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ON LOCALLY DUALLY FLAT SPECIAL FINSLER (α,β)-METRICS

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ABSTRACT. In this paper, we characterize locally dually flat (α,β)-metrics $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, $(m\neq 0,-1)$, with isotropic S-curvature and scalar flag curvature and show that these metrics reduce to locally Minkowskian metrics.

1. Introduction

The notion of dually flat metrics was first introduced by S.-I. Amari and H. Nagaoka [2] when they studied the information geometry on Riemannian spaces. Later on, Z. Shen extends the notion of dually flatness to Finsler metrics [10]. A geodesic curve $c=c(t)$ of a Finsler metric $F=F(x,y)$ on a smooth manifold M is given by $\ddot{c}^i(t)+2G^i\left(c(t),\dot{c}(t)\right)=0$, where the local functions $G^i=G^i(x,y)$ are called the spray coefficients given by

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}. \tag{1}$$

A Finsler metric $F=F(x,y)$ on a manifold is locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i=-\frac{1}{2}g^{ij}H_{y^j},$$

where $H=H(x,y)$ is a C^∞ scalar function on TM_0 satisfying $H(x,\lambda y)=\lambda^3H(x,y)$ for all $\lambda>0$. Such a coordinate system is called an adapted coordinate system. In [10], it is proved that a Finsler metric $F=F(x,y)$ on an open subset $U\subset\mathbb{R}^n$ is locally dually flat if and only if it satisfies the following PDE

$$\left[F^2\right]_{x^ky^l}y^k-2\left[F^2\right]_{x^l}=0. \tag{2}$$

In this case, H is given by $H=-\frac{1}{6}\left[F^2\right]_{x^m}y^m$.

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It is known that a Riemannian metric $F = \sqrt{a_{ij}(x)y^iy^j}$ is locally dually flat if and only if in an adapted coordinate system, $a_{ij}(x) = \frac{\partial^2\psi}{\partial x^i \partial x^j}(x)$, where $\psi = \psi(x)$ is a C^∞ function [2, 1]. The first example of non-Riemannian dually flat metrics is Funk metric, given in [10] as follows

$$F(x,y) = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{(1-|x|^2)} \pm \frac{\langle x,y\rangle}{(1-|x|^2)}. \tag{3}$$

Above metric is defined on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. Also it is locally projectively flat with constant flag curvature $K = -\frac{1}{4}$. More general, we have the following

Example 1. [5] Let $U \subset \mathbb{R}^n$ be a strongly convex domain, namely, there is a Minkowski norm $\phi(y)$ on $\mathbb{R}^n$ such that

$$U := \{y \in \mathbb{R}^n \mid \phi(y) < 1\}.$$

Define $\Theta = \Theta(x,y) > 0, y \neq 0$ by

$$x + \tfrac{y}{\Theta} \in \partial U,$$

$y \in T_xU = \mathbb{R}^n$. It is easy to show that Θ is a Finsler metric satisfying

$$\Theta_{x^k} = \Theta\Theta_{y^k}. \tag{4}$$

Using equation (4), it is easy to verify that $\Theta = \Theta(x,y)$ satisfies equation (2). Thus it is locally dually flat on U. Θ is called Funk metric on U. It is easy to see that Funk metric is of constant flag curvature $K = -\frac{1}{4}$. Also, Θ is of constant S-curvature, $S = \frac{n+1}{2}\Theta$. In particular, when $U = \mathbb{B}^n(1)$, the Funk metric is just the metric in the form of equation (3).

In fact, every locally dually flat and projectively flat metric on an open subset in $\mathbb{R}^n$ must be either a Minkowski metric or a Funk metric satisfying (4) after normalization.

A Finsler metric F is called projectively flat if F is projectively equivalent to a Minkowski/Euclidean metric. In this case, all geodesics of F are straight lines, namely, we can characterize geodesics of F as $\sigma(t) := f(t)a + b$ for some constant vectors $a, b \in \mathbb{R}^n$.

In [6], it is shown that a Finsler metric F on a manifold M is projectively flat if and only if F satisfies the following

$$F_{x^ky^l}y^k - F_{x^l} = 0.$$

In this case,

$$G^i = P(x,y)y^i,$$

with $P = \frac{F_{x^k}y^k}{2F}$. We call P the projective factor of F.

