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A minimization problem related to the principal frequency of the p-Bilaplacian with coupled Dirichlet–Neumann boundary conditions

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Abstract. For each fixed integer $N \geq 2$ let $\Omega \subset \mathbb{R}^N$ be an open, bounded and convex set with smooth boundary. For each real number $p \in (1, \infty)$ define

$$M(p;\Omega) = \inf_{u \in \mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\int_\Omega (\exp(|\Delta u|^p) - 1)\,dx}{\int_\Omega (\exp(|u|^p) - 1)\,dx},$$

where $\mathcal{W}_C^{2,\infty}(\Omega) := \cap_{1<p<\infty}\{u \in W_0^{2,p}(\Omega) : \ \Delta u \in L^\infty(\Omega)\}$. We show that if the radius of the largest ball which can be inscribed in Ω is strictly larger than a constant which depends on N then $M(p;\Omega)$ vanishes while if the radius of the largest ball which can be inscribed in Ω is strictly less than 1 then $M(p;\Omega)$ is a positive real number. Moreover, in the latter case when p is large enough we can identify the value of $M(p;\Omega)$ as being the principal frequency of the p-Bilaplacian on Ω with coupled Dirichlet–Neumann boundary conditions.

Keywords: p-Bilaplacian, principal frequency, Dirichlet–Neumann boundary conditions.

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1 Introduction

1.1 Notations

For each integer $N \geq 1$ we denote by $\mathbb{R}^N$ the N-dimensional Euclidean space. Let $|\cdot|$ denote the modulus on $\mathbb{R}$ and for each integer $N \geq 2$ let $|\cdot|_N$ denote the Euclidean norm on $\mathbb{R}^N$. For

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each open and bounded subset Ω of $\mathbb{R}^N$ denote by R_Ω the inradius of Ω (that is the radius of the largest ball which can be inscribed in Ω). Finally, for each integer $N \geq 1$ define

$$\mathbb{P}^N := \{\Omega \subset \mathbb{R}^N : \Omega \text{ is an open, bounded, convex set with smooth boundary } \partial\Omega\}.$$

1.2 Statement of the problem

For each $\Omega \in \mathbb{P}^N$ and each real number $p \in (1, \infty)$ we define

$$M(p;\Omega) := \inf_{u \in \mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\int_\Omega (\exp(|\Delta u|^p) - 1)dx}{\int_\Omega (\exp(|u|^p) - 1)dx} \tag{1.1}$$

where $\mathcal{W}_C^{2,\infty}(\Omega) := \cap_{1<p<\infty}\{u \in W_0^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega)\}$. The goal of this paper is to emphasize the following phenomena which appear in relation with the minimization problem (1.1): if R_Ω is large enough then $M(p;\Omega) = 0$ for each $p \in (1,\infty)$ while if R_Ω is small enough then $M(p;\Omega) > 0$ for each $p \in (1,\infty)$. Moreover, in the latter case we can identify the value of $M(p;\Omega)$ for each p large enough as being equal with the following quantity

$$\Lambda_C(p;\Omega) := \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p dx}{\int_\Omega |u|^p dx}, \tag{1.2}$$

(see Theorem 1.1 for the precise result on problem (1.1)). Regarding $\Lambda_C(p;\Omega)$ we recall the well-known fact that it represents the principal eigenvalue of the p-Bilaplacian with coupled Dirichlet–Neumann boundary conditions (see, e.g., N. Katzourakis & E. Parini [5, relation (1.6)]). In other words, $\Lambda_C(p;\Omega)$ is the smallest real number Λ for which the following equation has a nontrivial solution

$$\begin{cases} \Delta_p^2 u = \Lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = |\nabla u|_N = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ stands for the p-Bilaplacian. At this point we consider important to recall the fact that problem (1.3) with $p = 2$ represents the famous "*clamped plate*" problem, which was initially studied by Lord J. W. S. Rayleigh in his famous book *The Theory of Sound* (1877), and subsequently deeply investigated by G. Szegö (1950), G. Talenti (1981), M. Ashbaugh & R. Benguria (1995) and N. Nadirashvili (1995) from an isoperimetric point of view.

