ON THE REPETITION-FREE REALIZATION OF TRUTH FUNCTIONS BY TWO-TERMINAL GRAPHS I

by
ANDRÁS ÁDÁM

Introduction

The investigations described in the present paper join to a theorem of B. TRACHTENBROT ([8], Theorem 1). This theorem serves the purpose of determining a 2-graph if the truth function realized by it is known, and it solves the problem completely from theoretical point of view. However, in applying this theorem, there arises a so high number of tests which means a grave disadvantage in practical application. Our investigations are devoted to improving TRACHTENBROT's idea into a more determined method which seems to be able for being performed by an electronic computer or by a machine built for this special aim.

§ 1 gives a survey on the situation of researches in the investigated field. §§ 2—4 contain the description of the proposed algorithm.[1] The problems (arisen in boundaries of graph theory and logic) are investigated by terms of combinatorial set theory.

At the end of § 4 the questions of machine realization are touched (without the endeavour to perfectness).

The paper terminates with four appendices. App. 1 presents a proposition which has an auxiliary rôle in the algorithm. App. 2 considers a part of the main problem which requires to be treated separately. The two final appendices touch the immediate further questions of the theory, namely they give such mathematical formulations of the problems which will prove, it can be hoped, to be profitable in the future research.

§ 1. The notion of repetition-free realization. Preliminary theorems

It is supposed that the most important initial concepts (truth function, monotonic dependence, prime implicant, repetition-free superposition of truth functions; strongly connected two-terminal graph, canonical decomposition of such graphs, path) are already known to the reader. Concerning these notions we refer to the papers [1], [2], [3], [7], [8].

Let $e_1, \ldots, e_n$ denote (all) the edges of a 2-graph (i.e. strongly connected two-terminal graph) $\mathfrak{G}$; let us assign the truth variables $x_1, \ldots, x_n$ to these edges, respectively. Let that truth function $f(x_1, \ldots, x_n)$ be considered which has the value $\uparrow$ (on a place of its definition domain) if and only if there exists a path of $\mathfrak{G}$ whose every edge corresponds to a truth variable having the value

[1] The reader who is interested only in performing our procedure can omit the proofs of propositions of § 3. The ending of each proof is denoted by Q. E. D.

$\uparrow$. Then, usually, it is said that the graph $\mathfrak{G}$ *realizes the function f without repetition.* There are well-known the following fundamental facts:

f depends effectively and monotonic increasingly on its each variable,

a set of edges of $\mathfrak{G}$ is a path if and only if the conjunction of the variables corresponding to these edges is a prime implicant of f,

$\mathfrak{G}$ can be decomposed canonically if and only if f can be decomposed by repetition-free superposition[2] (KUZNECOV [7], Theorem on p. 197),

if the realizable function f cannot be decomposed by superposition, then f has essentially only one realization (i.e. any two 2-graphs realizing f can be connected by an isomorphism which maps the terminals of one of the graphs to the terminals of the other graph) (TRACHTENBROT [8], Theorem 2, p. 237).

The first fact exposed just now gives a necessary condition in order that a truth function should be realizable; however, this condition is not sufficient at all. So there arises the p r o b l e m o f r e a l i z a b i l i t y: let be stated for a function in order a necessary and sufficient condition for a function to be realizable (without repetition). Further, starting by a given truth function f, it is desirable to find a possibly simple procedure in order to construct the graph realizing f in the case when f has been found to have a realization.