Lemma 1. [4] *Let $F = F(x,y)$ be a Finsler metric on an open set $U \subset \mathbb{R}^n$. Then F is locally dually flat and projectively flat on U if and only if $F_{x^k} = CFF_{y^k}$, where C is a constant.*

For a Finsler metric F, the Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature is the trace of the Riemann curvature, Ric $:= R^m_m$. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. Riemannian metrics of constant sectional curvature were classified by E. Cartan a long time ago. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However, the local metric structure of a Finsler metric with constant flag curvature is much more complicated. For a flag $\{P, y\}$ in T_xM, where $P \subset T_xM$ is a tangent plane containing y, the flag curvature $K(x, y, P)$ is defined by

$$K(x, y, P) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)g_y(y, u)},$$

where $u \in P$ such that $P = \text{span}\{y, u\}$. A Finsler metric F is said to be of scalar flag curvature if $K(x, y, P) = K(x, y)$ is independent of P containing $y \in T_xM$. F is said to be of isotropic scalar flag curvature if $K(x, y, P) = K(x)$ and of constant flag curvature if $K(x, y, P) =$ constant.

The S-curvature $S = S(x, y)$ in Finsler geometry is introduced by Shen [11] as a non-Riemannian quantity, defined as

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0},$$

where $\tau = \tau(x, y)$ is a scalar function on $T_xM \backslash \{0\}$, called distortion of F and $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$.

A Finsler metric F is called of isotropic S-curvature if

$$S = (n + 1)cF$$

for some scalar function $c = c(x)$ on M. One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature with isotropic S-curvature.

In [14], the author studied Finsler metrics in the form $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ with non-zero constants ϵ and k and found that there is no locally dually flat Finsler metric in this form with constant flag (even of scalar flag curvature) or isotropic S-curvature unless it is Minkowskian. Sevaral geometer [12, 9, 8] also have studied different class of (α, β)-metrics and found that these class of locally dually flat Finsler (α, β)-metric with isotropic S-curvature and constant flag curvature again reduces to Minkowskian. These facts inspire us to study more general Finsler (α, β)-metrics $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, with isotropic S-curvature and flag curvature.

This class of (α, β)-metrics $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ contains Randers metric $F = \alpha + \beta$ for $m = 0$; Riemannian metric $F = \alpha$ for $m = -1$; Matsumoto metric $F =$

$\frac{\alpha^2}{(\alpha-\beta)}$, if we replace β by $-\beta$ and take $m=-2$; and Square metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ for $m=1$.

More precisely we have the following theorems.

Theorem 1. *Let $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m\neq 0,-1)$ be the (α,β)-metric on a manifold M and F is locally dually flat metric with isotropic S-curvature.*

(i) *If $\tau[s(k_2-k_3s^2)(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')]=0$, then F is locally projectively flat in adapted coordinate system with $G^i=C\tau\beta y^i$.*

(ii) *If $s(k_2-k_3s^2)(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0$, then F is locally projectively flat in adapted coordinate system with $G^i=0$.*

Theorem 2. *Let $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m\neq 0,-1)$ be a non-Riemannian (α,β)-metric on a manifold M. Then F is locally dually flat with isotropic S-curvature $S=(n+1)cF$ and scalar flag curvature if and only if it is locally Minkowskian.*

2. Preliminaries

Let M be an n-dimensional C^∞-manifold. T_xM denotes the tangent space of M at x and the tangent bundle TM is the disjoint union of tangent spaces $TM:=\bigcup_{x\in M}T_xM$. We denote the elements of TM by (x,y) where $y\in T_xM$. Let $TM_0=TM\setminus\{0\}$.

A Finsler metric on M is a function $F:TM\to[0,\infty)$ with the following properties:

(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous, and
(iii) the Hessian of $\frac{F^2}{2}$, with elements $g_{ij}(x,y)=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}$, is positive definite on TM_0.

The pair (M,F) is then called a Finsler space.

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, $\beta=b_iy^i$ be a 1-form and let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$, where $\phi=\phi(s)$ is a positive C^∞ function defined in a neighbourhood of the origin $s=0$. It is well known that $F=\alpha\phi(s)$ is a Finsler metric for any α and β with $b=||\beta||_\alpha<b_0$ if and only if $\phi(s)>0$, $\phi(s)-s\phi'(s)+(b^2-s^2)\,\phi''(s)>0$ $(|s|\leq b<b_0)$.