1.3 Motivation

For each $\Omega \in \mathbb{P}^N$ and each real number $p \in (1,\infty)$ we recall the eigenvalue problem for the p-Laplacian under homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where λ is a real parameter and $\Delta_p u := \text{div}(|\nabla u|_N^{p-2}\nabla u)$ is the p-Laplace operator. It is well-known (see, e.g., P. Lindqvist [7]) that the first eigenvalue of problem (1.4) has the following variational characterization

$$\lambda_1(p;\Omega) := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p dx}{\displaystyle\int_\Omega |u|^p dx}.$$

Defining

$$\Lambda_1(p;\Omega) := \inf_{u \in X_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\exp(|\nabla u|_N^p) - 1)\, dx}{\displaystyle\int_\Omega (\exp(|u|^p) - 1)\, dx}, \tag{1.5}$$

where $X_0(\Omega) := W^{1,\infty}(\Omega) \cap \big(\cap_{1<p<\infty} W_0^{1,p}(\Omega)\big)$, we recall that by [2, Theorem 2] (see also [1] for similar results) we know that $\Lambda_1(p;\Omega) = 0$ if $R_\Omega > 1$ while $\Lambda_1(p;\Omega) > 0$ if $R_\Omega \le 1$. Moreover, there exists a constant $M \in [e^{-1}, 1]$ such that if $R_\Omega \le M$ we have $\Lambda_1(p;\Omega) = \lambda_1(p;\Omega)$, for all $p \in (1,\infty)$. Furthermore, by [1, Theorem 2] we have that if $R_\Omega < 1$ then there exists a constant $P \in (1,\infty)$ such that $\Lambda_1(p;\Omega) = \lambda_1(p;\Omega)$, for all $p \in [P,\infty)$.

Motivated by these results regarding $\Lambda_1(p;\Omega)$ and $\lambda_1(p;\Omega)$ in this paper we show that we can arrive to a similar conclusion in relation with $M(p;\Omega)$ and $\Lambda_C(p;\Omega)$.

1.4 Main result

The main result of this paper is given by the following theorem.

Theorem 1.1. *Assume $N \ge 2$ is a given integer and let C_N be the constant given by*

$$C_N := \begin{cases} \dfrac{4}{\ln 2}, & \text{if } N = 2, \\ \dfrac{2^{\frac{2}{N}}(N-2)}{1 - 2^{\frac{2}{N}-1}}, & \text{if } N \ge 3. \end{cases} \tag{1.6}$$

Then for each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have that $M(p;\Omega) > 0$, if $R_\Omega < 1$ and $M(p;\Omega) = 0$ if $R_\Omega > C_N^{1/2}$. Moreover, if $\Omega \in \mathbb{P}^N$ with $R_\Omega < 1$ then there exists a constant $P^\star > 1$ such that $M(p;\Omega) = \Lambda_C(p;\Omega)$ for all $p \in [P^\star, \infty)$.

Actually, a careful look at the proof of Theorem 1.1 (more precisely, observing the fact that relation (3.1) holds true for a ball with the radius strictly smaller that $C_N^{1/2}$) shows that it can be improved in the particular case when Ω is a ball, in the following sense.

Corollary 1.2. *Assume $N \ge 2$ is a given integer and let B_R be a ball of radius R from $\mathbb{R}^N$ centered at the origin. Then for each $p \in (1,\infty)$ we have that $M(p;B_R) > 0$, if $R < C_N^{1/2}$ and $M(p;B_R) = 0$ if $R > C_N^{1/2}$. Moreover, if $R < C_N^{1/2}$ then there exists a constant $P^\star > 1$ such that $M(p;B_R) = \Lambda_C(p;B_R)$ for all $p \in [P^\star,\infty)$.*

Note that, unfortunately, our proof of Theorem 1.1 cannot fill the gap which occurs when $R_\Omega \in [1, C_N^{1/2}]$. In the case of Corollary 1.2 this gap reduces to an uncovered case when $R = C_N^{1/2}$.

