

Detecting non-locality in multipartite quantum systems with two-body correlation functions

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Bell inequalities define experimentally observable quantities to detect non-locality. In general, they involve correlation functions of all the parties. Unfortunately, these measurements are hard to implement for systems consisting of many constituents, where only few-body correlation functions are accessible. Here we demonstrate that higher-order correlation functions are not necessary to certify nonlocality in multipartite quantum states by constructing Bell inequalities from one- and two-body correlation functions for an arbitrary number of parties. The obtained inequalities are violated by some of the Dicke states, which arise naturally in many-body physics as the ground states of the two-body Lipkin-Meshkov-Glick Hamiltonian.

Local measurements on entangled composite quantum systems may lead to correlations that cannot be simulated by any local deterministic strategy assisted by shared randomness [1, 2]. This phenomenon is known as nonlocality. Apart from its fundamental interest, non-locality has also turned into a key resource for certain information-theoretic tasks, such as key distribution [3] or certified quantum randomness generation [4]. Hence, revealing the nonlocality of a given composite quantum state, or, in other words, certifying that it can be used to generate nonlocal correlations upon local measurements, is one of the central problems of quantum information theory.

Due to the structure of the set of classical correlations (see below), the natural way of tackling this problem is to use Bell inequalities [1]. These are linear inequalities formulated in terms of expectation values (correlators) of tensor products of measurements performed by the observers, and their violation signals nonlocality. Many constructions of Bell inequalities have been proposed (see e.g. Refs. [5]), however, most of them involve full-order correlators, that is, expectation values of observables of all parties). Intuitively, the latter carry most of the information about correlations, and consequently Bell inequalities based on them are the strongest ones, or even tight (see, e.g., Refs. [6, 7]). But are these all-partite mean values necessary to reveal nonlocality? It was recently shown in Ref. [8, 9] that this is not the case, although expectation values with all but one parties are still involved. Hence, one is led to the more demanding question of whether certification of non-locality is possible from the minimal information achievable in this type of experiments, i.e. two-body expectation values.

This question also arises naturally in the context of experimental implementations of Bell tests, in particular, in multipartite systems. It should be stressed that several interesting multipartite states are already within reach of current experimental technology. In particular, four-qubit Smolin state [10], eight-qubit GHZ state [11], and various Dicke states [12, 13] were experimentally generated. However, in the case of large systems determining experimentally expectation values of high-order is a hard task. Designing nonlocality tests that rely solely on low-order correlators would facilitate their experimental implementation.

It should also be stressed that an analogous question was already explored in the case of entanglement, which next to

nonlocality is a key resource of quantum information theory [14]. Several entanglement criteria relying solely on two-body expectation values have been proposed [15]. In particular, in [16] the possibility of addressing two-body statistics (although not individually) *via* collective observables was exploited.

In this letter we address the above question and propose a class of Bell inequalities constructed only from one and two-body correlators that are violated by quantum states. We simplify the problem by considering a subclass of symmetric Bell inequalities, i.e., those that are invariant under a swap of any pair of parties and characterize the corresponding polytope of classical correlations. We also show that our inequalities are powerful enough to certify nonlocality of the Dicke states that are ground states of the two-body Lipkin-Meshkov-Glick Hamiltonian [17], making our results promising from the experimental point of view.

Preliminaries.—Let us consider the standard Bell-type experiment in which N spatially separated observers perform measurements on their shares of some N -partite composite quantum state ρ . In what follows we focus on the simplest case where each party freely chooses one between two dichotomic measurements, whose outcomes we denote ± 1 . A convenient way of describing the established correlations in the two-outcome case is to use the collection of expectation values (also called correlators)

$$\{\langle \mathcal{M}_{j_1}^{(i_1)} \dots \mathcal{M}_{j_k}^{(i_k)} \rangle \mid k = 1, \dots, N\} \quad (1)$$

with $i_l = 1, \dots, N$ and $j_l = 0, 1$ ($l = 1, \dots, k$). We will refer to these collections as to ordered real vectors of dimension $3^N - 1$, and by saying correlations we mean the corresponding vector. Also, the *order* of a correlator is the number of parties k it involves [cf Eq. (1)], and, in particular, those with $k = N$ we call the *highest-order* correlators, while those with $k = 2$ the lowest-order or two-body correlators.

Within this framework, we say that the correlations represented by (1) are *classical* (or *local*) whenever, even if obtained from composite quantum states, they can be simulated by the observers with some shared classical information as the only resource. Such correlations form a polytope \mathbb{P} , whose vertices are those collections (1) in which every correlator takes the product form $\langle \mathcal{M}_{j_1}^{(i_1)} \dots \mathcal{M}_{j_k}^{(i_k)} \rangle =$

$\langle \mathcal{M}_{j_1}^{(i_1)} \rangle \cdot \dots \cdot \langle \mathcal{M}_{j_k}^{(i_k)} \rangle$ with individual mean values $\langle \mathcal{M}_{j_i}^{(i_i)} \rangle$ being ± 1 .

Bell was the first to recognize that the set of classical correlations can be constrained by certain inequalities, referred to as Bell inequalities [1]. In fact, since classical correlations form a polytope, \mathbb{P} can be fully determined by a finite number of *tight* Bell inequalities, i.e., those corresponding to the facets of \mathbb{P} . Correlations that fall outside of \mathbb{P} are called non-local. Consequently, the problem of characterizing all classical correlations reduces to finding all tight Bell inequalities for a given scenario. And, even if it sounds simple, the problem is difficult to resolve as the number of facets of the local polytope grows rapidly with the number of parties.

Bell inequalities from one- and two-body correlators.—Most of the known constructions of multipartite Bell inequalities contain highest-order correlators, i.e., those with $k = N$ in Eq. (1). In the following, we will see that one can design Bell inequalities that witness nonlocality only from one and two-body expectation values. A general form of such a Bell inequality is

$$\sum_{i=1}^N (\alpha_i \langle \mathcal{M}_0^{(i)} \rangle + \beta_i \langle \mathcal{M}_1^{(i)} \rangle) + \sum_{i < j}^N \gamma_{ij} \langle \mathcal{M}_0^{(i)} \mathcal{M}_0^{(j)} \rangle + \sum_{i \neq j}^N \delta_{ij} \langle \mathcal{M}_0^{(i)} \mathcal{M}_1^{(j)} \rangle + \sum_{i < j}^N \varepsilon_{ij} \langle \mathcal{M}_1^{(i)} \mathcal{M}_1^{(j)} \rangle + \beta_C \geq 0, \quad (2)$$

where $\alpha_i, \beta_j, \gamma_{ij}, \delta_{ij}$, and ε_{ij} are some real parameters, while β_C is the so-called classical bound. The corresponding polytope \mathbb{P}_2 of classical correlations is one constructed from the elements of \mathbb{P} by neglecting correlators of order higher than two. In other words, we take all elements (vectors) of \mathbb{P} and simply remove those with $k \geq 3$ [cf. Eq. (1)]. Analogously, the vertices of \mathbb{P}_2 are those collections of correlators for which $\langle \mathcal{M}_k^{(i)} \mathcal{M}_l^{(j)} \rangle = \langle \mathcal{M}_k^{(i)} \rangle \cdot \langle \mathcal{M}_l^{(j)} \rangle$, while the individual mean values are ± 1 .