Let G^i_α denote the spray coefficients of α given by

$$G^i_\alpha=\frac{1}{4}a^{il}\left\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\right\}, \tag{5}$$

where $(a^{ij})=(a_{ij})^{-1}$. In view of equations (1) and (5), we have

$$G^i=G^i_\alpha+Ry^i+Q^i, \tag{6}$$

where

$$R=\alpha^{-1}\Theta\{-2Q\alpha s_0+r_{00}\}, \tag{7}$$

$$Q^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\} b^i, \tag{8}$$

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

Consider the following notations [11]

$$r_{ij} = \frac{1}{2}\{b_{i;j} + b_{j;i}\}, \quad r^i_j = a^{ih} r_{hj}, \quad r_j = b_i r^i_j, \quad s_{ij} = \frac{1}{2}\{b_{i;j} - b_{j;i}\},$$
$$s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad b^i = a^{ih} b_h, \quad b^2 = b^i b_i,$$

where $b_{i;j}$ is the covariant derivative of b_i with respect to Levi-Civita connection of α.

In [13], Q. Xia have studied a class of locally dually flat (α,β)-metrics and obtained the following results.

Theorem 3. [13] *Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$ be an (α,β)-metric on n-dimensional manifold M where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_iy^i \neq 0$. Suppose that F is not Riemannian and $\phi'(0) \neq 0$. Then F is locally dually flat on M if and only if α, β and $\phi = \phi(s)$ satisfy*

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{9}$$

$$r_{00} = \frac{2}{3}\theta\beta + [\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\tau\beta^2,$$

$$G^l_\alpha = \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l,$$

$$\tau[s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0,$$

where $\tau := \tau(x)$ is a scalar function, $\theta := \theta_i(x)y^i$ is a 1-form on M, $\theta^l := a^{lm}\theta_m$ and $k_1 := \Pi(0), k_2 := \frac{\Pi'(0)}{Q(0)}$, $k_3 = \frac{1}{6Q(0)^2}[3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)|\Pi'''(0)]$, $Q := \frac{\phi'}{\phi - s\phi'}, \Pi := \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}$.

Corollary 1. *Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$ be an (α,β)-metric on n-dimensional manifold M with the same assumptions as in above Theorem 3. If ϕ satisfies*

$$s(k_2 - k_3s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0,$$

then F is locally dually flat on M if and only if α and β satisfy

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l),$$

$$r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2],$$

$$G^l_\alpha = \frac{1}{3}[2\theta y^l + \theta^l\alpha^2].$$

In [3], Cheng-Shen have studied the class of (α,β)-metrics of non-Randers type $\phi \neq t_1\sqrt{1+t_2s^2}+t_3s$ with isotropic S-curvature and obtained the following.

Theorem 4. [3] *Let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$ be a non-Riemannian (α,β)-metric on a manifold and $b=||\beta_x||_\alpha$. Suppose that $\phi \neq t_1\sqrt{1+t_2s^2}+t_3s$ for any constant $t_1>0, t_2$ and t_3. Then F is of isotropic S-curvature $S=(n+1)cF$, if and only if one of the following holds.*

(i) *β satisfies*

$$r_{ij}=\epsilon\{b^2a_{ij}-b_ib_j\}, s_j=0,$$

where $\epsilon=\epsilon(x)$ is a scalar function and $c=c(x)$ satisfies

$$\Phi=-2(n+1)k\frac{\phi\Delta^2}{b^2-s^2}, \tag{10}$$

where k is a constant. In this case $S=(n+1)cF$ with $c=k\epsilon$.

(ii) *β satisfies*

$$r_{ij}=0, s_j=0.$$

In this case, $S=0$, regardless of choices of a particular ϕ.

