots, t_m) \in C^m$ *and* $p_1, \ldots, p_m$ *are nonnegative numbers such that*

(a) $s_1 \geq \ldots \geq s_m$,

(b) $\sum_{i=1}^{k} p_i s_i \leq \sum_{i=1}^{k} p_i t_i$ $(k = 1, \ldots, m-1)$,

(c) $\sum_{i=1}^{m} p_i s_i = \sum_{i=1}^{m} p_i t_i$,

then inequality

$$\sum_{i=1}^{m} p_i f(s_i) \leq \sum_{i=1}^{m} p_i f(t_i) \tag{17}$$

holds for every $f \in F_C$.

Obviously, this result is also included in Theorem 1, but the direct proof of it is much simpler than for the Fusch inequality.

Lemma 13 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$. Let $\mathbf{s} := (s_1, \ldots, s_m) \in \mathbb{R}^m$ such that $s_1 \geq \ldots \geq s_m$, and let $\mathbf{t} := (t_1, \ldots, t_m) \in \mathbb{R}^m$. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and $\mathbf{t}$ in decreasing order $(1 \leq o \leq 2m)$. Let $p_1, \ldots, p_m$ be nonnegative numbers. If*

$$\sum_{i=1}^{k} p_i s_i \leq \sum_{i=1}^{k} p_i t_i, \quad k = 1, \ldots, m, \tag{18}$$

are satisfied, then

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i s_i - \sum_{\{j \in X \mid t_j \geq u_l\}} p_j t_j \leq u_l \left(\sum_{\{i \in X \mid s_i \geq u_l\}} p_i - \sum_{\{j \in X \mid t_j \geq u_l\}} p_j \right), \quad l = 1, \ldots, o. \tag{19}$$

Proof. Let $l \in \{1, \ldots, o\}$ be fixed, and let

$$\{1, \ldots, n_1\} := \{i \in X \mid s_i \geq u_l\}.$$

Since the numbers $p_1, \ldots, p_m$ are nonnegative, (19) obviously holds if $\{i \in X \mid s_i \geq u_l\}$ is empty, so it can be supposed that $n_1 \geq 1$. Then (19) can be written in the form

$$\sum_{i=1}^{n_1} p_i (s_i - u_l) \leq \sum_{i=1}^{n_1} p_i (t_i - u_l) - \sum_{\{j \in \{1, \ldots, n_1\} \mid t_j < u_l\}} p_j (t_j - u_l) + \sum_{\{j \in \{n_1+1, \ldots, m\} \mid t_j \geq u_l\}} p_j (t_j - u_l). \tag{20}$$

This is valid, since the members of the sums in (20) are nonnegative, and by (18),

$$\sum_{i=1}^{n_1} p_i (s_i - u_l) \leq \sum_{i=1}^{n_1} p_i (t_i - u_l).$$

The proof is complete. ■

Remark 14 *The findings of Remark 6 also remain valid here. Again, we only emphasize that the conditions of the Theorem 12 are only sufficient but not necessary for the inequality (17) to hold for all $f \in F_C$. This is illustrated only a simple example: let $s_1 = t_2 := 2$, $s_2 = t_1 := 1$ and $p_1 = p_2 := 1$. Then $p_1 s_1 = 2 > 1 = p_1 t_1$, and hence condition (b) in Theorem 12 is not satisfied. Despite this, $p_1 f(s_1) + p_2 f(s_2) = f(2) + f(1) = p_1 f(t_1) + p_2 f(t_2)$ for all functions whose domain contains 1 and 2.*

Just as in the case of Fusch inequality, the previous result and Theorem 1 give the weighted version of Hardy-Littlewood-Pólya inequality for increasing and convex functions.

Corollary 15 *Let $C \subset \mathbb{R}$ be an interval with nonempty interior. If $(s_1, \ldots, s_m) \in C^m$, $(t_1, \ldots, t_m) \in C^m$ and $p_1, \ldots, p_m$ are nonnegative numbers such that*

(a) $s_1 \geq \ldots \geq s_m$,

(b) $\sum_{i=1}^{k} p_i s_i \leq \sum_{i=1}^{k} p_i t_i$ $(k = 1, \ldots, m)$,

then for every $f \in F_C^i$ inequality (17) holds.