The rest of the paper comprises two more sections offering the following pieces of information: in Section 2 we recall the asymptotic behaviour of $\Lambda_C(p;\Omega)^{1/p}$, as $p \to \infty$, and we give a lower bound for $\Lambda_C(p;\Omega)$; Section 3 is devoted to the proof of the main result.

2 Auxiliary results on $\Lambda_C(p;\Omega)$

2.1 The asymptotic behaviour of $\Lambda_C(p;\Omega)^{1/p}$, as $p\to\infty$

Define

$$\Lambda_\infty^C(\Omega) := \inf_{u\in W_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}. \tag{2.1}$$

By [5, Theorem 1.1] we know that

$$\lim_{p\to\infty} \Lambda_C(p;\Omega)^{1/p} = \Lambda_\infty^C(\Omega). \tag{2.2}$$

Note that in general an explicit expression of $\Lambda_\infty^C(\Omega)$ is not available in the literature but when $\Omega = B_R$, where B_R stands for a ball of radius R from $\mathbb{R}^N$ centered at the origin, we have (by [5, Proposition 3.5]) that $\Lambda_\infty^C(B_R) = C_N R^{-2}$, where C_N is given by relation (1.6). Moreover, by [5, Proposition 3.5] we have that the minimizer realising the infimum in the definition of $\Lambda_\infty^C(B_R)$ is the positive, radially symmetric function $u_0(x) := w_1\left(\frac{x}{R}\right)$ with w_1 being the solution of the problem

$$\begin{cases} -\Delta w_1(x) = f(x), & \text{for } x\in B_1,\\ w_1(x) = 0, & \text{for } x\in\partial B_1,\end{cases}$$

where

$$f(x) := \begin{cases} 1, & \text{if } |x|_N \le 2^{-\frac{1}{N}},\\ -1, & \text{if } 2^{-\frac{1}{N}} < |x|_N < 1.\end{cases}$$

Actually, by [5, Lemma 3.3]) we know that for $N=2$ we have

$$w_1(x) = \begin{cases} \dfrac{\ln 2}{4} - \dfrac{|x|_2^2}{4}, & \text{for } |x|_2 \le 2^{-\frac{1}{2}},\\[2mm] \dfrac{|x|_2^2}{4} - \dfrac{\ln(|x|_2)}{2} - \dfrac{1}{4}, & \text{for } 2^{-\frac{1}{2}} < |x|_2 < 1,\end{cases}$$

while for $N\ge 3$ we have

$$w_1(x) = \begin{cases} \dfrac{2^{-\frac{2}{N}}}{N} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)} + \dfrac{2^{1-\frac{2}{N}}}{N(N-2)} - \dfrac{|x|_N^2}{2N}, & \text{for } |x|_N \le 2^{-\frac{1}{N}},\\[2mm] \dfrac{|x|_N^2}{2N} + \dfrac{|x|_N^{2-N}}{N(N-2)} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)}, & \text{for } 2^{-\frac{1}{N}} < |x|_N < 1.\end{cases}$$

Consequently, we have that the function $u_0 : B_R \to \mathbb{R}$, given by $u_0(x) := w_1\left(\frac{x}{R}\right)$, has the following expressions:

- if $N=2$ then

$$u_0(x) = \begin{cases} \dfrac{\ln 2}{4} - \dfrac{|x|_2^2}{4R^2}, & \text{for } |x|_2 \le 2^{-\frac{1}{2}}R,\\[2mm] \dfrac{|x|_2^2}{4R^2} - \dfrac{\ln(|x|_2) - \ln(R)}{2} - \dfrac{1}{4}, & \text{for } 2^{-\frac{1}{2}}R < |x|_2 < R.\end{cases}$$

