GENERALIZATION OF A RESULT OF ACZÉL, GHERMĂNESCU AND HOSSZÚ

by

D. Ž. DJOKOVIĆ[1]

1. Introduction

J. ACZÉL, M. GHERMĂNESCU and M. HOSSZÚ in their paper [1] have considered the basic cyclic functional equation

$$F(x_1, x_2, \ldots, x_n) + F(x_2, x_3, \ldots, x_n, x_1) + \ldots + F(x_n, x_1, \ldots, x_{n-1}) = 0 \tag{1.1}$$

and the derived equation

$$\begin{aligned} &F(x_1, x_2, \ldots, x_p) + F(x_2, x_3, \ldots, x_{p+1}) + \ldots \\ &+ F(x_{n-p+1}, x_{n-p+2}, \ldots, x_n) + F(x_{n-p+2}, x_{n-p+3}, \ldots, x_n, x_1) + \ldots \\ &+ F(x_n, x_1, \ldots, x_{p-1}) = 0 , \end{aligned} \tag{1.2}$$

where p and n $(> p)$ are two arbitrary positive integers.

In the mentioned paper they have formulated three theorems, the second being:

The most general solution of (1.2) *in the case when* $n \geqq 2p - 1$ *is given by*

$$F(x_1, x_2, \ldots, x_p) = f(x_1, x_2, \ldots, x_{p-1}) - f(x_2, x_3, \ldots, x_p) \tag{1.3}$$

where f *is an arbitrary function.*

This theorem (and two others) are proved under the following assumptions:

1° $x_i \in S$, where S is an arbitrary non-empty set;

2° The values of the function F lie in an additive abelian group $\mathbf{M}$;

3° The group $\mathbf{M}$ is such that the equation $mX = A$ $(X, A \in \mathbf{M})$ has a unique solution $X = A/m$ for every $m \leqq n$ $(m \in N)$.

In the proof of first and second theorems it is sufficient to take $m = n$ in 3°.

The idea of the proof of the third theorem, concerning equation (1.2) in case $p < n < 2p - 1$, is shown in [1] for two particular values of n and p. For details of the proof in the general case and for similar equations of. M. HOSSZÚ [2] (also there the same assumptions 1°, 2° and 3° are made).

In the second paragraph of this paper we shall solve the equation (1.2) for $n \geqq 2p - 1$, under the assumptions 1° and 2° only. In the third paragraph we shall solve the generalized equation (1.2) when all functions are taken to be different. Finally, we shall investigate the equation (1.2) with 1° and 2° in the case $p < n < 2p - 1$ for some special values of difference $2p - 1 - n$.

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[1] Belgrade.

2. Equation (1.2) for $n \geqq 2p - 1$

We have the following

Theorem 1. *If* $n \geqq 2p - 1$, *then the general solution of the functional equation* (1.2), *under the hypotesis* 1° *and* 2°, *is given by*

$$F(x_1, x_2, \ldots, x_p) = f(x_1, x_2, \ldots, x_{p-1}) - f(x_2, x_3, \ldots, x_p) + A\,, \tag{2.1}$$

where f is an arbitrary function and A an arbitrary element of **M** *such that* $nA = 0$.

Proof. Direct calculation shows that every function F of the form (2.1) satisfies the functional equation (1.2). We have to prove the converse, i.e. that it follows from (1.2) that F has the form (2.1).

Let c be a fixed element from S. Assuming that $n \geqq 2p - 1$ and putting $x_{p+1} = x_{p+2} = \ldots = x_n = c$ in (1.2), we obtain

$$\begin{aligned} &F(x_1, x_2, \ldots, x_p) + F(x_2, x_3, \ldots, x_p, c) + \ldots + \\ &+ F(x_p, c, c, \ldots, c) + (n - 2p + 1)\, F(c, c, \ldots, c) + \\ &+ F(c, c, \ldots, c, x_1) + F(c, c, \ldots, c, x_1, x_2) + \ldots + \\ &+ F(c, x_1, x_2, \ldots, x_{p-1}) = 0. \end{aligned} \tag{2.2}$$

If we substitute $x_p = c$ in the last equation, we get

$$\begin{aligned} &F(x_1, x_2, \ldots, x_{p-1}, c) + F(x_2, x_3, \ldots, x_{p-1}, c, c) + \ldots + \\ &+ F(x_{p-1}, c, c, \ldots, c) + (n - 2p + 2)\, F(c, c, \ldots, c) + \\ &+ F(c, c, \ldots, c, x_1) + F(c, c, \ldots, c, x_1, x_2) + \ldots + \\ &+ F(c, x_1, x_2, \ldots, x_{p-1}) = 0. \end{aligned} \tag{2.3}$$