Finally, we give a multiplicative version of the majorization type inequalities, which has its origins in Weyl's paper [11]. Theorem 1 can be reformulated in this form, let's give it first.

Theorem 16 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$, let $Y := \{1, \ldots, n\}$ for some $n \in \mathbb{N}_+$, and let $C \subset]0, \infty[$ be an interval with nonempty interior. Assume $(p_i)_{i=1}^m$ and $(q_j)_{j=1}^n$ are real sequences, and $\mathbf{s} := (s_1, \ldots, s_m) \in C^m$ and $\mathbf{t} := (t_1, \ldots, t_n) \in C^n$. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and $\mathbf{t}$ in decreasing order $(1 \leq o \leq m + n)$. If*

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j \quad \text{and} \quad \prod_{i=1}^{m} s_i^{p_i} = \prod_{j=1}^{n} t_j^{p_j}$$

and

$$\frac{\prod\limits_{\{i \in X | s_i \geq u_l\}} s_i^{p_i}}{\prod\limits_{\{j \in Y | t_j \geq u_l\}} t_j^{q_j}} \leq u_l^{\sum\limits_{\{i \in X | s_i \geq u_l\}} p_i - \sum\limits_{\{j \in Y | t_j \geq u_l\}} q_j}, \quad l = 1, \ldots, o,$$

then for every $f : C \to \mathbb{R}$ for which $f \circ \exp$ is convex inequality

$$\sum_{i=1}^{m} p_i f(s_i) \leq \sum_{i=1}^{n} q_j f(t_j) \tag{21}$$

holds.

Proof. Let $\ln(\mathbf{s}) := (\ln(s_1), \ldots, \ln(s_m))$ and $\ln(\mathbf{t}) := (\ln(t_1), \ldots, \ln(t_n))$.

Since the function ln is strictly increasing, exactly $\ln(u_1) > \ln(u_2) > \ldots > \ln(u_o)$ are the different elements of $\ln(\mathbf{s})$ and $\ln(\mathbf{t})$ in decreasing order and

$$\{i \in X \mid s_i \geq u_l\} = \{i \in X \mid \ln(s_i) \geq \ln(u_l)\}$$

and

$$\{j \in Y \mid t_j \geq u_l\} = \{j \in Y \mid \ln(t_j) \geq \ln(u_l)\}.$$

Now Theorem 1 can be applied to $\ln(\mathbf{s})$, $\ln(\mathbf{t})$ and $f \circ \exp$.

The proof is complete. ■

For the sake of completeness, we also give the variant related to the Fusch inequality, which is a special case of the previous one, but more similar to the original form of Weyl.

Theorem 17 *Let* $C \subset \left]0,\infty\right[$ *be an interval with nonempty interior. If* $(s_1,\ldots,s_m) \in C^m$, $(t_1,\ldots,t_m) \in C^m$ *and* $p_1,\ldots,p_m$ *are real numbers such that*

(a) $s_1 \geq \ldots \geq s_m$ *and* $t_1 \geq \ldots \geq t_m$,

(b) $\prod_{i=1}^{k} s_i^{p_i} \leq \prod_{i=1}^{k} t_i^{p_i}$ $(k = 1,\ldots,m-1)$,

(c) $\prod_{i=1}^{m} s_i^{p_i} = \prod_{i=1}^{m} t_i^{p_i}$,

then for every $f : C \to \mathbb{R}$ *for which* $f \circ \exp$ *is convex inequality*

$$\sum_{i=1}^{m} p_i f(s_i) \leq \sum_{i=1}^{m} p_i f(t_i)$$

holds.