- if $N\ge 3$ then

$$u_0(x) = \begin{cases} \dfrac{1 - 2^{\frac{2}{N}-1}}{2^{\frac{2}{N}}(N-2)} - \dfrac{|x|_N^2}{2NR^2}, & \text{for } |x|_N \le 2^{-\frac{1}{N}}R,\\[2mm] \dfrac{|x|_N^2}{2NR^2} + \dfrac{|x|_N^{2-N}}{N(N-2)R^{2-N}} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)}, & \text{for } 2^{-\frac{1}{N}}R < |x|_N < R.\end{cases}$$

Remark 2.1. Simple computations show that when $N = 2$ the function u_0 satisfies $\|u_0\|_{L^\infty(B_R)} = \frac{\ln 2}{4}$ and $\|\Delta u_0\|_{L^\infty(B_R)} = R^{-2}$. Similarly, when $N \geq 3$ the function u_0 verifies $\|u_0\|_{L^\infty(B_R)} = \frac{1-2^{\frac{2}{N}-1}}{2^{\frac{2}{N}}(N-2)}$ and $\|\Delta u_0\|_{L^\infty(B_R)} = R^{-2}$. Consequently, in both cases u_0 is a minimizer for $\Lambda_\infty^C(B_R)$ with $\|u_0\|_{L^\infty(B_R)} = C_N^{-1}$, where C_N is given by relation (1.6).

2.2 A lower bound for $\Lambda_C(p;\Omega)$

The goal of this section is to prove the following result:

Proposition 2.2. *Let $N \geq 2$ be an integer and $\Omega \in \mathbb{P}^N$ be a set. Then we have*

$$\Lambda_C(p;\Omega) \geq p^{-1} R_\Omega^{-2p}, \qquad \forall\, p \in (1,\infty)\,.$$

The main ingredient in proving Proposition 2.2 is a Hardy-type inequality due to E. Mitidieri [8, Corollary 2.2]. We recall this inequality below.

Theorem 2.3. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $\phi : \Omega \to (0,\infty)$ is a superharmonic function such that $\phi \in C^2(\overline{\Omega})$ and it satisfies $-\Delta\phi \geq a|\nabla\phi|_N^2\phi^{-1}$, in Ω, for some constant $a > 0$ then for each real number $p \in (1,\infty)$ the following inequality holds true*

$$\frac{(p-1)a+p}{p^2}\int_\Omega |\Delta\phi||u|^p dx \leq \int_\Omega \phi^p|\Delta\phi|^{1-p}|\Delta u|^p dx, \qquad \forall\, u \in C_0^\infty(\Omega)\,. \tag{2.3}$$

2.2.1 Proof of Proposition 2.2.

For each $\Omega \in \mathbb{P}^N$ let v be the unique function satisfying

$$\begin{cases} -\Delta v = 1, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

In particular, we have that $v \in C^2(\overline{\Omega})$. Letting $M_2(\Omega) := \max_{x\in\overline{\Omega}} v(x)$, we have by [4, Theorem 1.2 with $p = q = 2$] that

$$M_2(\Omega) \leq \frac{R_\Omega^2}{2}\,.$$

On the other hand, by [4, Theorem 3.2] (with $p = 2$ and F being the Euclidean norm on $\mathbb{R}^N$) we know that

$$2^{-1}|\nabla v(x)|_N^2 + v(x) \leq M_2(\Omega), \qquad \forall\, x \in \Omega\,.$$

Thus, defining $\phi : \Omega \to (0,\infty)$ by

$$\phi(x) := v(x) + M_2(\Omega), \qquad \forall\, x \in \Omega\,,$$

we have that $\phi \in C^2(\overline{\Omega})$ and since $-\Delta\phi(x) = -\Delta v(x) = 1$ for all $x \in \Omega$, by the above estimate we deduce that

$$2^{-1}\phi^{-1}(x)|\nabla\phi(x)|_N^2 \leq -\Delta\phi(x), \qquad \forall\, x \in \Omega\,.$$

In other words, ϕ given above satisfies the hypothesis from Theorem 2.3 with $a = 2^{-1}$ and, consequently, the following inequality holds true

$$\frac{3p-1}{2p^2}\int_\Omega |u|^p\,dx \leq \int_\Omega (v + M_2(\Omega))^p|\Delta u|^p\,dx, \qquad \forall\, u \in C_0^\infty(\Omega)\,. \tag{2.4}$$

Since $v(x) \leq M_2(\Omega) \leq 2^{-1}R_\Omega^2$ for each $x \in \Omega$ inequality (2.4) implies the conclusion of Proposition 2.2.

