ON SOME PROBLEMS OF APPROXIMATIONS[1]

by
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To Paul Erdős on his 50th birthday

§ 1. Introduction

The problems dealt with in the present Note are connected with the classical inequalities of A. MARKOV and S. BERNSTEIN (cf. [3]). We formulate them as follows. Let $f(x)$ be a polynomial of degree n satisfying the condition $|f(x)| \leqq 1$ in the finite real interval $a \leqq x \leqq b$. We have then in the same interval

$$(1.1)\qquad \begin{cases} |f'(x)| \leqq \dfrac{2}{b-a} \cdot n^2 , \\ |f'(x)| \leqq [(x-a)(b-x)]^{-1/2} \cdot n . \end{cases}$$

Both bounds are sharp as it can be shown by the example $f(x) = T_n\left(2\,\dfrac{x-a}{b-a} - 1\right)$ where T_n is Chebyshev's polynomial.

We shall deal with the following four problems.

Problem 1: *Let us consider all polynomials $f(x)$ of a fixed degree n not vanishing identically. Introducing the norm*

$$(1.2)\qquad \|f\| = \max e^{-x} |f(x)| , \qquad x \geqq 0 ,$$

we seek the maximum M_n of the ratio $\|f'\| : \|f\|$.

Problem 2: *Let x_0 be a fixed constant, $x_0 \geqq 0$. Considering the same class of polynomials and the same norm as in Problem 1, we seek the maximum $M_n(x_0)$ of the ratio $|f'(x_0)| : \|f\|$.*

Problem 3: *Again we consider the set of all polynomials $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ and the same norm $\|f\|$ as in the previous Problems. We seek the maximum G_n of the ratio $|a_n| : \|f\|$.*

Problem 4: *Let us consider all polynomials $f(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of the fixed degree n satisfying the condition $|f(x)| \leqq 1$ in the interval $-1 \leqq x \leqq 1$. We seek the maximum H_n of*

$$(1.3)\qquad \max \left| \sum_{\nu=0}^{n} \log(\nu+1)\, a_\nu x^\nu \right| , \qquad -1 \leqq x \leqq 1 .$$

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In this Note we derive upper and lower bounds for the maxima defined in these Problems; especially in the cases 1, 2, 4 we determine the correct order of magnitude as $n \to \infty$. The explicit evaluation of these maxima seems to be rather difficult. Our results can be formulated as follows:

$$(1.4)\qquad \begin{cases} \text{Problem 1: } M_n \sim n. \\ \text{Problem 2: } M_n(0) \sim n\,,\ M_n(x_0) \sim n^{1/2} \text{ if } x_0 > 0. \\ \text{Problem 3: } An^{1/3} < 2^{-n}\, n!,\ G_n < B\, n^{1/2}. \\ \text{Problem 4: } H_n \sim \log n\,. \end{cases}$$

Here A and B are positive constants independent of n. The symbol $a_n \sim b_n$ means always that the ratio $|a_n/b_n|$ is bounded away from 0 and ∞. Occasionally, we use the symbol $a_n \simeq b_n$ if the ratio a_n/b_n tends to 1 as $n \to \infty$.

These Problems (except 3) arose in conversations with Professor PAUL TURÁN during his stay at Stanford University in the first half of the year 1963. I owe him also some simplifications and other valuable comments to the proofs. Problem 4 has originated in the joint research of S. KNAPOWSKI—P. TURÁN on primes in certain arithmetic progressions [2].

Problem 3 is slightly different in character from the inequalities (1.1); it is the analog of the famous problem of CHEBYSHEV characterizing the Chebyshev polynomials $T_n(x)$.