We shall now consider these problems in the particular case when f is assumed to have no decomposition by (repetition-free) superposition.[3] A remarkable way in order to attempt our problems is shown by a theorem of B. TRACHTENBROT ([8], Theorem 1, p. 236) which will be recapitulated here in a somewhat different formulation. The correspondence between the paths of a 2-graph $\mathfrak{G}$ and the prime implicants of the function f realized by $\mathfrak{G}$ implies that, if we are searching the graph $\mathfrak{G}$, the paths of $\mathfrak{G}$ (as sets of edges, regardless to the ordering) can be given easily, therefore it remains to be determined how these edges are incident to the vertices. The set of (all) edges incident to an inner vertex is called an *inner star*. The theorem of Trachtenbrot states that *a set $\mathfrak{H}$ of edges of $\mathfrak{G}$ is an inner star if and only if $\mathfrak{H}$ satisfies each of the following three requirements:*

A) *any edge beside $\mathfrak{H}$ is contained in some path which contains no edge of $\mathfrak{H}$,*

B) *for any path, the number of the common edges of $\mathfrak{H}$ and this path is either* 0 *or* 2,

C) *to each pair of edges of $\mathfrak{H}$ there exists a path containing both of these edges.*[4]

For any set of edges of $\mathfrak{G}$ it can be controlled whether these conditions A), B), C) are fulfilled or not. So, theoretically, the theorem of Trachtenbrot gives a procedure for solving our construction problem.[5] However, the producere got by direct, "rough" application of this theorem is not satisfactory from two points of view either. Firstly, it does not give an elegant solution for the problem of realizability. Namely, if we apply this procedure for a non-realizable function f, then there are two possibilities: either it arises an obvious irregularity already

[2] Naturally, under the presupposition that f admits a (repetition-free) realization.

[3] The particular case of the problem of realizability was formulated as Problem 2 in [4] (p. 36). — Concerning the relation of the general problem to this particular case, we refer — beside the mentioned theorem of KUZNECOV — to § 2 of [7] and to [4].

[4] Our assumption on the indecomposability of f is essential in order that this theorem should be true. In the contrary case, the notion of inner star must be replaced by a more special and complicated notion.

[5] For the sake of completeness, one has to determine finally the edges incident to the one and the other terminal. Concerning this simple task see [8].

in the course of applying the procedure (but, in general, not at the beginning), or we reach regularly a 2-graph which realizes a truth function different from the function f given previously. Secondly, even in the case of realizability, the procedure requires a great number (2^n, where n is the number of the edges) of tests each of which decides whether a special subset satisfies the conditions or not. Consequently, it is desirable to seek a more definite algorithm in the sense that it has a reduced number of steps in comparation to the direct method which consists of testing all sets of edges concerning they fulfil A), B), C) or not. The main parts of the present paper are devoted to improve TRACHTENBROT's theorem into a more constructive (therefore more mechanizable) procedure which serves the purpose of determining the inner stars concerning a function, indecomposable by superposition, supposed to be realizable.[6]

The first imperfectness exposed above makes justifiable to look for a solution of the problem of realizability consisting of a possibly explicit criterion which can be applied for a truth function f still before constructing the graph realizing f (if such a graph exists). This difficult problem will be touched in Appendix 3.

§ 2. Formulation of the problem in terms of combinatorial set theory

If a truth function f is given which depends monotonic increasingly on its each variable, then the prime implicants can be regarded as certain subsets of the set E consisting of the variables of f. Also the properties A), B), C) occurring in TRACHTENBROT's theorem can be expressed in terms of combinatoral set theory. Using this idea, the questions can be raised as transformed followingly.[7]

Let a finite set E be given, further a collection $\mathcal{P}$ of subsets of E such that $\mathcal{P}$ satisfies the following condition: if $P_1 \in \mathcal{P}$, $P_2 \in \mathcal{P}$ and $P_1 \neq P_2$, then neither $P_1 \supset P_2$ nor $P_1 \subset P_2$ holds.

A subset F of E is called a W-*set* if it satisfies all the three statements (I), (II), (III):

(I) To any element x of $E-F$ there exists a set $P(\in \mathcal{P})$ such that $x \in P \subseteq E - F$.

(II) If an intersection $P \cap F$ is not empty, then $\overline{\overline{P \cap F}} = 2$ (for the members P of $\mathcal{P}$).

(III) To any pair x, y of elements of F there exists a set $P(\in \mathcal{P})$ satisfying $P \supseteq \{x, y\}$.