3 Proof of the main result

We start by establishing three lemmas which will be helpful in the proof of our main result.

Lemma 3.1. *Assume $N \geq 2$ is an integer. For each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have $M(p;\Omega) \leq \Lambda_C(p;\Omega)$.*

Proof. Assume $p \in (1,\infty)$ is arbitrary but fixed. Taking into account relation (1.1) for any $u \in C_0^\infty(\Omega) \setminus \{0\} \subset \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$ and $t \in (0,1)$ we have

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega (\exp(|\Delta(tu)|^p)-1)dx}{\displaystyle\int_\Omega (\exp(|tu|^p)-1)dx} = \frac{\displaystyle\int_\Omega |\Delta u|^p dx + \sum_{k=2}^\infty t^{(k-1)p}\int_\Omega \frac{|\Delta u|^{kp}}{k!}dx}{\displaystyle\int_\Omega |u|^p dx + \sum_{k=2}^\infty t^{(k-1)p}\int_\Omega \frac{|u|^{kp}}{k!}dx}.$$

Letting $t \to 0^+$ in the above inequality we get

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega |\Delta u|^p\, dx}{\displaystyle\int_\Omega |u|^p\, dx}, \qquad \forall\, u \in C_0^\infty(\Omega) \setminus \{0\}\,.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{2,p}(\Omega)$ and $\Lambda_C(p;\Omega)$ is defined by relation (1.2) we deduce that the conclusion of Lemma 3.1 holds true. □

Lemma 3.2. *Assume $N \geq 2$ is an integer. For each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have $M(p;\Omega) \geq \inf_{k\in\mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega)$.*

Proof. Assume $p \in (1,\infty)$ is arbitrary but fixed. Using the definition of $\Lambda_C(p;\Omega)$ given by relation (1.2) we deduce that for each $u \in \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$ (which, in particular, ensures that $u \in W_0^{2,q}(\Omega) \setminus \{0\}$ for any $q > 1$), we have

$$\frac{\displaystyle\int_\Omega (\exp(|\Delta u|^p)-1)\, dx}{\displaystyle\int_\Omega (\exp(|u|^p)-1)\, dx} \geq \frac{\displaystyle\sum_{k=1}^\infty \frac{\Lambda_C(kp;\Omega)}{k!}\int_\Omega |u|^{kp}\, dx}{\displaystyle\sum_{k=1}^\infty \frac{1}{k!}\int_\Omega |u|^{kp}\, dx} \geq \inf_{k\in\mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega)\,.$$

Passing above to the infimum over all $u \in \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$, we arrive at the conclusion of Lemma 3.2. □

Lemma 3.3. *Assume that $\Omega \in \mathbb{P}^N$ satisfies $\Lambda_\infty^C(\Omega) > 1$. Define*

$$\mathcal{O} := \{p \in (1,\infty):\ \Lambda_C(p;\Omega) \leq \Lambda_C(kp;\Omega),\ \forall\, k \geq 1\}\,.$$

Then there exists an integer $L \geq 1$ such that $(L,\infty) \subset \mathcal{O}$.

Proof. The proof of this lemma follows the ideas used in the proof of Step 5 from the proof of Theorem 2 in [1, p. 10]. We recall it just for the reader's convenience.