§ 2. Upper bounds

1. *Problems* 1 *and* 2. We use the familiar inequality $e^{-x} > (1 - x/n)^n$ valid for $0 < x < n$. Thus assuming $||f|| = 1$, we have $|(1 - x/n)^n f(x)| \leqq 1$ in the interval $0 \leqq x \leqq n$. Applying to the polynomial $(1 - x/n)^n f(x)$ of degree $2n$ the first inequality (1.1), we find $(a = 0,\ b = n)$

$$|-(1 - x/n)^{n-1} f(x) + (1 - x/n)^n f'(x)| < \frac{2}{n}(2\,n)^2 = 8\,n\,.$$

In particular for $x = 0$ we find

$$|-f(0) + f'(0)| \leqq 8\,n\,,$$

and since $|f(0) \leqq 1$, we have $|f'(0)| \leqq 8n + 1$, i.e., $M_n(0) \leqq 8n + 1$. Hence, if $f(x)$ is *any* polynomial of degree n not vanishing identically, we have

$$(2.1)\qquad M_n(0) \leqq (8\,n + 1)\,||f||.$$

Let $x_0 > 0$; we assume again that $f(x)$ is not vanishing identically, i.e., $||f|| > 0$. We apply (2.1) to the polynomial $f(x + x_0)$. Since $||f(x + x_0)|| \leqq e^{x_0}\,||f||$, we obtain $|f'(x_0)| \leqq e^{x_0}\,||f||\cdot(8n + 1)$. This being the case for every $x_0 > 0$, we have $||f'|| \leqq (8n + 1)\ ||f||$ so that

$$(2.2)\qquad M_n \leqq 8\,n + 1 \text{ and } M_n(x_0) \leqq e^{x_0}\,(8\,n + 1)\,.$$

A better bound for $M_n(x_0), x_0 > 0$ (as a matter of fact the best one so far as the order of magnitude is concerned) can be obtained with the aid of the second inequality (1.1):

$$|-(1 - x_0/n)^{n-1} f(x_0) + (1 - x_0/n)^n f'(x_0)| \leqq 2n\,(x_0(n - x_0))^{-1/2}.\ ||f||\,,$$

$$(1 - x_0/n)^n\,|f'(x_0)| \leqq (1 - x_0/n)^{n-1}\,|f(x_0)| + 2n\,(x_0(n - x_0))^{-1/2}.\ ||f||.$$

We have $|f(x_0)| \leq e^{x_0} \|f\|$. Hence for a fixed x_0, as $n \to \infty$, we have $M_n(x_0) < A n^{1/2}$, $A > 0$.

2. *Problem* 3. Let ε be positive. We have

$$\|f\|^2 = \max e^{-x} \left| f\left(\frac{x}{2}\right) \right|^2 \geqq \varepsilon \int_0^\infty e^{-\varepsilon x} \cdot e^{-x} \left| f\left(\frac{x}{2}\right) \right|^2 dx . \tag{2.3}$$

Now

$$\int_0^\infty e^{-(1+\varepsilon)x} \left| f\left(\frac{x}{2}\right) \right|^2 dx = (1+\varepsilon)^{-1} \int_0^\infty e^{-x} \left| f\left(\frac{x}{2(1+\varepsilon)}\right) \right|^2 dx .$$

We set

$$f\left(\frac{x}{2(1+\varepsilon)}\right) = \sum_{\nu=0}^{n} c_\nu L_\nu(x) \tag{2.4}$$

where $L_\nu(x)$ denotes the Laguerre polynomials (cf. [4], p. 100, (5.1.6)). Consequently,

$$\frac{a_n}{2^n(1+\varepsilon)^n} = (-1)^n \frac{c_n}{n!} \tag{2.5}$$

so that ([4], p. 99, (5.1.1))

$$\|f\|^2 \geqq \varepsilon(1+\varepsilon)^{-1} \sum_{\nu=0}^{n} |c_\nu|^2 \geqq \varepsilon(1+\varepsilon)^{-1} |c_n|^2 =$$

$$= \varepsilon(1+\varepsilon)^{-1} (n!)^2 2^{-2n} (1+\varepsilon)^{-2n} \cdot |a_n|^2 .$$

This yields the following inequality:

$$2^{-n} n! \cdot \frac{|a_n|}{\|f\|} \leqq (1+\varepsilon)^{n+1/2} \varepsilon^{-1/2} . \tag{2.6}$$

Writing $\varepsilon = 1/n$, the right-hand expression will be $\sim n^{1/2}$.