The characterization of \mathbb{P}_2 reduces to finding all its facets, i.e., *tight two-body Bell inequalities*. Although $\dim \mathbb{P}_2 = 2N^2$ is much smaller than the one of \mathbb{P} , $3^N - 1$, it still grows with N , thus difficulting the task of determining facets of \mathbb{P}_2 . One way to overcome this problem (and keep the dimension constant irrespectively of N) is to consider Bell inequalities that obey some symmetries. For instance, one could consider translationally invariant Bell inequalities consisting of correlators involving only nearest neighbours, or, in the spirit of Ref. [7], those that are invariant under any permutation of the parties. While we leave the first case for further studies, below we focus on the second case and construct symmetric Bell inequalities with one and two-body correlators.

By imposing the permutational symmetry, one requires that the expectation values $\langle \mathcal{M}_k^{(i)} \rangle$ and $\langle \mathcal{M}_k^{(i)} \mathcal{M}_l^{(j)} \rangle$, with fixed k, l and different i, j , appear in the Bell inequality (2) with the same “weights”, i.e., $\alpha_i = \alpha, \beta_i = \beta$, etc. This means that the general form of a symmetric Bell inequality with one-

and two-body correlators is

$$I := \alpha \mathcal{S}_0 + \beta \mathcal{S}_1 + \frac{\gamma}{2} \mathcal{S}_{00} + \delta \mathcal{S}_{01} + \frac{\varepsilon}{2} \mathcal{S}_{11} \geq -\beta_C, \quad (3)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are real parameters. Then, by \mathcal{S}_k and \mathcal{S}_{kl} with $k, l = 0, 1$ we denote the one- and two-body correlators symmetrized over all observers, i.e.,

$$\mathcal{S}_k = \sum_{i=1}^N \langle \mathcal{M}_k^{(i)} \rangle, \quad \mathcal{S}_{kl} = \sum_{i \neq j=1}^N \langle \mathcal{M}_k^{(i)} \mathcal{M}_l^{(j)} \rangle. \quad (4)$$

Geometrically, under this symmetry the polytope \mathbb{P}_2 is mapped to a simpler one \mathbb{P}_2^S , which, independently of N , is always five-dimensional and its elements are vectors $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{11})$. Accordingly, \mathbb{P}_2^S is fully characterized if one knows all its facets, which we call *tight symmetric two-body Bell inequalities*. Moreover, the number of vertices is significantly reduced from 2^{2N} of \mathbb{P}_2 to $2(N^2 + 1)$ of \mathbb{P}_2^S , and, as we will see below, vertices of the latter can be conveniently parameterized by three natural numbers (see appendix A for more details). Precisely, for a given local deterministic model, let us denote by a, b, c , and d the amount of parties whose local expectation values $\langle \mathcal{M}_k^{(i)} \rangle$ ($k = 0, 1$) are $\{1, 1\}$, $\{1, -1\}$, $\{-1, 1\}$, and $\{-1, -1\}$, respectively. By definition $a + b + c + d = N$, and therefore all vertices of \mathbb{P}_2 are mapped under the symmetry to four-tuples (a, b, c, d) forming a tetrahedron \mathbb{T}_N in \mathbb{N}^3 whose facets are determined by vanishing one of a, b, c , or d . One can then prove (see appendix A) that all vertices of \mathbb{P}_2^S are uniquely represented by those four-tuples that belong to the boundary $\partial \mathbb{T}_N$ of \mathbb{T}_N .

Then, for any local deterministic model the one-body symmetrized expectation values can be expressed within this parametrization as $\mathcal{S}_k = a + (-1)^k (b - c) - d$ with $k = 0, 1$. Moreover, since for any vertex of \mathbb{P}_2 it holds that $\mathcal{S}_{kl} = \mathcal{S}_k \mathcal{S}_l - \sum_{i=1}^N \langle \mathcal{M}_k^{(i)} \rangle \langle \mathcal{M}_l^{(i)} \rangle$ ($k, l = 0, 1$), the two-body expectation values are given by $\mathcal{S}_{ll} = \mathcal{S}_l^2 - N$, with $l = 0, 1$, and $\mathcal{S}_{01} = \mathcal{S}_0 \mathcal{S}_1 - (a - b - c + d)$. As a consequence, computing the classical bound of the Bell inequality (3) is equivalent to minimizing I being a function of a, b, c , and d over the boundary of \mathbb{T}_N , i.e., $\beta_C = -\min_{\partial \mathbb{T}_N} I$.

A class of symmetric two-body Bell inequalities.—Using the above characterization of the symmetric polytope of two-body local models, we can now search for particular Bell inequalities violated by multipartite quantum states. For sufficiently low number of parties all Bell inequalities corresponding to the facets of \mathbb{P}_2^S can be listed with the aid of a computer algorithm and they will be presented elsewhere [18]. Here we present a general class of few-parameter symmetric tight Bell inequalities and show that they reveal nonlocality in quantum states for any N .

To this end, in Eq. (3) we substitute $\gamma = x^2$ and $\varepsilon = y^2$ with x, y being positive natural numbers, and $\delta = \sigma xy$, where $\sigma = \pm 1$ stands for the sign of δ . Moreover, let $\alpha_{\pm} = x[\sigma \mu \pm (x + y)]$ with $\mu \equiv \beta/y$, and assume that μ is an integer with opposite parity to ε (γ) for odd N (even N). Exploiting

N	# Bell inequalities in the class	Total # of tight Bell inequalities
5	16	152
10	272	2018
15	1208	7744
20	3592	21274

TABLE I. The number of facets (second column) of \mathbb{P}_2^S that are grasped by our class of Bell inequalities for various numbers of parties N (first column). For comparison the third column contains the total number of facets of \mathbb{P}_2^S .

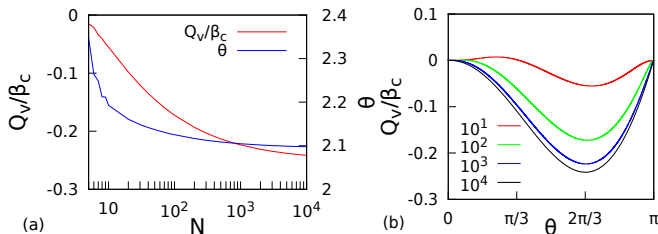


FIG. 1. (a) The effective (divided by the classical bound) maximal violation of Ineq. (6) (red line) and the corresponding angle θ in \mathcal{M}_1 (blue line) as functions of N . (b) Effective violation of Ineq. (6) as a function of θ for $N = 10^k$ with $k = 1, 2, 3, 4$. For large N the violation is robust against misalignments of the second observable.

the above parameterization one then proves that the classical bound of the resulting Bell inequality is (see appendix B)

$$\beta_C = \frac{1}{2}[N(x+y)^2 + (\sigma\mu \pm x)^2 - 1]. \quad (5)$$

Interestingly, under the additional assumption that x and y are coprimes, one can also analyze their tightness by hand (see Ref. [18] for more details), and it follows that this class contains a significant amount of facets of \mathbb{P}_2^S (see Table I). Specifically, this class contains those tight Bell inequalities that are tangent to \mathbb{P}_2^S at vertices belonging to a single facet of \mathbb{T}_N . A particular example of a Bell inequality of this form, arises from $x = y = -\sigma = 1$, and $\alpha_- = -2$. According to (5), $\beta_C = 2N$ and the resulting Bell inequality is

$$-2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} + 2N \geq 0. \quad (6)$$