Proof. Fusch inequality can be applied to the n-tuples $(\ln(s_1),\ldots,\ln(s_m))$ and $(\ln(t_1),\ldots,\ln(t_m))$ and to the function $f \circ \exp$. ■

Remark 18 *(a) Theorem 16 is much more general than Weyl's original inequality. Note that in Theorem 16, in general* $m \neq n$ *and the weights* $(p_i)_{i=1}^{m}$ *and* $(q_j)_{j=1}^{n}$ *are different.*

(b) Crucially, inequality (21) is true for functions $f : C \to \mathbb{R}$ *for which* $f \circ \exp$ *is convex, but it is not true in general if we only assume that* f *is convex. For example, let*

$$s_1 := 2, \quad s_2 := 1, \quad s_3 := 1/2, \quad t_1 := 3, \quad t_2 := 1, \quad t_3 := 1/3$$

and

$$p_1 = p_2 = p_3 := 1$$

and $f : \left]0,\infty\right[\to \mathbb{R}$, $f(x) := (x-3)^2$. *Then conditions (a), (b) and (c) in Theorem 17 are satisfied and* f *is convex (*$f \circ \exp$ *is not convex), but*

$$\sum_{i=1}^{3} f(s_i) = \frac{45}{4} > \frac{100}{9} = \sum_{i=1}^{3} f(t_i).$$

4 Application

The following inequality comes from Petrović [9].

Theorem 19 *Let* $C \subset \left[0,\infty\right[$ *be an interval with nonempty interior and with* $0 \in C$. *If* $(s_1,\ldots,s_m) \in C^m$ *such that*

$$\sum_{i=1}^{m} s_i \in C,$$

then for every $f \in F_C$ *we have*

$$\sum_{i=1}^{m} f(s_i) \leq f\left(\sum_{i=1}^{m} s_i\right) + (m-1) f(0).$$

The previous result has many generalizations, see for example papers [2], [7] and [10].

For the sake of clarity and ease of calculation, the Popoviciu's version is considered below.

First, we give a simple proof of Popoviciu's version using Theorem 1, illustrating its applicability.

Theorem 20 *(see [10]) Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$. Let $C \subset [0, \infty[$ be an interval with nonempty interior and with $0 \in C$. If $(s_1, \ldots, s_m) \in C^m$ and $p_1, \ldots, p_m$ are nonnegative numbers such that*

$$S := \max_{1 \leq i \leq m} s_i \leq \sum_{i=1}^{m} p_i s_i \in C, \tag{22}$$

then for every $f \in F_C$ we have

$$\sum_{i=1}^{m} p_i f(s_i) \leq f\left(\sum_{i=1}^{m} p_i s_i\right) + \left(\sum_{i=1}^{m} p_i - 1\right) f(0). \tag{23}$$

Proof. Let $u_1 > u_2 > \ldots > u_o$ be the different elements of $s_1, \ldots, s_m$, $t_1 := \sum\limits_{i=1}^{m} p_i s_i$ and $t_2 := 0$ in decreasing order ($1 \leq o \leq m+2$), and let $q_1 := 1$ and $q_2 := \sum\limits_{i=1}^{m} p_i - 1$. By (22),

$$u_1 = t_1 \quad \text{and} \quad u_o = t_2.$$

Since

$$\sum_{i=1}^{m} p_i = q_1 + q_2 \quad \text{and} \quad \sum_{i=1}^{m} p_i s_i = q_1 t_1 + q_2 t_2,$$

Theorem 1 and Remark 2 show that it is enough to prove that

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i s_i - \sum_{i=1}^{m} p_i s_i \leq u_l \left(\sum_{\{i \in X \mid s_i \geq u_l\}} p_i - 1\right), \quad l = 2, \ldots, o-1. \tag{24}$$

This is obviously true if

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i - 1 \geq 0.$$

Assume that

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i < 1 \tag{25}$$

and (24) is not satisfied that is

$$\sum_{i=1}^{m} p_i s_i < \sum_{\{i \in X \mid s_i \geq u_l\}} p_i (s_i - u_l) + u_l.$$

Then by (25),

$$\sum_{i=1}^{m} p_i s_i < \sum_{\{i \in X \mid s_i \geq u_l\}} p_i \left(S - u_l\right) + u_l \leq \left(S - u_l\right) + u_l = S,$$

which contradicts to (22).