Subtracting (2.3) from (2.2), we find

$$\begin{aligned} F(x_1, x_2, \ldots x_p) = {} & F(x_1, x_2, \ldots, x_{p-1}, c) - F(x_2, x_3, \ldots, x_p, c) + \\ &+ F(x_2, x_3, \ldots, x_{p-1}, c, c) - F(x_3, x_4, \ldots, x_p, c, c) + \ldots + \\ &+ F(x_{p-1}, c, c, \ldots, c) - F(x_p, c, c, \ldots, c) + F(c, c, \ldots, c). \end{aligned} \tag{2.4}$$

Putting

$$\begin{aligned} f(x_1, x_2, \ldots, x_{p-1}) = {} & F(x_1, x_2, \ldots, x_{p-1}, c) + F(x_2, x_3, \ldots, x_{p-1}, c, c) + \\ &+ \ldots + F(x_{p-1}, c, c, \ldots, c)\,, \end{aligned} \tag{2.5}$$

$$A = F(c, c, \ldots, c)\,, \tag{2.6}$$

the equality (2.4) takes on the form (2.1). For $x_1 = x_2 = \ldots = x_n = c$, the equation (1.2) gives $nA = 0$.

Thus the theorem is proved.

3. Generalization

Let us consider following functional equation

$$\begin{aligned} &F_1(x_1, x_2, \ldots, x_p) + F_2(x_2, x_3, \ldots, x_{p+1}) + \ldots + \\ &+ F_{n-p+1}(x_{n-p+1}, x_{n-p+2}, \ldots, x_n) + F_{n-p+2}(x_{n-p+2}, x_{n-p+3}, \ldots, x_n, x_1) + \\ &+ \ldots + F_n(x_n, x_1, \ldots, x_{p-1}) = 0 \qquad (p < n), \end{aligned} \tag{3.1}$$

under the hypothesis 1° and 2° of paragraph 1 (the hypothesis 2° holds for every function F_i).

Theorem 2. *The general solution of the functional equation* (3.1) *in the case* $n \geq 2p - 1$ *is given by*

$$F_i(x_1, x_2, \ldots, x_p) = \tag{3.2}$$
$$= f_i(x_1, x_2, \ldots, x_{p-1}) - f_{i+1}(x_2, x_3, \ldots, x_p) \quad (i = 1, 2, \ldots, n-1),$$
$$F_n(x_1, x_2, \ldots, x_p) =$$
$$= f_n(x_1, x_2, \ldots, x_{p-1}) - f_1(x_2, x_3, \ldots, x_p),$$

where f_i $(i = 1, 2, \ldots, n)$ *are arbitrary functions defined on* S *with values in* $\mathbf{M}$.

Proof. Using the conventions $F_i \equiv F_{i+n}$, $x_i \equiv x_{i+n}$, the equation (3.1) can be written in the form

$$\begin{aligned} &F_i(x_i, x_{i+1}, \ldots, x_{i+p-1}) + F_{i+1}(x_{i+1}, x_{i+2}, \ldots, x_{i+p}) + \ldots + \\ &+ F_{i+n-p}(x_{i+n-p}, x_{i+n-p+1}, \ldots, x_{i+n-1}) + \\ &+ F_{i+n-p+1}(x_{i+n-p+1}, x_{i+n-p+2}, \ldots, x_{i+n-1}, x_i) + \ldots + \\ &+ F_{i+n-1}(x_{i+n-1}, x_i, \ldots, x_{i+p-2}) = 0. \end{aligned} \tag{3.3}$$