3. Locally dually flatness and isotropic S-curvature

For the Finsler metric $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m\neq 0,-1)$, we obtain

$$\begin{aligned}\phi &= (1+s)^{m+1},\\ \phi' &= (m+1)(1+s)^m,\\ \phi'' &= m(m+1)(1+s)^{m-1},\\ \phi''' &= m(m^2-1)(1+s)^{m-2},\end{aligned}$$

$$\Pi = -\frac{2m^2+3m+1}{(1+s)(-1+sm)},$$

$$\Pi' = \frac{2m^3+m^2+4m^3s-2m+6sm^2-1+2sm}{(1+s)^2(-1+sm)^2},$$

$$\Pi'' = \frac{-2(1+2m+6s^2m^4+9s^2m^3+3s^2m^2+6sm^4+2m^4-3sm)}{(1+s)^3(-1+sm)^3} + \frac{-2(m^3-6sm^2+3m^3s)}{(1+s)^3(-1+sm)^3},$$

$$\Pi''' = \frac{6(-1+8sm^5-2m+2m^5+6s^2m^4-12s^2m^3-6s^2m^2+4sm^4)}{(1+s)^4(-1+sm)^4} + \frac{6(12s^2m^5+8m^5s^3+12m^4s^3+m^4+4s^3m^3+4sm+8sm^2)}{(1+s)^4(-1+sm)^4},$$

$$Q = -\frac{m+1}{-1+sm},$$

$$Q' = \frac{(m+1)m}{(-1+sm)^2},$$

$$Q'' = -2\,\frac{(m+1)m^2}{(-1+sm)^3},$$

$$k_1 = 2m^2+3m+1,$$

$$k_2 = 2m^2-m-1,$$

$$k_3 = -m(4\,m^3-4\,m^2-m+1).$$

By using above values in Theorem (3), we have the following two lemmas.

Lemma 2. *If* $s\,(k_2-k_3s^2)\,(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')=0$, *then* F *is locally dually flat on* M *if and only if* α,β *and* $\phi=\phi(s)$ *satisfy the following equations*

$$s_{l0} = \frac{1}{3}(\beta\theta_l-\theta b_l),$$

$$r_{00} = \frac{2}{3}\theta\beta+[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)]\alpha^2+\frac{1}{3}(3k_2-2-3k_3b^2)\tau\beta^2, \tag{11}$$

$$G^l_\alpha = \frac{1}{3}[2\theta+(3k_1-2)\tau\beta]y^l+\frac{1}{3}(\theta^l-\tau b^l)\alpha^2+\frac{1}{2}k_3\tau\beta^2b^l, \tag{12}$$

$$\tau[s(k_2-k_3s^2)(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')]=0,$$

where $k_1=2m^2+3m+1, k_2=2m^2-m-1, k_3=-m(4\,m^3-4\,m^2-m+1)$.

Lemma 3. *If* $s\,(k_2-k_3s^2)\,(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0$, *then* F *is locally dually flat on* M *if and only if* α *and* β *satisfy*

$$s_{l0} = \frac{1}{3}(\beta\theta_l-\theta b_l),$$

$$r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2],$$

$$G^l_\alpha = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2].$$

Further the given Finsler metrics $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, we have the following proposition.

Proposition 1. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m \neq 0, -1)$ be the (α, β)-metric on a manifold M. Then F is of isotropic S-curvature $S = (n+1)cF$ if and only if β satisfies*

$$r_{ij} = 0, \quad s_j = 0.$$

Proof. Let $\phi = \phi(s)$ be a positive C^∞ function defined on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'', \tag{13}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q'$$

and

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

Now equation (13) can be written as

$$\Phi = -(Q - sQ')(n+1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}. \tag{14}$$

We suppose that the case (i) of Theorem (4) holds. For the metric $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, we have

$$\Delta = \frac{1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2}{(ms-1)^2}.$$

It follows that $(ms-1)^2\,\Delta$ is a polynomial in s of degree 2. On the other hand, we have

$$\phi\Delta^2 = \frac{\phi(1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2)^2}{(ms-1)^4}. \tag{15}$$

Substituting equation (15) into equation (10), we get

(16)

$$(b^2 - s^2)(ms-1)^4\Phi =$$
$$-2\,(n+1)\,k(1+s)^{m+1}\left(1 + s - ms - 2ms^2 - s^2m^2 + m^2b^2 + mb^2\right)^2.$$