3. *Problem* 4. We use the formula

$$\log m = \int_0^\infty \frac{e^{-t} - e^{-mt}}{t} dt , \qquad m > 0 , \tag{2.7}$$

(cf. [1], p. 17, (18)), as it can be verified by differentiation with respect to m. (The formula is obvious for $m = 1$.) Hence we have the identity

$$\sum_{\nu=0}^{n} \log(\nu+1) a_\nu x^\nu = \int_0^\infty e^{-t} \frac{f(x) - f(e^{-t}x)}{t} dt . \tag{2.8}$$

Let ε and ω be positive numbers, $\varepsilon < \omega$. We divide the integral in (2.8) in three parts: I, II, III corresponding to the intervals $[0, \varepsilon]$, $[\varepsilon, \omega]$, $[\omega, \infty]$. In the first part we use the mean-value theorem combined with the first inequality (1.1); in the second and third part we use only the bound $|f(x)| \leqq 1$. Then we find

$$|\mathrm{I}| \leqq \int_0^{\varepsilon} e^{-t} \frac{x - e^{-t} x}{t} n^2 \, dt \leqq n^2 \int_0^{\varepsilon} \frac{1 - e^{-t}}{t} \, dt \leqq n^2 \varepsilon$$

since $e^{-t} \geqq 1 - t$. Further

$$|\mathrm{II}| < \int_{\varepsilon}^{\omega} e^{-t} \frac{2}{t} \, dt < \int_{\varepsilon}^{\omega} \frac{2\, dt}{t} = 2 \log \frac{\omega}{\varepsilon},$$

$$|\mathrm{III}| < \int_{\omega}^{\infty} e^{-t} \frac{2}{t} \, dt < \frac{2}{\omega} \int_{\omega}^{\infty} e^{-t} \, dt = \frac{2}{\omega} e^{-\omega}.$$

We choose $\varepsilon = 1/n^2$ and $\omega = n$ so that for $-1 \leqq x \leqq 1$

$$\left| \sum_{\nu=0}^{n} \log (\nu + 1) \, a_\nu x^\nu \right| < 1 + 6 \log n + \frac{2}{n} e^{-n} < 2 + 6 \log n . \tag{2.9}$$

It is easy to prove that $\limsup H_n/\log n \leqq 4$ as $n \to \infty$.

A similar argument leads to an upper estimate for $\left| \sum_{\nu=0}^{n} \lambda_\nu a_\nu x^\nu \right|$ provided that $|f(x)| = \left| \sum_{\nu=0}^{n} a_\nu x^\nu \right| \leqq 1$. Here we assume that the constants λ_ν are the moments of a positive distribution,

$$\lambda_\nu = \int_a^b t^\nu \, d\,\alpha(t), \qquad \nu = 0, 1, \ldots, n, \tag{2.10}$$

or else that the sequence $\{\lambda_\nu\}$ is the difference of two sequences of the form (2.10).

§ 3. Lower bounds

In order to obtain lower bounds for the quantities M_n, $M_n(x_0)$, G_n, H_n defined above, we have to exhibit certain special polynomials $f(x)$. We denote by $L_n^{(\alpha)}(x)$ and by $L_n^{(0)}(x) = L_n(x)$ the Laguerre polynomials (cf. [4], p. 100, (5.1.6)).

1. The polynomial $L_n(x)$ satisfies the inequality ([4], p. 162, (7.21.3))

$$e^{-x/2} |L_n(x)| \leqq 1, \qquad x \geqq 0, \tag{3.1}$$

with the equality sign for $x = 0$. Hence the function $f(x) = L_n(2x)$ has the norm $||f|| = 1$. Now ([4], p. 101, (5.1.13) and (5.1.14))

$$\frac{d}{dx} L_n(x) = -L_{n-1}^{(1)}(x) = -L_0(x) - L_1(x) - \ldots - L_{n-1}(x)$$

so that $| e^{-x/2} L_n'(x) | \leqq n$ with equality for $x = 0$; with other words $|| L_n'(2x) || = n$. Hence $M_n \geqq n$, $M_n(O) \geqq n$ so that $M_n \sim n$, $M_n(O) \sim n$.