To search for quantum violations of (6), we assume that all parties have the same pairs of observables, i.e., $\mathcal{M}_j^{(i)} = \mathcal{M}_j$ for every i . Without loss of optimality, these observables can be taken to be equal to $\mathcal{M}_0 = \sigma_z$ and $\mathcal{M}_1 = \cos\theta\sigma_z + \sin\theta\sigma_x$ for $\theta \in [0, \pi]$ [19]. Denoting then by $\mathcal{B}_N(\theta)$ the Bell operator constructed from the above observables, the Bell inequality (6) is violated if there is θ such that $\mathcal{B}_N(\theta) \not\geq 0$. We have numerically searched for the lowest negative eigenvalue of $\mathcal{B}_N(\theta)$ for various values of N and the obtained results are presented on Fig. 1. Clearly, the effective violation (divided by the classical bound) grows with N , and becomes more robust against misalignments of θ for large N . Also, the corresponding eigenstate of $\mathcal{B}_N(\theta)$, i.e., the state maximally

violating (6) is always symmetric (but there are also antisymmetric states violating this inequality), that is, it is invariant under permutation of any two particles. Consequently, since any (also mixed) N -qubit symmetric state is entangled if, and only if, it is genuinely multipartite entangled (see e.g. Refs. [20]), our Bell inequalities detect states that have genuine multipartite entanglement.

Nonlocality of physically relevant states.—As we have just demonstrated, the two-body multipartite Bell inequalities detect nonlocality of multipartite quantum states. But are these inequalities powerful enough to reveal nonlocality in “physically relevant” states, as for instance ground states of spin models that naturally appear in many-body physics? Here we show that this is the case by constructing a class of two-body symmetric Bell inequalities that are violated by the Dicke states [21]. These are N -qubit states spanning the $(N+1)$ -dimensional symmetric subspace of $(\mathbb{C}^2)^{\otimes N}$ and read

$$|D_N^k\rangle = \mathcal{S}(|\{0, N-k\}, \{1, k\}\rangle) \quad (k = 0, \dots, N), \quad (7)$$

where $|\{0, N-k\}, \{1, k\}\rangle$ is any pure product vector with $N-k$ qubits in the state $|0\rangle$ and k in the state $|1\rangle$, while \mathcal{S} denotes symmetrization over all parties. It is worth mentioning that $|D_N^k\rangle$ are genuinely multipartite entangled for any $k \neq 0, N$. Moreover, their entanglement properties have been extensively studied in the literature (see e.g. Refs. [22, 23] and references therein), and the state $|D_6^3\rangle$ was recently generated experimentally [12].

In many-body physics, the Dicke states arise naturally as the lowest-energy eigenstates of the isotropic Lipkin-Meshkov-Glick Hamiltonian [17]:

$$H = -\frac{\lambda}{N} \sum_{\substack{i,j=1 \\ i < j}}^N (\sigma_x^{(i)} \sigma_x^{(j)} + \sigma_y^{(i)} \sigma_y^{(j)}) - h \sum_{i=1}^N \sigma_z^{(i)}, \quad (8)$$

which describes N spins interacting through the two-body ferromagnetic coupling ($\lambda > 0$), embedded into the magnetic field acting along the z direction of strength $h \geq 0$. Again, $\sigma_a^{(i)}$ ($a = x, y, z$) are the Pauli matrices acting at site i .

In what follows we consider the case of weak magnetic field applied to the system, precisely $h \leq \lambda/N$. Then, the ground state of H is $|D^{N/2}\rangle$ for even N and $|D^{\lceil N/2 \rceil}\rangle$ for odd N , except for the case of $h = 0$ and odd N , for which the lowest energy is two-fold degenerate and the corresponding subspace is spanned by $|D_N^k\rangle$, with $k = \lfloor N/2 \rfloor$ and $k = \lceil N/2 \rceil$.

The class of tight two-body symmetric Bell inequalities that we use to detect nonlocality of the above Dicke states is obtained by taking $\alpha_N = N(N-1)(\lfloor N/2 \rfloor - N/2)$, $\beta_N = \alpha_N/N$, $\gamma_N = N(N-1)/2$, $\delta_N = N/2$, and $\varepsilon_N = -1$ in Eq. (3). This choice of parameters allows us to compute analytically the classical bounds of the resulting Bell inequalities for any N . Precisely, the minimization over $\partial\mathbb{T}_N$ gives

$$\beta_C(N) = \frac{1}{2}N(N-1) \left[\frac{N+2}{2} \right] \quad (9)$$

and also allows one to find five vertices at which these Bell inequalities are tangent to \mathbb{P}_2^S , ensuring their tightness. (cf. appendix C for the proof). It should be noticed that these Bell inequalities are independent of the class presented previously, and for $N = 2$ they reproduce the CHSH Bell inequality [24].

To prove that the resulting Bell inequalities are indeed violated by the Dicke states, let again $\mathcal{M}_j^{(i)}$ ($j = 0, 1$) be the qubit dichotomic observables at site i . Denoting by \mathcal{B}_N the resulting Bell operator, the direct way to reach the goal is to minimize the mean value $\langle D_N^k | \mathcal{B}_N | D_N^k \rangle$ with $k = \lfloor N/2 \rfloor, \lceil N/2 \rceil$. As before, we assume for simplicity that all observers measure the same pair of observables, i.e., $\mathcal{M}_j^{(i)} = \mathcal{M}_j$. This makes \mathcal{B}_N permutationally invariant, which together with the fact that the Dicke states are symmetric significantly simplifies the above problem. In fact, it follows that $\langle D_N^k | \mathcal{B}_N | D_N^k \rangle = \text{Tr}(\rho_N^k \tilde{\mathcal{B}}_N)$, where $\tilde{\mathcal{B}}_N$ stands for the two-qubit “reduced” Bell operator

$$\begin{aligned} \tilde{\mathcal{B}}_N &= \beta_C(N) \mathbb{1}_4 + \frac{N}{2} \alpha_N (\mathcal{M}_0 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \mathcal{M}_0) \\ &\quad + \frac{N(N-1)}{2} [\gamma_N \mathcal{M}_0 \otimes \mathcal{M}_0 + \varepsilon_N \mathcal{M}_1 \otimes \mathcal{M}_1 \\ &\quad \quad + \delta_N (\mathcal{M}_0 \otimes \mathcal{M}_1 + \mathcal{M}_1 \otimes \mathcal{M}_0)], \\ &\quad + \frac{N}{2} \beta_N (\mathcal{M}_1 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \mathcal{M}_1) \end{aligned} \quad (10)$$

with $\mathbb{1}_d$ being a $d \times d$ identity matrix, and ρ_N^k denotes any two-qubit subsystem of $|D_N^k\rangle$. Notice that the latter can be computed by hand for any N and k (see appendix C).