Putting $x_{i+p} = x_{i+p+1} = \ldots = x_{i+n-1} = c$ in (3.3), where c is a fixed element of S, we obtain

$$\begin{aligned} &F_i(x_i, x_{i+1}, \ldots, x_{i+p-1}) + F_{i+1}(x_{i+1}, x_{i+2}, \ldots, x_{i+p-1}, c) + \\ &+ F_{i+2}(x_{i+2}, x_{i+3}, \ldots, x_{i+p-1}, c, c) + \ldots + \\ &+ F_{i+p-1}(x_{i+p-1}, c, c, \ldots, c) + F_{i+p}(c, c, \ldots, c) + \\ &+ F_{i+p+1}(c, c, \ldots, c) + \ldots + F_{i+n-p}(c, c, \ldots, c) + \\ &+ F_{i+n-p+1}(c, c, \ldots, c, x_i) + F_{i+n-p+2}(c, c, \ldots, c, x_i, x_{i+1}) + \\ &+ \ldots + F_{i+n-1}(c, x_i, x_{i+1}, \ldots, x_{i+p-2}) = 0. \end{aligned} \tag{3.4}$$

The substitution $x_{i+p-1} = c$ in (3.4) gives

$$\begin{aligned} &F_i(x_i, x_{i+1}, \ldots, x_{i+p-2}, c) + F_{i+1}(x_{i+1}, x_{i+2}, \ldots, x_{i+p-2}, c, c) + \ldots + \\ &+ F_{i+p-2}(x_{i+p-2}, c, c, \ldots, c) + F_{i+p-1}(c, c, \ldots, c) + \\ &+ F_{i+p}(c, c, \ldots, c) + \ldots + F_{i+n-p}(c, c, \ldots, c) + \\ &+ F_{i+n-p+1}(c, c, \ldots, c, x_i) + F_{i+n-p+2}(c, c, \ldots, c, x_i, x_{i+1}) + \\ &+ \ldots + F_{i+n-1}(c, x_i, x_{i+1}, \ldots, x_{i+p-2}) = 0. \end{aligned} \tag{3.5}$$

Subtracting (3.5) from (3.4), we get the formula

$$\begin{aligned} &F_i(x_i, x_{i+1}, \ldots, x_{i+p-1}) = \\ &= F_i(x_i, x_{i+1}, \ldots, x_{i+p-2}, c) - F_{i+1}(x_{i+1}, x_{i+2}, \ldots, x_{i+p-1}, c) + \\ &+ F_{i+1}(x_{i+1}, x_{i+2}, \ldots, x_{i+p-2}, c, c) - F_{i+2}(x_{i+2}, x_{i+3}, \ldots, x_{i+p-1}, c, c) + \ldots + \\ &+ F_{i+p-2}(x_{i+p-2}, c, c, \ldots, c) - F_{i+p-1}(x_{i+p-1}, c, c, \ldots, c) + \\ &+ F_{i+p-1}(c, c, \ldots, c) \end{aligned} \tag{3.6}$$

which holds for every $i = 1, 2, \ldots, n$.

Let us put now

$$g_i(x_1, x_2, \ldots, x_{p-1}) = F_i(x_1, x_2, \ldots, x_{p-1}, c) + \tag{3.7}$$
$$+ F_{i+1}(x_2, x_3, \ldots, x_{p-1}, c, c) + \ldots +$$
$$+ F_{i+p-2}(x_{p-1}, c, c, \ldots, c),$$

$$A_i = F_{i+p-1}(c, c, \ldots, c). \tag{3.8}$$

Since $F_i \equiv F_{i+n}$, we conclude that $g_i \equiv g_{i+n}$. Putting in (3.1) $x_1 = x_2 = \ldots = x_n = c$, we find

$$A_1 + A_2 + \ldots + A_n = 0. \tag{3.9}$$

According to (3.7) and (3.8), the formula (3.6) can be written in the form

$$F_i(x_1, x_2, \ldots, x_p) = g_i(x_1, x_2, \ldots, x_{p-1}) - g_{i+1}(x_2, x_3, \ldots, x_p) + A_i \tag{3.10}$$
$$(i = 1, 2, \ldots, n).$$

Finally, with the notations

$$f_1 = g_1,$$
$$f_2 = g_2 - A_1,$$
$$f_3 = g_3 - A_1 - A_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_n = g_n - A_1 - A_2 - \ldots - A_{n-1},$$

(3.10) can be written in the form

$$F_i(x_1, x_2, \ldots, x_p) = f_i(x_1, x_2, \ldots, x_{p-1}) - f_{i+1}(x_2, x_3, \ldots, x_p) \tag{3.11}$$
$$(i = 1, 2, \ldots, n;\ f_1 = f_{n+1}).$$

Since the form (3.11) is identical with the form (3.2) we have proved that (3.2) is the consequence of (3.1). Conversely, the direct calculation shows that the function (3.2) satisfies the equation (3.1) for arbitrary f_i.