2. In order to obtain a lower bound for $M_n(x_0)$, $x_0 > 0$, we use ([4], p.239, (8.91.7)) where we write $\lambda = 0$ and we choose $\alpha \geqq -1/6$. Thus

$$\max e^{-x/2} \, | L_n^{(\alpha)}(x) | \sim n^{\alpha/2-1/4} , \qquad x \geqq a , \tag{3.2}$$

where $a > 0$ is arbitrary and fixed. Now let $c = c(n, \alpha, x_0) = c_n$ be a positive constant to be determined later; $\lim\limits_{n\to\infty} c_n$ will exist and it will be finite and positive. We write

$$f(x) = n^{1/4-\alpha/2} L_n^{(\alpha)}(2\,x + 2\,c_n) , \qquad 2c_n \geqq a , \tag{3.3}$$

so that $||f||$ depends on n and $||f|| \sim 1$ as $n \to \infty$. Hence (see above)

$$\frac{1}{2} f'(x_0) = - n^{1/4-\alpha/2} L_{n-1}^{(\alpha+1)}(2\,x_0 + 2\,c_n) \sim n^{1/4-\alpha/2} \cdot n^{\frac{\alpha+1}{2} - \frac{1}{4}} = n^{1/2}.$$

Here we used FEJÉR's classical asymptotic formula for the Laguerre polynomials (cf. [4], p. 196, (8.22.1)); the oscillatory part of the main term is

$$\cos \left\{ 2(n-1)^{1/2} (2\,x_0 + 2\,c_n)^{1/2} - \frac{(\alpha+1)\,\pi}{2} - \frac{\pi}{4} \right\} = \cos x' .$$

We determine c_n in such a way that x' satisfies the condition $x' \equiv 0 \pmod{\pi}$. For this purpose we set

$$2(n-1)^{1/2} (2x_0 + 2c_n)^{1/2} - \frac{(\alpha+1)\pi}{2} - \frac{\pi}{4} = [2(n-1)^{1/2} \, (3\,x_0)^{1/2} \cdot \pi^{-1}] \, \pi = k\,\pi$$

where $[y]$ is the greatest integer $\leq y$; k is an integer. Hence $\lim\limits_{n\to\infty} (2x_0 + 2c_n)^{1/2} = (3x_0)^{1/2}$ so that $\lim\limits_{n\to\infty} c_n = x_0/2$. Fejér's formula holds uniformly in every fixed positive interval.

3. Formula (3.2) is based on rather complicated considerations. A simpler approach to the same result $M_n(x_0) \sim n^{1/2}$, $x_0 > 0$, is the following. Let c be again a positive constant to be determined later. We choose

$$f(x) = T_n\left(\frac{x+c}{n} - 1\right) = (-1)^n \, T_n\left(1 - \frac{x+c}{n}\right),$$

so that $e^{-x} \, |f(x)| \leqq 1$ in the interval $0 \leqq x \leqq 2n - c$. Now we write $1 - \frac{x_0 + c}{n} = \cos\varphi = 1 - \frac{\varphi^2}{2} + \ldots$ so that $\varphi \simeq \left(\frac{2(x_0+c)}{n}\right)^{1/2}$ and $T_n(\cos\varphi) = \cos n\,\varphi = \cos \left(2n(x_0 + c)\right)^{1/2}$. We determine $c = c(n, x_0) = c_n$ in such a way that $\left(2n(x_0 + c_n)\right)^{1/2} = (k + 1/4)\pi$, k an appropriate integer. For this purpose we may follow a similar procedure as above; $\lim\limits_{n\to\infty} c_n$ will exist and it will be again finite and positive. Hence $e^{-x_0} \, |f(x_0)| \sim 1$.