As before, let us finally set the measurements to $\mathcal{M}_0 = \sigma_z$ and $\mathcal{M}_1 = \cos \theta \sigma_z + \sin \theta \sigma_x$ with $\theta \in [0, \pi]$, and choose the particular Dicke state with $k = \lceil N/2 \rceil$ excitations, for any N . Then, $\langle D_N^k | \mathcal{B}_N | D_N^k \rangle$ can be computed to be $4 \lfloor N/2 \rfloor \sin^2(\theta/2) [(\lceil N/2 \rceil + 1) \sin^2(\theta/2) - 1]$, and the latter attains its minimum for

$$\theta_{\min}^N = \pm \arccos \left(\frac{\lfloor N/2 \rfloor}{\lceil N/2 \rceil + 1} \right), \quad (11)$$

resulting in the following quantum violations

$$\langle D_N^{\lceil N/2 \rceil} | \mathcal{B}_N | D_N^{\lceil N/2 \rceil} \rangle = - \frac{\lfloor N/2 \rfloor}{\lceil N/2 \rceil + 1}. \quad (12)$$

Similar values can be obtained for $k = \lfloor N/2 \rfloor$: it is enough to rename (flip) the outcomes of \mathcal{M}_j ($j = 0, 1$), because $|D_N^{\lfloor N/2 \rfloor}\rangle$ is obtained from $|D_N^{\lceil N/2 \rceil}\rangle$ by swapping the elements of the computational basis $\{|0\rangle, |1\rangle\}$.

To summarize, our Bell inequalities are violated by the Dicke states for any N , although the effective violation decays with N as $1/N^3$ (see Fig. 2). It should be stressed that, even though from the previous analysis one may conclude that this violation is purely bipartite, this is certainly not the case. The Dicke states are symmetric, and therefore any marginal bipartite correlations obtained from them in a Bell experiment with the same two dichotomic observables per site are local; otherwise all bipartite marginal correlations would be nonlocal, contradicting the fact that in this case quantum correlations are monogamous [25]. This also means that our results provide further examples, after [26], of local marginal bipartite

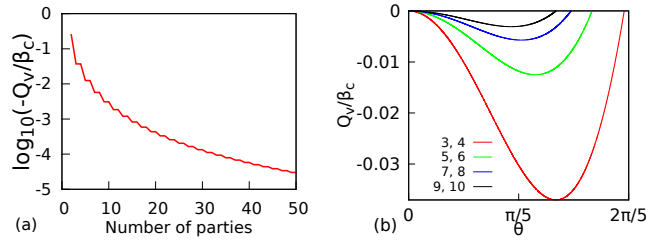


FIG. 2. (a) Effective violation of Ineq. (6) by the Dicke states $|D_N^{\lceil N/2 \rceil}\rangle$ as a function of N . The violation decays with N as $1/N^3$. (b) Effective violation as a function of θ for various values of N .

correlations that are only compatible with global multipartite nonlocal correlations (see also Ref. [18]).

Discussion and conclusion.—Bell inequalities allow detecting nonlocality in composite quantum states. They involve expectation values of products of local measurements performed by the observers. A natural question is about the minimal amount of knowledge (in terms of the size of correlators) that is needed to reveal nonlocality. Here we have demonstrated that multipartite two-body Bell inequalities are enough to witness nonlocality for an arbitrary number of parties.

Interestingly, our inequalities are violated by symmetric states, making our result feasible from the experimental point of view. First, such states appear naturally as ground states of models that can be realized with ultracold atoms or ions, such as Lipkin-Meshkov-Glick like models with long range interactions (for ionic spin 1/2 and spin 1 realizations see [28, 29], for cold atoms in nanophotonic waveguides see [30]), or degenerated ground states of the ferromagnetic Heisenberg model [31]. Second, in multipartite symmetric states one-body and two-body expectation values can be addressed via collective measurements of total spin operators $S_\alpha = (1/2) \sum_{i=1}^N \sigma_\alpha^{(i)}$ ($\alpha = x, y, z$) and simple second-order functions thereof, respectively (see e.g. Ref. [27]). We have shown, for instance, that the nonlocality of some of the half-filled Dicke states can be certified from only two correlators $\langle \sigma_z^{(1)} \sigma_z^{(2)} \rangle$ and $\langle \sigma_z^{(1)} \sigma_x^{(2)} \rangle$, which within the above framework can be expressed as $\langle \sigma_z^{(1)} \sigma_z^{(2)} \rangle = (4\langle S_z^2 \rangle - N)/N(N-1)$ and $\langle \sigma_z^{(1)} \sigma_x^{(2)} \rangle = ([S_z, S_x]_+)/N(N-1)$ with $[\cdot]_+$ denoting the anticommutator. Such measurements are routinely realized in atomic systems with current experimental technologies, such as spin polarization spectroscopy [32, 33].

Acknowledgments.—Discussions with J. Stasińska are greatly acknowledged. This work is supported by Spanish DIQIP CHIST-ERA, FIS2010-14830 projects and AP2009-1174 FPU PhD grant, EU IP SIQS, ERC AdG QUAGATUA and StG PERCENT. R. A. also acknowledges the Spanish MINECO for the Juan de la Cierva scholarship.

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APPENDIX

Appendix A: Characterization of vertices of \mathbb{P}_2^S

Let \mathbb{P}_2 be the polytope of all local models constructed from \mathbb{P} by forgetting the correlators of order higher than two, and let V be a set containing 2^N vertices of \mathbb{P}_2 . Recall that the latter are those local strategies for which $\langle \mathcal{M}_k^{(i)} \mathcal{M}_l^{(j)} \rangle = \langle \mathcal{M}_k^{(i)} \rangle \cdot \langle \mathcal{M}_l^{(j)} \rangle$ and $\langle \mathcal{M}_k^{(i)} \rangle = \pm 1$, for $i, j = 1, \dots, N$ and $k, l = 0, 1$. Moreover, by \mathbb{P}_2^S we denote the image of \mathbb{P}_2 under symmetrization, i.e., the set of five-dimensional vectors

$$(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{11}) \quad (13)$$

with \mathcal{S}_k and \mathcal{S}_{kl} ($k, l = 0, 1$) defined by

$$\mathcal{S}_k = \sum_{i=1}^N \langle \mathcal{M}_k^{(i)} \rangle, \quad \mathcal{S}_{kl} = \sum_{\substack{i,j=1 \\ i \neq j}}^N \langle \mathcal{M}_k^{(i)} \mathcal{M}_l^{(j)} \rangle, \quad (14)$$

that are computed for all elements of \mathbb{P}_2 . Analogously, by V_S we denote the set of vertices of \mathbb{P}_2^S . Our aim now is to characterize the elements of V_S and identify those vertices of \mathbb{P}_2 that are mapped onto the vertices of $\mathbb{P}_{2,S}$. In particular, we will demonstrate that $|V_S| = 2(N^2 + 1)$, which is a significantly smaller number than $|V| = 2^{2N}$.