Now by considering another form of Φ defined by equation (14), we have

$$\Phi = \frac{2m^4nb^2s - 2m^4ns^3 + 4m^3nb^2s - 6m^3ns^3 - m^3nb^2 - 2m^3b^2s + 2m^2nb^2s}{(ms-1)^4} - \frac{m^3ns^2 - 4m^2ns^3 - 2b^2m^3 - 2m^2nb^2 - 2m^2b^2s + 3m^2ns^2 - 2b^2m^2 + nb^2m}{(ms-1)^4} - \frac{3s^2m^2 + 4mns^2 + 3m^2ns + 3m^2s + 3s^2m}{(ms-1)^4} - \frac{-mn + 2mns - ns - m + 2ms - n - s - 1}{(ms-1)^4}.$$

Equation (16) can be rewritten as

$$\begin{aligned}(b^2 - s^2)(ms-1)^4\Phi = &-2b^4m^2n - b^4nm - 2b^4m^2 - 2b^4m^3 - nb^2 - nb^2m \\ &-b^4m^3n - b^2 - b^2m + s(2b^4m^4n + 4b^4m^3n - 2b^4m^3 + 2b^4m^2n - 2b^4m^2 \\ &+3m^2nb^2 + 3b^2m^2 + 2nb^2m + 2b^2m - nb^2 - b^2) + s^2(2b^2m^3 + 5m^2nb^2 \\ &+5b^2m^2 + 5nb^2m + 3b^2m + mn + m + n + 1) + s^3(-4m^4nb^2 - 10m^3nb^2 \\ &+2b^2m^3 - 6m^2nb^2 + 2b^2m^2 - 3m^2n - 3m^2 - 2mn - 2m + n + 1)\end{aligned}$$

(17)

In view of equation (16) and (17), $(b^2 - s^2)(ms - 1)^4\Phi$ does have same degree of polynomial in s and b only if $m = 0$ which contradicts our assumptions. Therefore case (ii) of Theorem (4) holds. In this case, we have

$$r_{00} = 0, \tag{18}$$

$$s_j = 0. \tag{19}$$

□

4. Projectively Flatness and scalar flag curvature

In this section, we have studied projective flatness of locally dually flat Finsler metric F for both Lemma 2 and 3.

For lemma 2 of locally dually flat:

Proposition 2. *Let* $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m \neq 0, -1)$ *be the* (α,β)*-metric on a manifold* M*. Then locally dually flat metric* F *with isotropic S-curvature is locally projectively flat in adapted coordinate system with* $G^i = C\tau\beta y^i$.

Proof. From Proposition (1), we have $r_{ij} = 0$. Then using equation (11), we get $[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)]\alpha^2 = [-\frac{2}{3}\theta - \frac{1}{3}(3k_2 - 2 - 3k_3b^2)\tau\beta]\beta$. Since α^2 is irreducible polynomial of y^i, we conclude that

$$\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) = 0 \tag{20}$$

and

$$2\theta + (3k_2 - 2 - 3k_3 b^2)\tau\beta = 0.$$

Now contracting equation (9) with b^l, we get

$$s_0 = \frac{1}{3}(\beta b^l \theta_l - \theta b^2).$$

Since $s_0 = 0$ the above equation can be written as

$$b^l \theta_l = \frac{\theta b^2}{\beta}. \tag{21}$$

Using equation (20) and (21), we have

$$\theta = \tfrac{3+2b^2}{2b^2}\tau\beta \qquad \text{and} \qquad \theta_l = \tfrac{3+2b^2}{2b^2}\tau b_l.$$

Using the values of θ and θ_l in equation (9), we get $s_{ij} = 0$. Thus β is closed.

Let

$$\frac{1}{3}(\theta^l - \tau b^l)\alpha^2 = \frac{1}{3}\left[\left(\frac{3+2b^2}{2b^2} - 1\right)\tau\beta\right] y^l \tag{22}$$

and

$$\frac{1}{2}k_3 \tau \beta^2 b^l = \frac{1}{2}k_3 b^2 \tau \beta y^l. \tag{23}$$

Using equation (22), (23), and (12), we get

$$G^l_\alpha = C\tau\beta y^l,$$

where $C = \left[\frac{1+k_1}{b^2} + \frac{1}{2b^2} + \frac{k_3 b^2}{2}\right]$ is a constant. Thus α is projectively flat.