Let us fix two elements a, b (chosen arbitrarily) of E. Our aim is in §§ 3–4 to determine those W-sets with at least four elements which are supersets of $\{a, b\}$; especially, to decide whether a such W-set exists or not. During these investigations a W-set F which fulfils $F \supseteq \{a, b\}$ and $\overline{\overline{F}} \geqq 4$ will be called a *desired* W-*set*.

Some of the propositions stated in the course of our investigations give sufficient conditions in order that no desired W-set exists. After proving a proposition of this character, it will be supposed that the condition is not fulfilled (even if this supposition is not formulated explicitly).

[6] Although the procedure proposed here is more definite in the mentioned sense, it has a more complicated description than the "rough method".

[7] If the elements of a set $\mathfrak{A}$ are sets themselves, then we shall say that $\mathfrak{A}$ is the *collection* of its *members*.

§ 3. Foundation of the algorithm

Proposition 1. *If one of the following statements α), β), γ) is true, then there exists no desired W-set:*

α) $a \in P$, $b \in P$ *holds for no member of* $\mathscr{P}$,
β) $a \in P$, $b \notin P$ *holds for no member of* $\mathscr{P}$,
γ) $a \notin P$, $b \in P$ *holds for no member of* $\mathscr{P}$.

Proof. If α) is true, then no superset of $\{a, b\}$ can fulfil (III). If e.g. β) holds and $c(\neq a, b)$ is an element of an arbitrary superset F of $\{a, b\}$, then $\mathscr{P}$ cannot have a member P satisfying $P \cap F = \{a, c\}$, hence F does not fulfil (III). Q. E. D.

Now we define four collections $\mathscr{P}_{ab}^{(0)}$, $\mathscr{P}_a^{(0)}$, $\mathscr{P}_b^{(0)}$, $\mathscr{P}_0^{(0)}$ of subsets of E by the following rules:

$\mathscr{P}_{ab}^{(0)}$ consists of the sets of form $P - \{a, b\}$ where P runs through the members of $\mathscr{P}$ containing both of a and b.

$\mathscr{P}_a^{(0)}$ consists of the sets of form $P - \{a\}$ where P runs through the members of $\mathscr{P}$ satisfying $a \in P$ and $b \notin P$.

$\mathscr{P}_b^{(0)}$ consists of the sets of form $P - \{b\}$ where P runs through the members of $\mathscr{P}$ satisfying $b \in P$ and $a \notin P$.

$\mathscr{P}_0^{(0)}$ consists of those members of $\mathscr{P}$ which contain neither a nor b.

We form further collections[8] of subsets of E. Each of $\mathscr{P}_a^{(1)}$, $\mathscr{P}_b^{(1)}$, $\mathscr{P}_0^{(1)}$ consists of the (distinct) sets of form $P - H_{ab}^{(0)}$ where P runs through the members of $\mathscr{P}_a^{(0)}$, $\mathscr{P}_b^{(0)}$ $\mathscr{P}_0^{(0)}$, respectively, and $H_{ab}^{(0)}$ is the union of the members of $\mathscr{P}_{ab}^{(0)}$. Let $H_a^{(1)}$, $H_b^{(1)}$, $H_0^{(1)}$ denote the unions of the members of $\mathscr{P}_a^{(1)}$, $\mathscr{P}_b^{(1)}$, $\mathscr{P}_0^{(1)}$, respectively.

Proposition 2. *If*

$$(H_a^{(1)} \cup H_b^{(1)}) - H_0^{(1)} \tag{1}$$

is not empty, then there exists no desired W*-set.*

Proof. We verify the proposition indirectly. Firstly, we show that any desired W-set F is a superset of the difference (1). In the contrary case, since (I) assures that any element of $E - F$ occurs in some member of $\mathscr{P}_0^{(0)}$, any element of the difference outside F would be contained in some member of $\mathscr{P}_0^{(1)}$. — In the further proof we separate two cases.

Case 1: the difference (1) contains two or more elements. Let x, y be distinct elements of (1). By (III) there exists a member P of $\mathscr{P}$ containing both x and y, but this P must contain one of a, b too, what contradicts to (II).