To this end, for every element of V we denote by

$$x_i = \langle \mathcal{M}_0^{(i)} \rangle, \quad y_i = \langle \mathcal{M}_1^{(i)} \rangle, \quad (15)$$

the pair of local deterministic expectation values (those that assume values ± 1) for party i , and by $\{x_i, y_i\}$ the corresponding local strategy. Then, we notice that the values of \mathcal{S}_0 and \mathcal{S}_1 do not depend on particular local strategies applied by the parties but rather on their amount. This suggests introducing the following parametrization:

$$\begin{aligned} a &= \#\{i \in \{1, \dots, N\} \mid x_i = 1, y_i = 1\} \\ b &= \#\{i \in \{1, \dots, N\} \mid x_i = 1, y_i = -1\}, \\ c &= \#\{i \in \{1, \dots, N\} \mid x_i = -1, y_i = 1\}, \\ d &= \#\{i \in \{1, \dots, N\} \mid x_i = -1, y_i = -1\}. \end{aligned} \quad (16)$$

In other words, for a given vertex of \mathbb{P}_2 , a , b , c , and d stand for the number of parties who apply one of the four different local strategies $\{1, 1\}$, $\{1, -1\}$, $\{-1, 1\}$, or $\{-1, -1\}$, respectively. Clearly, $a + b + c + d = N$, and by means of these four numbers, the symmetrized local expectation values \mathcal{S}_k ($k = 0, 1$) can be expressed as

$$\mathcal{S}_0 = a + b - c - d, \quad \mathcal{S}_1 = a - b + c - d. \quad (17)$$

Furthermore, since for every element of V it holds that

$$\mathcal{S}_{xy} = \mathcal{S}_x \mathcal{S}_y - \sum_{i=1}^N \langle \mathcal{M}_x^{(i)} \rangle \langle \mathcal{M}_y^{(i)} \rangle \quad (x, y = 0, 1), \quad (18)$$

by using the parametrization (16) together with Eqs. (17), we can rewrite the two-body symmetrized expectation values as

$$\begin{aligned} \mathcal{S}_{00} &= \mathcal{S}_0^2 - N = (a + b - c - d)^2 - N, \\ \mathcal{S}_{11} &= \mathcal{S}_1^2 - N = (a - b + c - d)^2 - N, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \mathcal{S}_{01} &= \mathcal{S}_0 \mathcal{S}_1 - \sum_{i=1}^N \langle \mathcal{M}_x^{(i)} \rangle \langle \mathcal{M}_y^{(i)} \rangle \\ &= (a + b - c - d)(a - b + c - d) - (a - b - c + d). \end{aligned} \quad (20)$$

Thus, all vertices of \mathbb{P}_2 are mapped under symmetrization onto elements of \mathbb{P}_2^S that can later be parameterized by elements of the following set $\{(a, b, c, d) \in \mathbb{N}^4 \mid a + b + c + d = N\}$, which is isomorphic to a tetrahedron in \mathbb{N}^3

$$\mathbb{T}_N = \{(a, b, c) \in \mathbb{N}^3 \mid a + b + c \leq N\}. \quad (21)$$

In addition, the facets of \mathbb{T}_N contain those three-tuples (a, b, c) for which either $abc = 0$ or $a + b + c = N$ (equivalently $d = 0$). The cardinality of \mathbb{T}_N , which is basically the number of all possible choices of four natural numbers summing up to N , and it amounts to $(1/6)(N+1)(N+2)(N+3)$.

In what follows, we show that vertices of \mathbb{P}_2^S are uniquely represented by all those 4-tuples from \mathbb{T}_N that belong to its boundary $\partial\mathbb{T}_N$, i.e., those for which the condition $abcd = 0$ is satisfied.

Theorem 1. *Let us denote by $\varphi : \mathbb{T}_N \mapsto \mathbb{P}_2^S$ the above parametrization, i.e.,*

$$\varphi((a, b, c, d)) = (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{11}). \quad (22)$$

Then $\varphi(p)$ is a vertex of \mathbb{P}_2^S iff $p \in \partial\mathbb{T}_N$.

Proof. We start from the ‘‘only if’’ part. Assume on the contrary that $p = (a, b, c, d)$ belongs to the interior of \mathbb{T}_N , which means that all its components are larger than zero, (i.e., $a, b, c, d \geq 1$). Then, let us consider a vector $v = (1, -1, -1, 1) \notin \mathbb{T}_N$ and notice that the values of \mathcal{S}_k and \mathcal{S}_{kk} with $k = 0, 1$ are constant along the line $p + \lambda v$ for any $\lambda \in \mathbb{R}$, while $\mathcal{S}_{01}(p + \lambda v) = \mathcal{S}_{01}(p) - 4\lambda$. Hence, for any $\alpha, \beta > 0$,

$$\begin{aligned} &\alpha\varphi(p + \beta v) + \beta\varphi(p - \alpha v) \\ &= \alpha(\mathcal{S}_0(p), \mathcal{S}_1(p), \mathcal{S}_{00}(p), \mathcal{S}_{01}(p) - 4\beta, \mathcal{S}_{11}(p)) \\ &\quad + \beta(\mathcal{S}_0(p), \mathcal{S}_1(p), \mathcal{S}_{00}(p), \mathcal{S}_{01}(p) + 4\alpha, \mathcal{S}_{11}(p)) \\ &= (\alpha + \beta)(\mathcal{S}_0(p), \mathcal{S}_1(p), \mathcal{S}_{00}(p), \mathcal{S}_{01}(p), \mathcal{S}_{11}(p)) \\ &= (\alpha + \beta)\varphi(p), \end{aligned} \quad (23)$$

which allows us to express $\varphi(p)$ as

$$\varphi(p) = \frac{\alpha}{\alpha + \beta}\varphi(p + \beta v) + \frac{\beta}{\alpha + \beta}\varphi(p - \alpha v). \quad (24)$$

Now choose $\alpha = \min\{a, d\}$ and $\beta = \min\{b, c\}$. Then, both $p + \beta v$ and $p - \alpha v$ belong to the boundary of \mathbb{T}_N , and consequently $\varphi(p) \in \mathbb{P}_2^S$ represented by p can be written as a convex combination of two other elements $p + \min\{a, d\}v$ and $p - \min\{b, c\}v$ of \mathbb{P}_2^S . This implies that $\varphi(p)$ cannot be extremal.

In order to prove the ‘‘if’’ part, assume that $p \in \partial\mathbb{T}_N$ and $\varphi(p) = (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_{00}, \mathcal{S}_{01}, \mathcal{S}_{11})$ is not a vertex of \mathbb{P}_2^S . Then, $\varphi(p)$ can be decomposed into a convex combination of vertices of \mathbb{P}_2^S that are represented within our parametrization by $p_i = (a_i, b_i, c_i, d_i) \in \mathbb{T}_N$, i.e.,

$$\varphi(p) = \sum_{i=0}^k \lambda_i \varphi(p_i) \quad (25)$$

with $0 < \lambda_i < 1$ summing up to unity, and

$$\varphi(p_i) = (\mathcal{S}_0^{(i)}, \mathcal{S}_1^{(i)}, \mathcal{S}_{00}^{(i)}, \mathcal{S}_{01}^{(i)}, \mathcal{S}_{11}^{(i)}). \quad (26)$$

By combining Eqs. (22) and (26), Eq. (25) is equivalent to the following five equations:

$$\mathcal{S}_l = \sum_{i=0}^k \lambda_i \mathcal{S}_l^{(i)}, \quad \mathcal{S}_{ll} = \sum_{i=0}^k \lambda_i \mathcal{S}_{ll}^{(i)} \quad (27)$$

for $l = 0, 1$, and

$$\mathcal{S}_{01} = \sum_{i=0}^k \lambda_i \mathcal{S}_{01}^{(i)}. \quad (28)$$

Since for all vertices of \mathbb{P}_2 it holds that $\mathcal{S}_{ll}^{(i)} = [\mathcal{S}_l^{(i)}]^2 - N$ [cf. Eqs. (19)], Eqs. (27) imply that $\mathcal{S}_l^{(i)}$ must satisfy

$$\sum_i \lambda_i \left(\mathcal{S}_l^{(i)}\right)^2 = \left(\sum_i \lambda_i \mathcal{S}_l^{(i)}\right)^2 \quad (l = 0, 1). \quad (29)$$