The proof is complete. ■

Next, we give a refinement of inequality (23) in the form

$$\sum_{i=1}^{m} p_i f\left(s_i\right) \leq r_1 f\left(w_1\right) + r_2 f\left(w_2\right) \leq f\left(\sum_{i=1}^{m} p_i s_i\right) + \left(\sum_{i=1}^{m} p_i - 1\right) f\left(0\right). \quad (26)$$

By Theorem 1, the system of linear equations

$$\left.\begin{array}{l} r_1 + r_2 = \sum\limits_{i=1}^{m} p_i \\ r_1 w_1 + r_2 w_2 = \sum\limits_{i=1}^{m} p_i s_i \end{array}\right\}$$

must be satisfied, and therefore

$$r_1 = \frac{\sum\limits_{i=1}^{m} p_i s_i - w_2 \sum\limits_{i=1}^{m} p_i}{w_1 - w_2} \quad \text{and} \quad r_2 = \frac{w_1 \sum\limits_{i=1}^{m} p_i - \sum\limits_{i=1}^{m} p_i s_i}{w_1 - w_2} \quad (27)$$

necessarily.

Theorem 21 *Let $X := \{1, \ldots, m\}$ for some $m \in \mathbb{N}_+$, and let $C := [0, \infty[$. Assume $\mathbf{s} := (s_1, \ldots, s_m) \in C^m$ and $p_1, \ldots, p_m$ are nonnegative numbers such that*

$$0 < s := \min_{1 \leq i \leq m} s_i \quad \text{and} \quad S := \max_{1 \leq i \leq m} s_i \leq \sum_{i=1}^{m} p_i s_i.$$

Then for every $0 < w_2 \leq s$ we can find a number $c \geq \sum\limits_{i=1}^{m} p_i s_i$ (depending on w_2) such that for any $w_1 \geq c$ and any $f \in F_C$ (26) is satisfied.

Proof. First we show that

$$r_1 f\left(w_1\right) + r_2 f\left(w_2\right) \leq f\left(\sum_{i=1}^{m} p_i s_i\right) + \left(\sum_{i=1}^{m} p_i - 1\right) f\left(0\right) \quad (28)$$

holds for all

$$0 < w_2 \leq \frac{\sum\limits_{i=1}^{m} p_i s_i}{\sum\limits_{i=1}^{m} p_i} < w_1$$

and for all $f \in F_C$.

Then either

$$0 < w_2 < w_1 \leq \sum_{i=1}^{m} p_i s_i$$

or

$$0 < w_2 \leq \sum_{i=1}^{m} p_i s_i < w_1.$$

In both cases, the fulfilment of condition (3) can be checked by elementary calculation, so Theorem 1 can be applied.

Second, we study the inequality

$$\sum_{i=1}^{m} p_i f(s_i) \leq r_1 f(w_1) + r_2 f(w_2) \tag{29}$$

under the conditions

$$0 < w_2 \leq s \quad \text{and} \quad \sum_{i=1}^{m} p_i s_i < w_1.$$

If $s_1 = \ldots = s_m$, then (29) is a simple Jensen's inequality, so we can suppose that $s < S$.

Let w_2 be fixed, and let $u_1 > u_2 > \ldots > u_o$ be the different elements of $\mathbf{s}$ and w_1 and w_2 in decreasing order ($3 \leq o \leq m+2$). Obviously,

$$w_2 = u_o, \quad u_1 = w_1.$$

By Theorem 1 and Remark 2, inequality (29) holds if

$$\sum_{\{i \in X \mid s_i \geq u_l\}} p_i s_i - r_1 w_1 \leq u_l \left(\sum_{\{i \in X \mid s_i \geq u_l\}} p_i - r_1 \right), \quad 1 < l < o.$$

This inequality is equivalent to

$$\sum_{\{i \in X \mid s_i > u_l\}} p_i (s_i - u_l) \leq \frac{\sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i}{w_1 - w_2} (w_1 - u_l), \quad 1 < l < o. \tag{30}$$

It is obvious that for every $1 < l < o$

$$\sum_{\{i \in X \mid s_i > u_l\}} p_i (s_i - u_l) \leq \sum_{\{i \in X \mid s_i > u_{o-1}\}} p_i (s_i - w_2) \tag{31}$$

and

$$\left(\sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i \right) \frac{w_1 - S}{w_1 - w_2} \leq \left(\sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i \right) \frac{w_1 - u_l}{w_1 - w_2}. \tag{32}$$