Case 2: the difference consists of one element x. If F is a desired W-set, then it contains a, b, x; let a further element of F be denoted by y. By (III) there is a $P(\in \mathscr{P})$ such that $P \cap F = \{x, y\}$. The fact that P contains neither a nor b contradicts to the definition of x. Q. E. D.

So we can suppose $H_a^{(1)} \subseteq H_0^{(1)}$ and $H_b^{(1)} \subseteq H_0^{(1)}$. Let $\mathscr{P}_a^{(2)}$, $\mathscr{P}_b^{(2)}$, $\mathscr{P}_0^{(2)}$ be defined as the collections of sets of form $Q \cap H_a^{(1)} \cap H_b^{(1)}$ where Q runs through the members of $\mathscr{P}_a^{(1)}$, $\mathscr{P}_b^{(1)}$, $\mathscr{P}_0^{(1)}$, respectively. Similarly, let $\mathscr{P}_a^{(3)}$, $\mathscr{P}_b^{(3)}$, $\mathscr{P}_0^{(3)}$, be defined as the collections of sets of form $Q - R$ where Q runs through the members of $\mathscr{P}_a^{(2)}$, $\mathscr{P}_b^{(2)}$, $\mathscr{P}_0^{(2)}$, respectively, and R is defined by the following induction. Let R_1 be the union of those members of $\mathscr{P}_0^{(2)}$ each of which consists of (exactly) one element. If R_{i-1} is already defined and there exists a set of

[8] It is allowed that the empty set occurs as a member of the collections to be defined.

form $P - R_{i-1}$ ($P \in \mathscr{P}_0^{(2)}$) consisting of one element, then let R_i be the union of R_{i-1} and the one-element sets of form $P - R_i$ ($P \in \mathscr{P}_0^{(2)}$). Let R denote the last R_i.

Let $H_a^{(2)}$, $H_b^{(2)}$, $H_0^{(2)}$, $H_a^{(3)}$, $H_b^{(3)}$, $H_0^{(3)}$ denote the union of the members of $\mathscr{P}_a^{(2)}$, $\mathscr{P}_b^{(2)}$, $\mathscr{P}_0^{(2)}$, $\mathscr{P}_a^{(3)}$, $\mathscr{P}_b^{(3)}$, $\mathscr{P}_0^{(3)}$ respectively.

The next proposition follows immediately from the above definitions.

Proposition 3. *There hold the equalities* $H_a^{(2)} = H_b^{(2)} = H_0^{(2)}$ *and* $H_a^{(3)} = H_b^{(3)} = H_0^{(3)}$.

Proposition 4. *Each desired* W*-set is a subset of* $H_0^{(3)} \cup \{a, b\}$.

Proof. Let x be an arbitrary element $\neq a, b$ of a desired W-set. (II) certifies that x cannot occur in some member of $\mathscr{P}_{ab}^{(0)}$, so x occurs in a member of one of $\mathscr{P}_a^{(1)}$, $\mathscr{P}_b^{(1)}$, $\mathscr{P}_0^{(1)}$. By our agreement after proving Proposition 2, x is an element of some member of $\mathscr{P}_0^{(1)}$. If no member of $\mathscr{P}_a^{(1)}$ contained x, then $\{a, x\}$ would not occur as a subset of some P ($\in \mathscr{P}$), this contradicts to (III). Therefore x must belong to $H_a^{(1)}$, and, by a similar inference, also to $H_b^{(1)}$. This implies that $x \in H_0^{(2)}$.

We have to prove finally that each element $\neq a, b$ of a desired W-set is contained in $H_0^{(3)}$. This is implied obviously by the following statement: if y is an element of some set R_i, then y cannot occur in any desired W-set. In proving this statement, we can assume that the similar statement was already verified for the elements of R_{i-1}. There exists a member P of $\mathscr{P}_0^{(2)}$ such that $P - R_{i-1} = \{y\}$. If y were contained in a desired W-set F, then $P \cap F = \{y\}$ would be true, what contradicts to (II). Q. E. D.