If we think of Eq. (29) as a quadratic equation for a particular $\mathcal{S}_l^{(m)}$, i.e., i.e.,

$$\begin{aligned} &\lambda_m(\lambda_m - 1) \left(\mathcal{S}_l^{(m)}\right)^2 + 2\lambda_m \mathcal{S}_l^{(m)} \sum_{i \neq m} \lambda_i \mathcal{S}_l^{(i)} \\ &+ \left(\sum_{i \neq m} \lambda_i \mathcal{S}_l^{(i)}\right)^2 - \sum_{i \neq m} \lambda_i \left(\mathcal{S}_l^{(i)}\right)^2 = 0 \end{aligned} \quad (30)$$

it has real solutions if and only if its discriminant is nonnegative, which in turn holds iff

$$-4\lambda_0 \sum_{\substack{i < j \\ i, j \neq m}} \lambda_i \lambda_j \left(\mathcal{S}_l^{(i)} - \mathcal{S}_l^{(j)}\right)^2 \geq 0. \quad (31)$$

Since all λ 's are positive, the above condition is fulfilled iff $\mathcal{S}_l^{(i)} = \mathcal{S}_l^{(j)}$ for all $i, j \neq m$ and $l = 0, 1$. Due to the fact that (31) must be obeyed for any m , we have eventually that

$$\mathcal{S}_l^{(i)} = \mathcal{S}_l^{(j)} = \mathcal{S}_l \quad (32)$$

for any $i, j = 1, \dots, k$ and $l = 0, 1$.

On the other hand, the assumption that $\varphi(p)$ is not a vertex of \mathbb{P}_2^S , i.e., that it can be decomposed as in (25), means that $\mathcal{S}_{01}^{(i)}$ cannot be equal, as otherwise p_i are all the same. If we then express $\mathcal{S}_{01} = \mathcal{S}_0\mathcal{S}_1 - (a - b - c + d)$ and $\mathcal{S}_{01}^{(i)} = \mathcal{S}_0^{(i)}\mathcal{S}_1^{(i)} - (a_i - b_i - c_i + d_i)$, this, in virtue of Eq. (32), implies

$$a - b - c + d = \sum_i \lambda_i (a_i - b_i - c_i + d_i). \quad (33)$$

If we further note that

$$a_i + b_i + c_i + d_i = N = a + b + c + d \quad (34)$$

must hold for any i , we infer that a, b, c , and d are convex combinations of a_i, b_i, c_i , and d_i , respectively, and, as a result

$$p = \sum_{i=1}^k \lambda_i p_i. \quad (35)$$

In order to complete the proof (that is, to reach the contradiction with the assumption) it is enough to notice that $p \in \text{int}\mathbb{T}_N$, since not all of p_i can belong to the same facet of \mathbb{T}_N . In fact, if all p_i belong to the same facet of the tetrahedron, one of their coordinates (the same one for all i) must be zero (for instance, $a_i = 0$). Then, it directly follows from Eqs. (32) and (34) that all p_i s are equal, contradicting the assumption that (25) is a proper convex combination. Consequently, p belongs to the interior of the tetrahedron, which contradicts the assumption that $p \in \partial\mathbb{T}_N$, completing the proof. \square

Appendix B: A class of symmetric Bell inequalities

In this appendix, we revisit the three-parameter class of symmetric multipartite Bell inequalities and compute in detail its classical bound. Recall for this purpose that $\gamma = x^2$ and $\varepsilon = y^2$, where x and y are positive integers. Assume that $\mu = \beta/y \in \mathbb{Z}$, $\delta = \sigma xy$ and $\alpha_{\pm} = x[\sigma\mu \pm (x+y)]$ with σ denoting the sign of δ , and further that the parity of μ is opposite to that of ε (γ) for odd N (even N). In what follows we will show that the classical bound of the resulting Bell inequality is

$$\beta_C = \frac{1}{2} [N(x+y)^2 + (\sigma\mu \pm x)^2] - \frac{1}{2}. \quad (36)$$

First, note that for all local deterministic models, the left-hand side of (3) can be rewritten as

$$I = \alpha\mathcal{S}_0 + \beta\mathcal{S}_1 + \frac{\gamma}{2}(\mathcal{S}_0^2 - N) + \delta(\mathcal{S}_0\mathcal{S}_1 - z) + \frac{\varepsilon}{2}(\mathcal{S}_1^2 - N), \quad (37)$$

where $z = a - b - c + d$. The above choice of parameters further implies that

$$I = \frac{x^2}{2}\mathcal{S}_0^2 + \sigma xy\mathcal{S}_0\mathcal{S}_1 + \frac{y^2}{2}\mathcal{S}_1^2 - \frac{N}{2}(x+y) - \sigma xyz + x[\sigma\mu \pm (x+y)]\mathcal{S}_0 + \beta\mathcal{S}_1, \quad (38)$$

which can be rewritten as

$$I = \frac{1}{2} (x\mathcal{S}_0 + \sigma y\mathcal{S}_1 + \sigma\mu \pm x)^2 - \frac{1}{2}(\sigma\mu \pm x)^2 + xy(\pm\mathcal{S}_0 \mp \sigma\mathcal{S}_1 - \sigma z) - \frac{1}{2}N(x+y). \quad (39)$$

Within the parameterization (17), $\pm\mathcal{S}_0 \mp \sigma\mathcal{S}_1 - \sigma z = 4r - N$, where r depends on α and the sign of δ (i.e., σ) as follows:

$$r = \begin{cases} b, & \text{for } \alpha_+, \sigma = 1 \\ a, & \text{for } \alpha_+, \sigma = -1 \\ c, & \text{for } \alpha_-, \sigma = 1 \\ d, & \text{for } \alpha_-, \sigma = -1 \end{cases}. \quad (40)$$

As a consequence

$$I = \frac{1}{2} (x\mathcal{S}_0 + \sigma y\mathcal{S}_1 + \sigma\mu \pm x)^2 + 4xyr - \frac{1}{2} [(\sigma\mu \pm x)^2 + N(x+y)^2]. \quad (41)$$

When comparing the above expression with Eq. (36), it follows that in order to prove the latter to be the classical bound of I , it is enough to show that

$$(x\mathcal{S}_0 + \sigma y\mathcal{S}_1 + \sigma\mu \pm x)^2 + 8xyr \geq 1. \quad (42)$$

Since both x and y are positive integers, the above inequality is trivially satisfied if $r \neq 0$. For $r = 0$ (i.e., when optimizing over the facets of \mathbb{T}_N), observe that the expression in the parentheses is integer. Therefore, the inequality (42) is not satisfied only if the parenthesis is equal to zero. To prove that this may not happen, we will show that the above assumptions guarantee that the parity of $x\mathcal{S}_0 + \sigma y\mathcal{S}_1 + \sigma\mu \pm x$ is always odd. Let us consider the cases of odd and even N separately. For odd N , one finds that both \mathcal{S}_0 and \mathcal{S}_1 are odd. Hence, $x\mathcal{S}_0 + \sigma y\mathcal{S}_1 + \sigma\mu \pm x$ has the same parity as $y + \mu$. Then, by assumption, μ has opposite parity to ε . Noting that $\varepsilon = y^2$, and that both y^2 and y have the same parity, we conclude that $y + \mu$ is odd, meaning that the above expression cannot be zero. For even N , \mathcal{S}_0 as well as \mathcal{S}_1 are even, implying that $x\mathcal{S}_0 + y\sigma\mathcal{S}_1$ is even. The assumptions further guarantee that $\sigma\mu \pm x$ is odd, and therefore the expression is nonzero.