Now we have $r_{ij} = 0, s_j = 0$ and $s_{ij} = 0$, using these values in equations (6), (7), and (8) we obtain $R = 0$ and $Q^i = 0$. Thus we have $G^i = Py^i$, where $P = C\tau\beta$. Thus we can say that locally dually flat metric F with isotropic S-curvature is locally projectively flat in adapted coordinate system. □

For lemma 3 of locally dually flat:

Proposition 3. *Let $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m} (m \neq 0, -1)$ be a locally dually flat non-Randers type (α, β)-metric on a manifold M. Suppose that F is of isotropic S-curvature $S = (n+1)cF$, where $c = c(x)$ is a scalar function on M. Then F is a locally projectively flat in adapted coordinate systems with $G^i = 0$.*

Proof. We have

$$s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l), \tag{24}$$

$$r_{00} = \frac{2}{3}[\theta\beta - (\theta_l b^l)\alpha^2], \tag{25}$$

$$G^l_\alpha = \frac{1}{3}[2\theta y^l + \theta^l \alpha^2]. \tag{26}$$

By (18) and (25), we obtain

$$(\theta_l b^l)\alpha^2 = \theta\beta.$$

Since α^2 is an irreducible polynomial the above equation reduces to

$$\theta = 0,$$

and

$$\theta_l b^l = 0.$$

Then equations (24), (25), and (26) become

$$\begin{aligned} s_{l0} &= 0, \\ G^m_\alpha &= 0, \\ r_{00} &= 0. \end{aligned} \tag{27}$$

By equations (19) and (27), we get $s_0 = 0$ and $s^l_0 = 0$ respectively. Thus by equation (6), we get $G^i = 0$. □

Proposition 4. *Let* $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m \neq 0, -1)$ *be locally projectively flat with zero flag curvature. If* $\tau = 0$, *then* F *is locally Minkowskian.*

Proof. Let us assume that F is locally projectively flat, so that in local coordinate system the spray coefficients of F are in the form $G^i = Py^i$, where in our case $P = C\tau\beta$. It is known that if the spray coefficients of F are in the form $G^i = Py^i$, then F is of scalar flag curvature with

$$K = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{C^2\tau^2\beta^2 - C\tau_{x^k}\beta y^k - C\beta_{x^k}\tau y^k}{F^2}. \tag{28}$$

□

Lemma 4. *Suppose that* $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ $(m \neq 0, -1)$ *is projectively flat with constant flag curvature* $K = \lambda = \text{constant}$, *then* $K = 0$.

Proof. From equation (28), we have $\lambda(\alpha+\beta)^{2m+2} = (C^2\tau^2\beta^2 - C\tau_{x^k}\beta y^k - C\beta_{x^k}\tau y^k)\alpha^{2m}$. Hence $\lambda\left[\alpha^{2m+2} + \binom{2m+2}{1}\alpha^{2m+1}\beta + \binom{2m+2}{2}\alpha^{2m}\beta^2 +, ..., \beta^{2m+2}\right] = (C^2\tau^2\beta^2 - C\tau_{x^k}\beta y^k - C\beta_{x^k}\tau y^k)\alpha^{2m}$. Comparing the different coefficients of α, we have

(i) $\lambda\binom{2m+2}{2} = C^2\tau^2$,
(ii) $\lambda\alpha\binom{2m+2}{1} = -C\tau_{x^k}y^k$, which is not possible, and
(iii) $\lambda = 0$.

Case (i) If $\lambda = 0$ then $\tau = 0$. Thus we have $K = 0$. In this case $G^i = G^i_\alpha = 0$.
Case (ii) If $\lambda\binom{2m+2}{2} = C^2\tau^2$ then $\lambda = \frac{1}{(m+1)(2m+1)}C^2\tau^2$ which is a function of x only. That is, flag curvature $K = K(x)$. Thus F is Riemannian metric. But we have already assumed that F is non-Riemannian. Thus $\lambda = \frac{1}{(m+1)(2m+1)}C^2\tau^2$ will be possible only if $\lambda = 0 = \tau$. Again we have $K = 0$. □

Proof of Theorem (2). By Propositions (2) and (3), we conclude that F is dually flat and projectively flat in any adapted coordinate system. By Lemma (1), we have

$$F_{xk} = CFF_{y^k}.$$

The spray coefficients $G^i = Py^i$ are given by $P = \frac{1}{2}CF$. Since $G^i = 0$, then $P = 0$ and thus $C = 0$. It implies that $F_{x^k} = 0$ and then F is a locally Minkowskian metric in the adapted coordinated system. This completes the proof. □

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