Proposition 5. *If one of* $\mathscr{P}_a^{(3)}$ *and* $\mathscr{P}_b^{(3)}$ *contains the empty set, then there exists no desired* W*-set.*

Proof. Let be assumed that $\emptyset$ occurs as a member of e.g. $\mathscr{P}_a^{(3)}$. This means that a suitable member P of $\mathscr{P}$ has an empty intersection with $H_a^{(3)}$ ($= H_0^{(3)}$). If F were a desired W-set, then $P \cap F$ would be equal to $\{a\}$ this contradicts (II). Q. E. D.

Proposition 6. *A subset* $F(\supset \{a, b\})$ *of* E *is a desired* W*-set if and only if all the following seven conditions are fulfilled* (F' denotes the set $F - \{a, b\}$):

(O′) $F' \subseteq H_0^{(3)}$, $\overline{\overline{F'}} \geq 2$.

(I′) (= (I)) *To any element* x *of* $E - F$ *there exists a set* $P(\in \mathscr{P})$ *such that* $x \in P \subseteq E - F$.

(II_1') *Each member of* $\mathscr{P}_a^{(3)}$ *and* $\mathscr{P}_b^{(3)}$ *has a non-empty intersection with* F'.

(II_2') *No proper subset of* F' *fulfils the property* (II_1').

(II_3') *Every intersection mentioned in* (II_1') *consists of exactly one element.*

(II_4') *If an intersection* $P \cap F'$ *is non empty (where* $P \in \mathscr{P}_0^{(3)}$*), then* $\overline{\overline{P \cap F}} = 2$.

(III′) *To any pair* x, y *of elements of* F' *there exists a member of* $\mathscr{P}_0^{(3)}$ *containing both* x *and* y.

Proof. Let us remember that the desired W-sets are characterized by the requirements (I), (II), (III), $\overline{\overline{F}} \geq 4$, $F \supset \{a, b\}$. Firstly we shall prove that any desired W-set satisfies the properties exposed in Proposition 6. (O′) was stated already in Proposition 4. (I′) is trivially satisfied. Owing (O′), (II) implies (II_1') and (II_3'); one can see easily that (II_2') follows from (O′), (II_1') and (II_3') (see also Proposition 3).[9] (II_4') and (III′) are immediate consequences of (II) and (III), respectively.

Conversely, let a subset F of E possess the properties enumerated in Proposition 6. (I) is satisfied by F trivially. Let $P \cap F \neq \emptyset$ be assumed for an arbitrary $P(\in \mathcal{P})$, in this case there are three possibilities:

if both of a, b is contained in $P \cap F$, then (O′) assures $P \cap F = \{a, b\}$ ($H_0^{(3)}$ is disjoint, by its definition, to any member of $\mathcal{P}$ containing a, b),

if exactly one of a, b belongs to $P \cap F$, then (O′) and (II_3') imply $\overline{\overline{P \cap F}} = 2$,

if neither a nor b is contained in $P \cap F$, then $\overline{\overline{P \cap F}} = 2$ follows from (O′) and (II_4');

so (II) is satisfied by F in each possible case.

We are now going to show that F fulfils (III). Let a pair x, y of elements of F be considered. If $x = a$ and $y = b$, then (III) holds since the assumption of Proposition 1 is supposed to be false. If exactly one of a, b occurs in the pair x, y, then (III) is implied by (O′) and (II_1') (indeed, if e.g. x differs from a and b, then (O′) and Proposition 3 assure that x occurs in some member of $\mathcal{P}_a^{(3)}$ and in some member of $\mathcal{P}_b^{(3)}$). Finally, if a and b do not occur in the investigated pair, then (III) is a consequence of (III′). Q. E. D.