Under the additional assumption that x and y are coprimes one is also able to determine analytically all vertices of \mathbb{P}_2^S saturating these Bell inequalities, and thus check their tightness. The detailed considerations will be presented elsewhere [18].

Appendix C: Violation of two-body Bell inequalities by the Dicke states

Here we compute in detail the classical bound of the Bell inequality violated by the Dicke states, and present the explicit form of the reduced bipartite subsystem of the Dicke state $|D_N^{[N/2]}\rangle$.

The classical bound

The explicit form of these Bell inequalities for even N reads

$$I_N^e = \frac{N(N-1)}{4} \mathcal{S}_{00} + \frac{N}{2} \mathcal{S}_{01} - \frac{1}{2} \mathcal{S}_{11} \geq -\beta_C, \quad (43)$$

while for odd N ,

$$I_N^o = \frac{1}{2} \binom{N}{2} \mathcal{S}_{00} + \frac{N}{2} \mathcal{S}_{01} - \frac{1}{2} \mathcal{S}_{11} + \frac{N(N-1)}{2} \mathcal{S}_0 + \frac{N-1}{2} \mathcal{S}_1 \geq -\beta_C. \quad (44)$$

Our aim is to prove that the minimal value of the Bell inequalities (43) and (44) over local models is given by

$$\min_{\partial \mathbb{T}_N} I_N^{e/o} = -\frac{1}{4} \begin{cases} N(N-1)(N+2), & N \text{ even} \\ N(N-1)(N+3), & N \text{ odd}, \end{cases} \quad (45)$$

and, their classical bound is then $\beta_C = -\min_{\partial \mathbb{T}_N} I_N^{e/o}$.

For this purpose, let us first notice that for classical correlations the following constraints hold:

$$-N \leq \mathcal{S}_{00}, \mathcal{S}_{11} \leq N(N-1), \quad |\mathcal{S}_{01}| \leq N(N-1), \quad (46)$$

and $|\mathcal{S}_k| \leq N$ with $k = 0, 1$. Hence, the first term in Eqs. (43) and (44) is the dominant one (it is of the fourth order in N , while the remaining ones are of the second or third orders in N). This means that in order to minimize $I_N^{e/o}$ over the local models, one needs to make the term containing \mathcal{S}_{00} small. Since $\mathcal{S}_{00} = \mathcal{S}_0^2 - N$, the above expression suggests treating \mathcal{S}_0 as a parameter with which to lower the number of variables in the optimization. Then, among all the solutions parameterized by \mathcal{S}_0 we can choose the smallest one.

Let us now switch to the parametrization in terms of (16). We already know that $\mathcal{S}_0 = a + b - c - d$ which together with $a + b + c + d = N$, allows us to decrease the number of free variables in the optimization down to two. That is, by taking, for instance,

$$a = \frac{1}{2}(N + \mathcal{S}_0) - b, \quad c = \frac{1}{2}(N - \mathcal{S}_0) - d, \quad (47)$$

we can express the two-body expectation values as

$$\mathcal{S}_{00} = \mathcal{S}_0^2 - N, \quad \mathcal{S}_{11} = [N - 2(b + d)]^2 - N, \quad (48)$$

and

$$\mathcal{S}_{01} = \mathcal{S}_0[N - 1 - 2(b + d)] + 2(b - d), \quad (49)$$

where we now consider b and d as free variables that are non-negative integers constrained as

$$0 \leq b \leq \frac{1}{2}(N + \mathcal{S}_0), \quad 0 \leq d \leq \frac{1}{2}(N - \mathcal{S}_0). \quad (50)$$

Notice finally that from theorem 1 it follows that, in order to find β_C , it suffices to minimize $I_N^{e/o}$ over the 4-tuples

(a, b, c, d) belonging to the boundary of the tetrahedron, i.e., those for which $abcd = 0$. Eqs. (47) imply that the cases of $a = 0$ or $d = 0$ are now equivalent to $b = (1/2)(N + \mathcal{S}_0)$ or $d = (1/2)(N - \mathcal{S}_0)$, respectively, in (50). Within this framework, treating \mathcal{S}_0 as a parameter means that we intersect the three-dimensional tetrahedron with hyperplanes of constant \mathcal{S}_0 and look for the minimal value of $I_N^{e/o}$ for points lying on the boundary of the resulting two-dimensional object. Then, we choose the optimal solution among those parametrized by \mathcal{S}_0 .

In what follows we will separate the proof into the cases of even and odd N , and in each one we consider all the facets of the tetrahedron separately.

Even N

When combining Eqs. (48) and (49) with Eq. (43), one obtains a function in b and d parameterized by \mathcal{S}_0 :

$$I_N^e(b, d; \mathcal{S}_0) = \frac{1}{2} \left\{ \frac{N(N-1)}{2} (\mathcal{S}_0^2 - N) - [N - 2(b + d)]^2 + N [\mathcal{S}_0(N - 2(b + d) - 1) + 2(b - d)] + N \right\}.$$

Case $a=0$. As commented before, this case is equivalent to $b = (N + \mathcal{S}_0)/2$, which when applied to Eq. (51) gives

$$I_N^e\left(\frac{N+\mathcal{S}_0}{2}, d; \mathcal{S}_0\right) = \frac{1}{4} (N^2 - 3N - 2)(\mathcal{S}_0^2 - N) - d[2d + \mathcal{S}_0(N + 2) + N]. \quad (51)$$

This is a quadratic function in d which, since its second derivative with respect to d is negative, it has a local maximum. Therefore, it attains its minimal value either at $d = 0$ or $d = (N - \mathcal{S}_0)/2$. In the first case, $I_N^e\left(\frac{N+\mathcal{S}_0}{2}, 0; \mathcal{S}_0\right) = (1/4)(\mathcal{S}_0^2 - N)(N^2 - 3N - 2)$, which is clearly minimal for $\mathcal{S}_0 = 0$. Hence $I_N^e\left(\frac{N+\mathcal{S}_0}{2}, 0; 0\right) = -(1/4)N(N^2 - 3N - 2)$.