We argue by contradiction. Indeed, assume that for each integer $m \geq 1$ there exists a real number $p_m \geq m$ and an integer $k_m \geq 2$ such that $\Lambda_C(p_m;\Omega) > \Lambda_C(k_m p_m;\Omega)$. Since $\Lambda_\infty^C(\Omega) > 1$ it follows that $\Lambda_\infty^C(\Omega) - \sqrt{\Lambda_\infty^C(\Omega)} > 0$. Let us now fix $\varepsilon \in (0, \Lambda_\infty^C(\Omega) - \sqrt{\Lambda_\infty^C(\Omega)})$. It is clear that $(\Lambda_\infty^C(\Omega)-\varepsilon)^2 > \Lambda_\infty^C(\Omega)$. On the other hand, by (2.2), $\lim_{q\to\infty}\sqrt[q]{\Lambda_C(q;\Omega)} = \Lambda_\infty^C(\Omega)$, and thus there exists a positive integer A_ε such that $1 < \Lambda_\infty^C(\Omega)-\varepsilon < \sqrt[q]{\Lambda_C(q;\Omega)}$, for all $q \geq A_\varepsilon$. Then,

$$(\Lambda_\infty^C(\Omega)-\varepsilon)^{2p_m} \leq (\Lambda_\infty^C(\Omega)-\varepsilon)^{k_m p_m} < \Lambda_C(k_m p_m;\Omega) < \Lambda_C(p_m;\Omega), \qquad \forall\, m > A_\varepsilon\,.$$

Hence, using again (2.2), we conclude that

$$(\Lambda_\infty^C(\Omega)-\varepsilon)^2 \leq \lim_{m\to\infty} \sqrt[p_m]{\Lambda_C(p_m;\Omega)} = \Lambda_\infty^C(\Omega)\,,$$

which is a contradiction. The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1.

• *Step 1.* We show that $M(p;\Omega)=0$, for each $\Omega\in\mathbb{P}^N$ with $R_\Omega > C_N^{1/2}$ and each $p\in(1,\infty)$.

Assume that $p\in(1,\infty)$ is arbitrary but fixed. Firstly, note that for each $\Omega\in\mathbb{P}^N$ we may assume without loss of generality, by a translation of the domain, that $0\in\Omega$ is exactly the center of the largest ball which can be inscribed in Ω, in other words $B_{R_\Omega}\subset\Omega$. Next, let u_0 be a minimizer for $\Lambda_\infty^C(B_{R_\Omega})$ with $\|u_0\|_{L^\infty(B_{R_\Omega})} = C_N^{-1}$, where C_N is given by relation (1.6), and $\|\Delta u_0\|_{L^\infty(B_{R_\Omega})} = R_\Omega^{-2}$ (see Remark 2.1 for details). Then we can define $U_0:\Omega\to\mathbb{R}$ by

$$U_0(x) := \begin{cases} u_0(x), & \text{if } x\in B_{R_\Omega},\\ 0, & \text{if } x\in\Omega\setminus B_{R_\Omega}. \end{cases}$$

Since $u_0\in\mathcal{W}_C^{2,\infty}(B_{R_\Omega})$ it follows that $u_0\in W_0^{2,q}(B_{R_\Omega})$ for each $q\in(1,\infty)$ and by [6, Lemma 5.2.5 & Theorem 5.4.4 & Section 5.5] we deduce that $U_0\in W_0^{2,q}(\Omega)$ for each $q\in(1,\infty)$. It follows that, actually, we have $nU_0\in\mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}$, for each positive integer n. Testing with nU_0 in the definition of $M(p;\Omega)$, and taking into account that $|\Delta U_0(x)|\leq R_\Omega^{-2}$, for a.a. $x\in B_{R_\Omega}$, we get

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega [\exp(|\Delta(nU_0(x))|^p)-1]\,dx}{\displaystyle\int_\Omega [\exp(|nU_0(x)|^p)-1]\,dx} \leq \frac{\displaystyle\int_{B_{R_\Omega}} [\exp(|nR_\Omega^{-2}|^p)-1]\,dx}{\displaystyle\int_{B_{R_\Omega}} [\exp(n^p|u_0(x)|^p)-1]\,dx}\,.$$