We discuss now the values $y = \frac{x+c}{n} - 1 \geqq 1$ so that we can write $y = \operatorname{ch} \alpha$, $\alpha \geqq 0$. We have then $f(x) = \operatorname{ch} n\alpha$ and

$$e^{-x} f(x) = e^{c-n(\operatorname{ch}\alpha+1)} \cdot \frac{e^{n\alpha} + e^{-n\alpha}}{2}.$$

In order to prove $\|f\| \sim 1$ it remains to show that $\max(-\operatorname{ch}\alpha - 1 + \alpha) < 0$, $\alpha \geqq 0$. This is indeed so, since $-\operatorname{ch}\alpha \leqq -1 - \frac{\alpha^2}{2} < 1 - \alpha$.

Now we consider

$$f'(x_0) = \frac{1}{n} T_n'\left(\frac{x_0+c}{n} - 1\right) = \frac{(-1)^{n-1}}{n} T_n'\left(1 - \frac{x_0+c}{n}\right).$$

We write again $1 - \frac{x_0+c}{n} = \cos\varphi$ so that $\varphi \cong \left(\frac{2(x_0+c)}{n}\right)^{1/2}$. But

$$n^{-1} T_n'(\cos\varphi) = \frac{\sin n\varphi}{\sin\varphi} \cong \left(\frac{n}{2(x_0+c)}\right)^{1/2} \sin\left(2n(x_0+c)\right)^{1/2}.$$

Since $(2n(x_0+c))^{1/2} = (k+1/4)\pi$ we have $\sin(2n(x_0+c))^{1/2} = \pm \sin\frac{\pi}{4}$.

This proves the assertion.

4. We seek a lower bound for G_n. For the function (3.3) we have $\|f\| \sim 1$ and

$$|a_n| = n^{1/4-\alpha/2} \cdot \frac{2^n}{n!}.$$

Choosing $\alpha = -1/6$ the lower bound $A \cdot 2^n (n!)^{-1} \cdot n^{1/3}$ follows.

5. Finally we seek a lower bound for H_n. In (2.8) we choose $f(x) = T_n(x)$ and we write $x = 1$. We obtain

$$\int_0^\infty e^{-t} \frac{1 - T_n(e^{-t})}{t} dt = \int_0^{\pi/2} \sin\varphi \frac{1 - \cos n\varphi}{\log \frac{1}{\cos\varphi}} d\varphi \geqq \int_0^{\pi/2} \sin\varphi \cos\varphi \frac{1 - \cos n\varphi}{1 - \cos\varphi} d\varphi$$

where the inequality $\log x \leqq x - 1$, $x \geqq 1$, was used. Now

$$\int_0^{\pi/4} \frac{1 - \cos n\varphi}{\varphi} d\varphi = \int_0^{n\pi/4} \frac{1 - \cos x}{x} dx$$

and the last integral is $\sim \log n$.

This establishes the proof of (1.4).

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О НЕСКОЛЬКИХ АППРОКСИМАЦИОННЫХ ПРОБЛЕМАХ

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Резюме

В настоящей работе рассматриваются четыре задачи приближения, в каждой из которых участвуют полиномы с заданной степенью n. Три первые задачи аналогичны определенным проблемам приближения, в которых вместо конечных интервалов, употребляющихся в классических случаях, участвует бесконечный интервал $x \geqq 0$. В работе приводятся два результата.

1) Определим для произвольного полинома $f(x)$ «норму» следующим образом:

$$\|f\| = \max e^{-x}\,|f(x)|\,,\ x \geqq 0\,. \text{ Тогда } \|f'\| \leqq A_n \|f\|\,,$$

где A_n — постоянная, зависящая только от степени n. A_n имеет порядок n.

2) Пусть $\left|\sum_{\nu=0}^{n} a_\nu x^\nu\right| \leqq 1$ в интервале $a-1 \leqq x \leqq 1$. Тогда

$$\left|\sum_{\nu=0}^{n} \log(\nu+1)\, a_\nu x^\nu\right| \leqq k \log n\,,$$

где k посто янное.

Работа посвящается P. ERDŐS в честь пятидесятилетия со дня рождения.