§ 4. Oversight of an algorithm for determining the W-sets

Owing the results of the preceding paragraph and Appendices 1, 2, we can propose an algorithm which determines the W-sets if the set E and the members of $\mathcal{P}$, belonging to a realizable truth function f indecomposable by superposition, are given. Before all, we determine the three-element W-sets (see App. 2). Afterwards we consider the pairs of elements of E (in arbitrary order), and we apply the following procedure for each of these pairs. Firstly, we look at whether the considered pair a, b occurs in a W-set, determined sooner, or not. If it does occur, then the investigation of the pair a, b is finished, there exists no other W-set which is a superset of $\{a, b\}$; we consider the next pair.[10] In the other case (i.e. if there was got no W-set sooner, containing both a and b) we form the sets according with the definitions of § 3. In the corresponding stages we control the fulfilment of the supposition of Proposition 1, 2, 5 respectively. (If one of these suppositions is fulfilled, then the investigation of the pair a, b is finished.) In the case when it is possible that $\{a, b\}$ is a subset of some W-set, it remains to decide which subset of $H_0^{(3)}$ satisfies (I′) — (III′). Proposition 7 (in App. 1) gives a method for determining the subsets satisfying (II_1') and (II_2') explicitly, we must choose the set from these sets (if it exists) which fulfils also (I′), (II_3'), (II_4'), (III′). This can be executed by tests.

The procedure presented just now seems to be convenient for being realized by a machine whose activities are, in great lines, similar to an electronic digital computer. Instead of arithmetical operations, it need perform some set-theoretical operations. The storage of the machine contains the subsets of E which occur in the procedure. The members of $\mathcal{P}$ are stored during the whole

[9] The reader can observe that the properties enumerated in Proposition 6 do not form an independent system, namely, we could get an equivalent system by omitting (II_2'). However, the admission of this property in our proposition will prove to be advantageous from the view point of our further aims.

[10] This test is made justified by the fact that the intersection of two distinct W-sets cannot have two or more elements since f is indecomposable and realizable. — If we omitted this step, then a W-set having m elements would appear in $\binom{m}{2}$ exemplars.

procedure. It seems to be a convenient method that each set is stored in one memory cell. The length of the cells (i.e. the number of digits in any cell) is greater or equal to the number of elements of E ($\bar{\bar{E}}$ digits correspond one-one to the elements of E). The value of the digits can be $+1$, -1 or 0. $+1$ denotes that the corresponding element of E occurs in the stored subset, -1 denotes that the corresponding element does not occur in the subset, and 0 is the value of the digits to whom no element of E is assigned. We must store also the W-sets constantly when they had been produced. The sets occurring only in investigating a fixed pair can be cancelled after the transition to an other pair. The members of e.g. $\mathscr{P}_0^{(3)}$ are stored similarly as the members of $\mathscr{P}$, but now also those digits have the value O which correspond to the elements of $E - H_0^{(3)}$; we must provide that no member of $\mathscr{P}_0^{(3)}$ should be stored in two exemplars. Furthermore, it is necessary to have a special unit which serves for executing the method justified by Proposition 7.

Appendices

1

In the present section a proposition of combinatorial set-theoretical nature is proved. The idea of this proposition is not essentially new, a thought of related character arises already in [5] and [6].[11] However, I believe that the equivalence stated here has not yet been formulated in such an explicit manner.

Let $M = \{r_1, r_2, \ldots, r_\alpha\}$ be a finite set, let a collection of its certain subsets $N_1, N_2, \ldots, N_\beta$ be given. We can suppose $\bigcup\limits_{\gamma=1}^{\beta} N_\gamma = M$. Let the truth variables $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\alpha$ be assigned to $r_1, r_2, \ldots, r_\alpha$, respectively. Let us form for each N_γ ($\gamma = 1, \ldots, \beta$) the disjunction $\mathfrak{N}_\gamma$ of those (unnegated) variables which correspond to the elements of N_γ.