On the other hand, we recall that by Remark 2.1 we know that $\|u_0\|_{L^\infty(B_{R_\Omega})} = C_N^{-1}$, where C_N is given by relation (1.6). We deduce that if we assume $R_\Omega > C_N^{1/2}$, then letting $\epsilon_0>0$ be such that $\epsilon_0 + R_\Omega^{-2} < C_N^{-1}$, we get that there exists a subset $\omega\subset B_{R_\Omega}$ with $|\omega|>0$ such that $|u_0(x)| > \epsilon_0 + R_\Omega^{-2}$, for all $x\in\omega$. It follows that, for each positive integer n we have

$$M(p;\Omega) \leq \frac{|B_{R_\Omega}|[\exp(|nR_\Omega^{-2}|^p)-1]}{\displaystyle\int_\omega [\exp(n^p|u_0(x)|^p)-1]\,dx} \leq \frac{|B_{R_\Omega}|[\exp(|nR_\Omega^{-2}|^p)-1]}{|\omega|\left[\exp\left[n^p\left(\epsilon_0+R_\Omega^{-2}\right)^p\right]-1\right]}\,.$$

Letting $n\to\infty$ we find $M(p;\Omega)=0$.

• *Step 2.* We show that $M(p;\Omega)>0$, for each $\Omega\in\mathbb{P}^N$ with $R_\Omega<1$ and each $p\in(1,\infty)$. Moreover, there exists $P^\star>1$ such that $M(p;\Omega)=\Lambda_C(p;\Omega)$ for all $p\geq P^\star$.

Let $\Omega\in\mathbb{P}^N$ with $R_\Omega<1$ and $p\in(1,\infty)$ be arbitrary but fixed. By Proposition 2.2 we know that

$$\Lambda_C(q;\Omega) \geq q^{-1}R_\Omega^{-2q}, \qquad \forall\, q\in(1,\infty)\,.$$

That fact and relation (2.2) yield

$$\Lambda_\infty^C(\Omega) = \lim_{q\to\infty} \Lambda_C(q;\Omega)^{1/q} \geq \lim_{q\to\infty} \sqrt[q]{q^{-1}R_\Omega^{-2q}} = R_\Omega^{-2} > 1\,. \tag{3.1}$$

Since $\Lambda_\infty^C(\Omega) > 1$ the hypothesis of Lemma 3.3 is fulfilled. Let $L \geq 1$ be the smallest integer number for which Lemma 3.3 holds true. It follows that

$$\Lambda_C(q;\Omega) \leq \Lambda_C(kq;\Omega), \qquad \forall\, k \geq 1,\ \forall\, q > L\,.$$

Taking $k_0 := \left[Lp^{-1}\right] + 2$ we get $k_0 p > L$ and consequently, by the above inequality we find that

$$\Lambda_C(k_0 p;\Omega) \leq \Lambda_C(kp;\Omega)\,,$$

for each integer $k \geq k_0$. Thus,

$$\Lambda_C(k_0 p;\Omega) \leq \inf_{k\geq k_0} \Lambda_C(kp;\Omega)\,.$$

On the other hand, by Lemma 3.2 we know that

$$M(p;\Omega) \geq \inf_{k\in\mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega)\,.$$

All the above pieces of information imply that

$$M(p;\Omega) \geq \inf_{k\in\{1,2,\ldots,k_0\}} \Lambda_C(kp;\Omega) > 0\,.$$

Finally, if we assume, in addition, that $p > L$ then similar arguments as above yield $M(p;\Omega) \geq \Lambda_C(p;\Omega)$. On the other hand, by Lemma 3.1 we have $M(p;\Omega) \leq \Lambda_C(p;\Omega)$, and, consequently, we conclude that $M(p;\Omega) = \Lambda_C(p;\Omega)$, for all $p \geq P^\star := L+1$. The proof of Theorem 1.1 is now complete. □

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