The proof of Theorem 2 is finished.

4. Equation (1.2) for $p < n < 2p - 1$

The case $p < n < 2p - 1$ of equation (1.2) is much more difficult. For that case we have proved the following three theorems.

Theorem 3. *If $n = 2p - 2 > p$ and if we admit the hypotheses 1°, 2° and 3° with $m = 2$, then the general solution of* (1.2) *is*

$$F(x_1, x_2, \ldots, x_p) = G_0(x_1, x_2, \ldots, x_{p-1}) - G_0(x_2, x_3, \ldots, x_p) + \tag{4.1}$$
$$+ G_1(x_1, x_p) - G_1(x_p, x_1) + A \qquad (nA = 0).$$

Theorem 4. *If $n = 2p - 3 > p$ and if we admit only the hypotheses 1° and 2°, the general solution of* (1.2) *is*

$$F(x_1, x_2, \ldots, x_p) = G_0(x_1, x_2, \ldots, x_{p-1}) - G_0(x_2, x_3, \ldots, x_p) + \\ + G_1(x_1, x_{p-1}, x_p) - G_1(x_p, x_1, x_2) + A \qquad (nA = 0). \tag{4.2}$$

Theorem 5. *If $n = 2p - 4 > p$ and if we admit the hypotheses 1°, 2° and 3° with $m = 2$, then the general solution of* (1.2) *is*

$$F(x_1, x_2, \ldots, x_p) = G_0(x_1, x_2, \ldots, x_{p-1}) - G_0(x_2, x_3, \ldots, x_p) + \\ + G_1(x_1, x_{p-2}, x_{p-1}, x_p) - G_1(x_p, x_1, x_2, x_3) + \\ + G_2(x_1, x_2, x_{p-1}, x_p) - G_2(x_{p-1}, x_p, x_1, x_2) + A \ (nA=0) \tag{4.3}$$

In formulas (4.1), (4.2) and (4.3) G_0, G_1 and G_2 are arbitrary functions.

These theorems suggest that the following conjecture is true:

If $p < n < 2p - 1$ and n is odd then under the hypoteses 1° and 2° the general solution of (1.2) *is*

$$F(x_1, x_2, \ldots, x_p) = A + G_0(x_1, x_2, \ldots, x_{p-1}) - G_0(x_2, x_3, \ldots, x_p) + \\ + \sum_{k=1}^{p-(n+1)/2} \{G_k(x_1, x_2, \ldots, x_k, x_{n-p+k+1}, \ldots, x_{p-1}, x_p) - \\ - G_k(x_{p-k+1}, \ldots, x_p, x_1, \ldots, x_{2p-n+k})\} \qquad (nA = 0). \tag{4.4}$$

If $p < n < 2p - 1$ and n is even then under the hypotheses 1°, 2° and 3° with $m = 2$ the general solution of (1.2) *is*

$$F(x_1, x_2, \ldots, x_p) = A + G_0(x_1, x_2, \ldots, x_{p-1}) - G_0(x_2, x_3, \ldots, x_p) + \\ + \sum_{k=1}^{p-n/2} \{G_k(x_1, x_2, \ldots, x_k, x_{n-p+k+1}, \ldots, x_{p-1}, x_p) - \\ - G_k(x_{p-k+1}, \ldots, x_p, x_1, \ldots, x_{2p-n-k})\} \qquad (nA = 0) \tag{4.5}$$

In the following two paragraphs we shall prove Theorems 3 and 4. We omit the proof of Theorem 5.

5. Proof of Theorem 3

Using the same procedure as in section 2, we find for this case

$$F(x_1, x_2, \ldots, x_p) = [F(x_1, x_2, \ldots, x_{p-1}, c) - F(x_2, x_3, \ldots, x_p, c) + \\ + F(x_2, x_3, \ldots, x_{p-1}, c, c) - F(x_3, x_4, \ldots, x_p, c, c) + \\ + \ldots + \\ + F(x_{p-2}, x_{p-1}, c, c, \ldots, c) - F(x_{p-1}, x_p, c, c, \ldots, c)] + \\ + F(x_{p-1}, c, c, \ldots, c) - F(x_p, c, c, \ldots, c, x_1) + \\ + F(c, c, \ldots, c, x_1). \tag{5.1}$$