Proposition 7. *The following two statements are equivalent for a subset M' of M:*

(i) *M' has a non-empty intersection with each of $N_1, N_2, \ldots, N_\beta$ and for any proper subset M'' of M' there exists a set N_γ ($1 \leq \gamma \leq \beta$) which satisfies $M'' \cap N_\gamma = \emptyset$.*

(ii) *The conjunction of the variables which correspond to the elements of M' is a prime implicant of the (monotonic increasing) truth function*

$$\mathfrak{f}(\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\alpha) = \mathfrak{N}_1 \,\&\, \mathfrak{N}_2 \,\&\ldots\&\, \mathfrak{N}_\beta.$$

Proof. Under a *full elementary conjunction* of $\mathfrak{f}$ we understand, usually, a conjunction in which each of $\mathfrak{r}_1, \ldots, \mathfrak{r}_\alpha$ exactly once occurs and which contains only these variables (the unique occurrence of a variable can be unnegated or negated). These conjunctions can be identified with the places of the definition domain of $\mathfrak{f}$ in the customary manner. Let $\mathfrak{A}$ and $\mathfrak{B}$ be full elementary conjunctions; $\mathfrak{A}$ is said to be *greater* as $\mathfrak{B}$ if there is no variable which occurs negated in $\mathfrak{A}$ and unnegated in $\mathfrak{B}$.

Let us assign to any subset M' of M a full elementary conjunction by what follows: the conjunction contains $\mathfrak{r}_\delta$ ($1 \leq \delta \leq \alpha$) unnegated if and only if $\mathfrak{r}_\delta$ occurs in the set M'. One can see that $M' \supseteq M''$ holds if and only if a greater conjunction is assigned to M' than to M''. The value of $\mathfrak{f}$ on the place

[11] Cf. Theorem on p. 337 and Footnote[2] in [6].

corresponding to M' shows whether M' has a non-empty intersection with each of $N_1, \ldots, N_\beta$ or not.

M' has the property (i) if and only if the full elementary conjunction $\mathfrak{A}$ (assigned to M') satisfies the following to statements: $\mathfrak{f}(\mathfrak{A}) = \uparrow$, and $\mathfrak{B} < \mathfrak{A}$, $\mathfrak{B} \neq \mathfrak{A}$ imply $\mathfrak{f}(\mathfrak{B})$ where $\mathfrak{B}$ is an arbitrary full elementary conjunction of $\mathfrak{f}$. Let $\mathfrak{A}'$ be the conjunction which results from $\mathfrak{A}$ by cancelling the variables which occur in $\mathfrak{A}$ negated. Since $\mathfrak{f}$ is monotonic increasing, $\mathfrak{A}$ fulfils the above statement if and only if $\mathfrak{A}'$ is a prime implicant of $\mathfrak{f}$. Q. E. D.

2

The presented procedure is able to determine those W-sets only which consist of at least four elements. It remains the task of determining the three-element W-sets. This can be performed as it follows.

Let a, b be fixed elements of E. The next proposition can be verified similarly to Proposition 1.

Proposition 8. *If one of the statements α), β), γ) occurring in Proposition 1 is true, then there exists no three-element W-set containing both a and b.*

Let the sets D_1, D_2 be defined by

$$D_1 = \bigcup_{\substack{P \ni a \\ P \ni b}} P, \qquad D_2 = \bigcup_{\substack{P \not\ni a \\ P \not\ni b}} P,$$

and let D_3, C be defined by $D_3 = \bigcup P$, $C = \bigcap P$ where P runs through those members of $\mathcal{P}$ which contain exactly one of a, b.

Proposition 9. *If either the set $C - (D_1 \cup D_2)$ is empty or it contains at least two elements, then there exists no three-element W-set containing a and b.*

Assume that $C - (D_1 \cup D_2)$ consists of one element c, in this case there are two possibilities:

(i) *$\{a, b, c\}$ is a W-set, there exists no other three-element W-set containing a and b, and $(D_1 \cup D_3) - \{a, b, c\} \subseteq D_2$,*

(ii) *there exists no three-element W-set containing a and b, and $\big((D_1 \cup D_3) - \{a, b, c\}\big) - D_2$ is not empty.*

Proof. One can see easily that $\{a, b, c\}$ fulfils (II) and (III) if and only if $c \in C - (D_1 \cup D_2)$. If $c' (\neq c)$ is an arbitrary other element of $C - (D_1 \cup D_2)$, then (I) cannot be fulfilled by $\{a, b, c\}$. If $C - (D_1 \cup D_2) = \{c\}$, then $\big((D_1 \cup D_3) - \{a, b, c\}\big) - D_2 = \emptyset$ is the necessary and sufficient condition in order that $\{a, b, c\}$ should fulfil (I). Q. E. D.