Since $\{i \in X \mid s_i = u_{o-1}\} \neq \emptyset$ and p_i $(i = 1, \ldots, m)$ is positive,

$$\sum_{\{i\in X|s_i>u_{o-1}\}} p_i (s_i - w_2) < \sum_{\{i\in X|s_i>u_{o-1}\}}^{m} p_i (s_i - w_2) + \sum_{\{i\in X|s_i=u_{o-1}\}}^{m} p_i (s_i - w_2)$$
$$+ \sum_{\{i\in X|s_i=u_o\}}^{m} p_i (s_i - w_2) = \sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i. \tag{33}$$

Inequalities (31) and (32) show that (30) is satisfied if

$$\sum_{\{i\in X|s_i>u_{o-1}\}} p_i (s_i - w_2) \le \left(\sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i \right) \frac{w_1 - S}{w_1 - w_2}, \tag{34}$$

which is true for any sufficiently large w_1, since (33) holds and $\frac{w_1-S}{w_1-w_2} \to 1$ as $w_1 \to \infty$.

The proof is complete. ■

Remark 22 *(a) Since the function* $w_1 \to \frac{w_1-S}{w_1-w_2}$ $(w_1 > w_2)$ *is increasing, the constant* c *associated with* w_2 *is given by*

$$\sum_{\{i\in X|s_i>u_{o-1}\}} p_i (s_i - w_2) = \left(\sum_{i=1}^{m} p_i s_i - w_2 \sum_{i=1}^{m} p_i \right) \frac{c - S}{c - w_2}.$$

(b) All I could find was paper [8], which deals with the refinement of the Petrovic inequality. It is likely that there are other papers on this topic that have escaped my attention, in any case there are not many such papers. In [8] the refinement of the Petrovic inequality is obtained by applying a refinement of the Jensen's inequality, which is obtained by applying a refinement of the Jensen inequality, and is not comparable to our result. We stress that (26) is a parameter dependent refinement.

(c) It is not hard to think that if (26) holds, then r_1 *and* r_2 *must be non-negative. It is easily follows from Theorem 1 that inequality (28) also holds if*

$$0 < w_2 < w_1 < \frac{\sum_{i=1}^{m} p_i s_i}{\sum_{i=1}^{m} p_i},$$

and in this case r_2 *is negative.*

For example, let

$$s_1 := 3,\ s_2 := 1,\ p_1 := 1,\ p_2 := 2,\ w_1 := \frac{4}{3},\ w_2 := \frac{1}{3}.$$

Then $r_1 := 4$ *and* $r_2 := -1$*, and inequality (28) is*

$$4f\left(\frac{4}{3}\right) - f\left(\frac{1}{3}\right) \le f(5) + 2f(0).$$

Acknowledgement 23 *Research supported by the Hungarian National Research, Development and Innovation Office grant no. K139346.*

References

[1] L. Fuchs, A new proof of an inequality of Hardy, Littlewood and Pólya, *Mat. Tidsskr. B.* 1947 53–54. (1947)

[2] F. Giaccardi, Su alcune disuguaglianze, *Giorn. Mat. Finanz.* **1** (4) (1955) 139-153.

[3] G. H. Hardy and J. E. Littlewood and G. Pólya, *Inequalities* (2nd ed. Cambridge University Press, 1952)

[4] L. Horváth, Integral inequalities using signed measures corresponding to majorization, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **117**, 80 (2023) No. 2.

[5] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications.* (vol. 143 of Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1979)

[6] C. P. Niculescu and L. E. Persson, *Convex functions and their applications. A contemporary approach.* (Springer, Berlin 2006)

[7] J. Pečarić, On the Petrović inequality for convex functions, *Glas. Mat.* **18** (1983) 77-85.

[8] J. Pečarić and J. Perić, Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means, *An. Univ. Craiova Ser. Mat. Inform.*, **39**(1) 2012 65–75.

[9] M. Petrović, Sur une fonctionnelle, *Publ. Math. Univ. Belgrade* **1** (1932) 149-156.

[10] T. Popoviciu, Les fonctions convexes, *Actualités Sci. Ind.* No 922, Paris, 1945.

[11] H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. Natl. Acad. Sci. U.S.A.* **35** (1949) no. 7 408-411.