The square bracket has the form (4.1). In order to reduce the remaining terms to this form, we put in (1.2)

$$\text{(i)}\ x_2 = x_3 = \ldots = x_{p-1} = x_{p+1} = x_{p+2} = \ldots = x_n = c;$$

$$\text{(ii)}\ x_2 = x_3 = \ldots = x_{p-1} = x_p = \ldots = x_n = c;$$

$$\text{(iii)}\ x_1 = x_2 = \ldots = x_{p-1} = x_{p+1} = x_{p+2} = \ldots = x_n = c,$$

and thus we obtain

$$\begin{aligned} &F(x_1, c, c, \ldots, c, x_p) + F(c, c, \ldots, c, x_p, c) + F(c, c, \ldots, c, x_p, c, c) + \ldots + \\ &+ F(c, x_p, c, c, \ldots, c) + F(x_p, c, c, \ldots, c, x_1) + F(c, c, \ldots, c, x_1, c) + \\ &+ F(c, c, \ldots, c, x_1, c, c) + \ldots + F(c, x_1, c, c, \ldots, c) = 0, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &F(x_1, c, c, \ldots, c) + F(c, x_1, c, c, \ldots, c) + \ldots + F(c, c, \ldots, c, x_1) + \\ &+ (p-2)\, F(c, c, \ldots, c) = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &F(x_p, c, c, \ldots, c) + F(c, x_p, c, c, \ldots, c) + \ldots + F(c, c, \ldots, c, x_p) + \\ &+ (p-2)\, F(c, c, \ldots, c) = 0. \end{aligned} \tag{5.4}$$

Subtracting (5.3) and (5.4) from (5.2) we obtain

$$\begin{aligned} 0 = {} & F(x_1, c, c, \ldots, c, x_p) + F(x_p, c, c, \ldots, c, x_1) - F(x_1, c, c, \ldots, c) - \\ & - F(c, c, \ldots, c, x_1) - F(x_p, c, c, \ldots, c) - F(c, c, \ldots, c, x_p) + \\ & + 2\, F(c, c, \ldots, c), \end{aligned} \tag{5.5}$$

since $nF(c, c, \ldots, c) = 0$. From (5.1) and (5.5) we get

$$\begin{aligned} 2F(x_1, x_2, \ldots, x_p) = {} & 2[F(x_1, x_2, \ldots, x_{p-1}, c) - F(x_2, x_3, \ldots, x_p, c) + \\ & + F(x_2, x_3, \ldots, x_{p-1}, c, c) - F(x_3, x_4, \ldots, x_p, c, c) + \\ & + \ldots + \\ & + F(x_{p-1}, c, c, \ldots, c) - F(x_p, c, c, \ldots, c)] + \\ & + [F(x_1, c, c, \ldots, c, x_p) - F(x_p, c, c, \ldots, c, x_1) + \\ & + F(c, c, \ldots, c, x_1) - F(c, c, \ldots, c, x_p) + \\ & + F(x_p, c, c, \ldots, c) - F(x_1, c, c, \ldots, c)] + \\ & + 2F(c, c, \ldots, c). \end{aligned}$$

Dividing by two, we find that F has the form (4.1).

It can be shown by simple calculation that (4.1) satisfies (1.2).

6. Proof of Theorem 4

By the same argumentation as in section 2 we obtain

$$\begin{aligned} &F(x_1, x_2, \ldots, x_p) = \\ &= [F(x_1, x_2, \ldots, x_{p-1}, c) - F(x_2, x_3, \ldots, x_p, c) + \\ &+ F(x_2, x_3, \ldots, x_{p-1}, c, c) - F(x_3, x_4, \ldots, x_p, c, c) + \end{aligned}$$

$$
\begin{aligned}
(6.1) \qquad & + \dots + \\
& + F(x_{p-3}, x_{p-2}, x_{p-1}, c, c, \dots, c) - F(x_{p-2}, x_{p-1}, x_p, c, c, \dots, c)] + \\
& + F(x_{p-2}, x_{p-1}, c, c, \dots, c) - F(x_{p-1}, x_p, c, c, \dots, c, x_1) + \\
& + F(x_{p-1}, c, c, \dots, c, x_1) - F(x_p, c, c, \dots, c, x_1, x_2) + \\
& + F(c, c, \dots, c, x_1, x_2).
\end{aligned}
$$

The square bracket is of the form (4.2). Let us substitute in (1.2)

(i) $x_3 = x_4 = \dots = x_{p-1} = x_{p+1} = x_{p+2} = \dots = x_n = c$;