3

Let f be a truth function as at beginning § 2. We have there formulated what the properties due to TRACHTENBROT mean in terms of combinatorial set theory. The exposed correspondence makes possible that also other properties of f should be expressed set-theoretically.

Namely, let the finite set E and the collection $\mathcal{P}$ be given as in § 2. $\mathcal{P}$ is called *indecomposable by superposition* if to any non empty subset $F (\subset E)$ there exist two members P_1, P_2 of $\mathcal{P}$ such that $(P_1 \cap F) \neq \emptyset$, $P_2 \cap F \neq \emptyset$, and the union $(P_1 \cap F) \cup (P_2 - F)$ is not a member of $\mathcal{P}$.[12]

[12] Concerning the equivalence of this property and the original definition of indecomposability, see Theorem 2 in [2] (p. 51) and Corollary in [4] (p. 36).

The collection $\mathscr{P}$ is called *realizable by a 2-graph* if there exist such subsets $V_P, V_Q, V_1, V_2, \ldots, V_k$ of E such that

α) any element of E occurs in exactly two of $V_P, V_Q, V_1, \ldots, V_k$ and

β) the following two properties are equivalent for a subset F of E:

1°) $F \in \mathscr{P}$

2°) each of the intersections $F \cap V_P$ and $F \cap V_Q$ consists of one element, and each of the intersections $F \cap V_1, \ldots, F \cap V_k$ consists of 0 or 2 elements, and no proper subset of F satisfies the statements exposed above in 2°).

Now TRACHTENBROT's theorem can be expressed in the following manner: if $\mathscr{P}$ is indecomposable by superposition and realizable by a 2-graph, then the sets $V_1, V_2, \ldots, V_k$ (are determined uniquely and) coincide with the W-sets.

4

We say that the truth function

$$g(y_1, y_2, \ldots, y_{n+r})$$

originates by *r-fold distribution* from the function

$$f(x_1, x_2, \ldots, x_n)$$

if there exists a partition of the set $\{y_1, y_2, \ldots, y_{n+r}\}$ such that

the number of the classes in n, and

in that case when we start with g and we put the variables equal to each other in any class, then the resulting function g' coincides with f essentially (i.e. there exists a one-to-one mapping of the set of the variables of g' onto the set of the variables of f such that the equality of all pairs of corresponding variables implies the equality of the values of g' and f).

For the sake of simplicity, we shall consider only such truth functions which depend monotonic increasingly from all the variables, and only such realizations in which always non-negated variables correspond to the edges of the of the realizing graphs.[13]

Let us consider a truth function f. The question of determining an optimal realization of f (by a 2-graph) can be expressed in the following manner: let us find a truth function g such that

g admits a repetition-free realization, and

g originates by r-fold distribution from f where the number r has the property that all the functions originating by 0-fold, 1-fold, $\ldots$, $(r-1)$-fold distribution from f are non-realizable (in the repetition-free sense).

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[13] Our idea could be explained also without these restrictions, but its formulation would be more complicated in the full generality.

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О БЕСПОВТОРНОЙ РЕАЛИЗАЦИИ ФУНКЦИЙ ИСТИННОСТИ СПОСОБОМ ДВУХПОЛЮСНЫХ ГРАФОВ, I

А. ÁDÁM

Разработан алгорифм, пригодный для машинного выполнения, для определения внутренних звезд на основе теоремы 1 цитированной работы ТРАХТЕНБРОТА, исходя из реализуемой функции, которая неразложима относительно суперпозиций.