(ii) $x_3 = x_4 = \dots = x_n = c$;

(iii) $x_1 = x_2 = \dots = x_{p-1} = x_{p+1} = x_{p+2} = \dots = x_n = c$.

then we obtain

$$
\begin{aligned}
& F(x_1, x_2, c, c, \dots, c, x_p) + F(x_2, c, c, \dots, c, x_p, c) + F(c, c, \dots, c, x_p, c, c) + \\
& \quad + F(c, c, \dots, c, x_p, c, c, c) + \dots + F(c, c, x_p, c, c, \dots, c) + \\
(6.2) \qquad & \quad + F(c, x_p, c, c, \dots, c, x_1) + F(x_p, c, c, \dots, c, x_1, x_2) + \\
& \quad + F(c, c, \dots, c, x_1, x_2, c) + F(c, c, \dots, c, x_1, x_2, c, c) + \dots + \\
& \quad + F(c, x_1, x_2, c, c, \dots, c) = 0 ,
\end{aligned}
$$

$$
\begin{aligned}
& F(x_1, x_2, c, c, \dots, c) + F(x_2, c, c, \dots, c) + (p - 4) F(c, c, \dots, c) + \\
(6.3) \qquad & + F(c, c, \dots, c, x_1) + F(c, c, \dots, c, x_1, x_2) + F(c, c, \dots, c, x_1, x_2, c) + \\
& + F(c, c, \dots, c, x_1, x_2, c, c) + \dots + F(c, x_1, x_2, c, c, \dots, c) = 0 ,
\end{aligned}
$$

$$
\begin{aligned}
(6.4) \qquad & F(x_p, c, c, \dots, c) + F(c, x_p, c, c, \dots, c) + \dots + F(c, c, \dots, c, x_p) + \\
& \qquad + (p - 3) F(c, c, \dots, c) = 0.
\end{aligned}
$$

Subtracting (6.3) and (6.4) from (6.2) we find

$$
\begin{aligned}
0 = {} & F(x_1, x_2, c, c, \dots, c, x_p) + F(x_p, c, c, \dots, c, x_1, x_2) + F(x_2, c, c, \dots, c, x_p, c) + \\
& + F(c, x_p, c, c, \dots, c, x_1) - F(x_1, x_2, c, c, \dots, c) - \\
(6.5) \qquad & - F(x_2, c, c, \dots, c) - F(c, c, \dots, c, x_1) - F(c, c, \dots, c, x_1, x_2) - \\
& - F(x_p, c, c, \dots, c) - F(c, x_p, c, c, \dots, c) - F(c, c, \dots, c, x_p, c) - \\
& - F(c, c, \dots, c, x_p) + 4F(c, c, \dots, c)
\end{aligned}
$$

since $nF(c, c, \dots, c) = 0$. Substituting

$$x_2 = x_3 = \dots = x_{p-2} = x_p = x_{p+1} = \dots = x_n = c$$

in (1.2) gives that

$$
\begin{aligned}
& F(x_1, c, c, \dots, c, x_{p-1}, c) + F(c, c, \dots, c, x_{p-1}, c, c) + \\
& + F(c, c, \dots, c, x_{p-1}, c, c, c) + \dots + F(c, x_{p-1}, c, c, \dots, c) +
\end{aligned}
$$

$$(6.6)\qquad + F(x_{p-1}, c, c, \ldots, c, x_1) + F(c, c, \ldots, c, x_1, c) + \\ + F(c, c, \ldots, c, x_1, c, c) + \ldots + F(c, c, x_1, c, c, \ldots, c) + \\ + F(c, x_1, c, c, \ldots, c, x_{p-1}) = 0.$$

Putting $x_{p-1} = c$ resp. $x_1 = c$ into (6.6) we get

$$(6.7)\qquad F(x_1, c, c, \ldots, c) + F(c, x_1, c, c, \ldots, c) + \ldots + F(c, c, \ldots, c, x_1) + \\ + (p-3)\, F(c, c, \ldots, c) = 0,$$

$$(6.8)\qquad F(x_{p-1}, c, c, \ldots, c) + F(c, x_{p-1}, c, c, \ldots, c) + \ldots + F(c, c, \ldots, c, x_{p-1}) + \\ + (p-3)\, F(c, c, \ldots, c) = 0.$$

Taking into account, (6.6), (6.7) and (6.8), we can write

$$(6.9)\qquad 0 = -F(x_1, c, c, \ldots, c, x_{p-1}, c) - F(x_{p-1}, c, c, \ldots, c, x_1) - \\ - F(c, x_1, c, c, \ldots, c, x_{p-1}) + F(c, c, \ldots, c, x_1) + \\ + F(c, x_1, c, c, \ldots, c) + F(x_1, c, c, \ldots, c) + \\ + F(c, c, \ldots, c, x_{p-1}) + F(c, c, \ldots, c, x_{p-1}, c) + F(x_{p-1}, c, c, \ldots, c) - \\ - 3\, F(c, c, \ldots, c).$$

By addition of (6.1), (6.5) and (6.9) we obtain

$$(6.10)\qquad F(x_1, x_2, \ldots, x_p) = \\ = F(x_1, x_2, \ldots, x_{p-1}, c) - F(x_2, x_3, \ldots, x_p, c) + \\ + F(x_2, x_3, \ldots, x_{p-1}, c, c) - F(x_3, x_4, \ldots, x_p, c, c) + \\ + \ldots + \\ + F(x_{p-1}, c, c, \ldots, c) - F(x_p, c, c, \ldots, c)] + \\ + [F(x_1, x_2, c, c, \ldots, c, x_p) - F(x_{p-1}, x_p, c, c, \ldots, c, x_1) + \\ + F(x_{p-1}, x_p, c, c, \ldots, c) - F(x_1, x_2, c, c, \ldots, c) + \\ + F(c, x_p, c, c, \ldots, c, x_1) - F(c, x_1, c, c, \ldots, c, x_{p-1}) + \\ + F(c, x_1, c, c, \ldots, c) - F(c, x_p, c, c, \ldots, c)] + \\ + [F(x_2, c, c, \ldots, c, x_p, c) - F(x_1, c, c, \ldots, c, x_{p-1}, c) + \\ + F(x_1, c, c, \ldots, c) - F(x_2, c, c, \ldots, c) + \\ + F(c, c, \ldots, c, x_{p-1}, c) - F(c, c, \ldots, x_p, c) + \\ + F(c, c, \ldots, c, x_{p-1}) - F(c, c, \ldots, c, x_p)] + \\ + F(c, c, \ldots, c).$$

Finally we put

$$G_0(x_1, x_2, \ldots, x_{p-1}) = F(x_1, x_2, \ldots, x_{p-1}, c) + F(x_2, x_3, \ldots, x_{p-1}, c, c) + \\ + \ldots + F(x_{p-1}, c, c, \ldots, c) - F(x_1, c, c, \ldots, c, x_{p-1}, c) + \\ + F(x_1, c, c, \ldots, c) + F(c, c, \ldots, c, x_{p-1}, c) + F(c, c, \ldots, c, x_{p-1}),$$

$$G_1(x_1, x_{p-1}, x_p) = - F(x_{p-1}, x_p, c, c, \ldots, c, x_1) + F(x_{p-1}, x_p, c, c, \ldots, c) -$$
$$- F(c, x_1, c, c, \ldots, c, x_{p-1}) + F(c, x_1, c, c, \ldots, c) ,$$
$$A = F(c, c, \ldots, c) ,$$

thus formula (6.10) reduces to (4.2).

On the other hand every function of the form (4.2) satisfies the equation (1.2) if $n = 2p - 3$, and thus Theorem 4 is proved.

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ОБОБЩЕНИЕ ОДНОГО РЕЗУЛЬТАТА ACZÉL, GERMĂNESCU и HOSSZÚ

D. Ž. DJOKOVIĆ

Резюме

Рассматриваются циклическое функциональное уравнение (1.2) и обобщенное уравнение (3.1). Предполагается, что независимые переменные $x_i \in S$, где S — произвольное непустое множество, и что значения функций принадлежат некоторой аддитивной абелевой группе **M**. Приведены общие решения этих уравнений для «лёгкого» случая $n \geqq 2p - 1$. В «тяжёлом» случае $p < n < 2p - 1$ рассматривается уравнение (1.2) только для $n = 2p - 2$ и $n = 2p - 3$. Что бы получить общее решение в случае $n = 2p - 2$ мы должны предположить выполнимость и единственность деления с 2 в группе **M**.

Общее решение функционального уравнения (1.2) получили раньше Aczél, Germănescu, Hosszú но при более сильных предположений об группе **M**.