In the second case, i.e., $d = (N - \mathcal{S}_0)/2$, Eq. (51) implies

$$I_N^e\left(\frac{N+\mathcal{S}_0}{2}, \frac{N-\mathcal{S}_0}{2}; \mathcal{S}_0\right) = -\frac{N(N-1)}{4} [N + 2 - \mathcal{S}_0(\mathcal{S}_0 - 2)]. \quad (52)$$

It is easy to check that this expression attains its lowest value at either $\mathcal{S}_0 = 0$ or $\mathcal{S}_0 = 2$, which corresponds to

$$I_N^e\left(\frac{N+\mathcal{S}_0}{2}, \frac{N-\mathcal{S}_0}{2}; \mathcal{S}_0\right) = -\frac{1}{4} N(N-1)(N+2) \quad (\mathcal{S}_0 = 0, 2), \quad (53)$$

i.e., the value in Eq. (45). Hence, for two different 4-tuples

$$\left(0, \frac{N}{2}, 0, \frac{N}{2}\right), \quad \left(0, \frac{N}{2} + 1, 0, \frac{N}{2} - 1\right), \quad (54)$$

we obtain the value of I_N^e given in Eq. (45).

Case $b=0$. It follows from Eq. (51)

$$I_N^e(0, d; \mathcal{S}_0) = \frac{1}{2} \left\{ \frac{N(N-1)}{2} (\mathcal{S}_0^2 - N) - (N - 2d)^2 + N [\mathcal{S}_0(N - 2d - 1) - 2d] \right\}. \quad (55)$$

It is again not difficult to see that the second derivative of $I_N^e(0, d; \mathcal{S}_0)$ with respect to d is negative, and therefore we look for its minimal value at the boundary of the range of d . For $d = 0$ the above expression reduces to the right-hand side of Eq. (52), which, as previously mentioned, has minima for $\mathcal{S}_0 = 0$ and $\mathcal{S}_0 = 2$. This results in two additional elements of \mathbb{T}_4 for which I_N^e attains the value in Eq. (45), i.e.,

$$\left(\frac{N}{2} - 1, 0, \frac{N}{2} + 1, 0\right), \quad \left(\frac{N}{2}, 0, \frac{N}{2}, 0\right) \quad (56)$$

For $d = (N - \mathcal{S}_0)/2$, it follows from Eq. (55) that

$$I_N^e(0, \frac{N-\mathcal{S}_0}{2}; \mathcal{S}_0) = \frac{1}{4}(\mathcal{S}_0^2 - N)(N^2 + N - 2), \quad (57)$$

which has a minimum at $\mathcal{S}_0 = 0$ giving the fifth point saturating our Bell inequality

$$\left(\frac{N}{2}, 0, 0, \frac{N}{2}\right). \quad (58)$$

Cases $c=0$ or $d=0$. First, we notice that the case $c = 0$ is equivalent to $d = (N - \mathcal{S}_0)/2$. Then, following exactly the same reasoning as above, one straightforwardly finds that assuming either $d = (N - \mathcal{S}_0)/2$ or $d = 0$, the lowest value of I_N^e is $-(1/4)N(N - 1)(N + 2)$ and is obtained for the same five vectors as before, i.e., (54), (56), and (58).

Odd N

By applying Eqs. (47), (48), and (49), we can turn I_N into a two-variable function

$$I_N^o(b, d; \mathcal{S}_0) = \frac{1}{4}N(N - 1)[\mathcal{S}_0(\mathcal{S}_0 + 4) - N] - 2(b^2 + d^2) - b[4d - 1 + N(\mathcal{S}_0 - 2)] - d(N\mathcal{S}_0 - 1) \quad (59)$$

parameterized by \mathcal{S}_0 . Notice that since N is odd, \mathcal{S}_0 is also odd.

Case $a=0$. This case is equivalent to $b = (N + \mathcal{S}_0)/2$, and the latter when applied to Eq. (59) gives

$$\begin{aligned} I_N^o(\frac{N+\mathcal{S}_0}{2}, d; \mathcal{S}_0) &= -d[2d + \mathcal{S}_0(N + 2) + 2N - 1] \\ &\quad + \frac{1}{4}(N^2 - 3N - 2)(\mathcal{S}_0^2 - N) \\ &\quad + \frac{1}{2}(N - 1)^2\mathcal{S}_0 \\ &= I_N^e(\frac{N+\mathcal{S}_0}{2}, d; \mathcal{S}_0) + d + \frac{1}{2}(N - 1)^2\mathcal{S}_0. \end{aligned} \quad (60)$$

Since the second derivative of $I_N^o(\frac{N+\mathcal{S}_0}{2}, d; \mathcal{S}_0)$ with respect to d is negative for any \mathcal{S}_0 , it has its minimal value either at $d = 0$ or $d = (N - \mathcal{S}_0)/2$. For the first case, one obtains

$$I_N^o(\frac{N+\mathcal{S}_0}{2}, 0; \mathcal{S}_0) = \frac{1}{4}(\mathcal{S}_0^2 - N)(N^2 - 3N - 2) + \frac{1}{2}(N - 1)^2\mathcal{S}_0 \quad (61)$$

which is minimal at $\mathcal{S}_0 = 0$, and it reads $I_N^o(\frac{N+\mathcal{S}_0}{2}, 0; \mathcal{S}_0) = -(1/4)N(N^2 - 3N - 2)$.

For $d = (N - \mathcal{S}_0)/2$, it follows from Eq. (60) that

$$I_N^o(\frac{N+\mathcal{S}_0}{2}, \frac{N-\mathcal{S}_0}{2}; \mathcal{S}_0) = -\frac{1}{4}N(N - 1)(N + 4 - \mathcal{S}_0^2). \quad (62)$$

Clearly, the above expression is minimal for $\mathcal{S}_0 = 0$, but since \mathcal{S}_0 must be odd, we obtain the lowest value for $\mathcal{S}_0 = \pm 1$. As a result, the Bell expression I_N^o attains the minimum value (45) at the following two elements of \mathbb{T}_N

$$\left(0, \frac{N \pm 1}{2}, 0, \frac{N \mp 1}{2}\right). \quad (63)$$

Case $b=0$. For $b = 0$, Eq. (59) simplifies to

$$I_N^o(0, d; \mathcal{S}_0) = \frac{1}{4}N(N - 1)[\mathcal{S}_0(\mathcal{S}_0 + 4) - N] - 2d^2 - d(N\mathcal{S}_0 - 1). \quad (64)$$

$I_N^o(0, d; \mathcal{S}_0)$ is thus a quadratic function in d that has a local maximum. Hence, it attains its minimal values at the boundary of the range of d , i.e., either for $d = 0$ or for $d = (N - \mathcal{S}_0)/2$. For $d = 0$,

$$I_N^o(0, 0; \mathcal{S}_0) = \frac{1}{4}N(N - 1)[\mathcal{S}_0(\mathcal{S}_0 + 4) - N], \quad (65)$$

which, since \mathcal{S}_0 must be odd, is minimal at $\mathcal{S}_0 = -3$ or $\mathcal{S}_0 = -1$, and the corresponding value is $-(1/4)N(N - 1)(N + 3)$. Consequently, at the following two vertices

$$\left(\frac{N - 1}{2}, 0, \frac{N + 1}{2}, 0\right), \quad \left(\frac{N - 3}{2}, 0, \frac{N + 3}{2}, 0\right) \quad (66)$$

I_N^o attains (45).

For $d = (N - \mathcal{S}_0)/2$, Eq. (64) reads

$$I_N^o(0, \frac{N-\mathcal{S}_0}{2}; \mathcal{S}_0) = -\frac{N - 1}{4}[(N + 2)(N - \mathcal{S}_0^2) - 2(N + 1)\mathcal{S}_0]. \quad (67)$$

Since \mathcal{S}_0 is odd, I_N^o is minimal at $\mathcal{S}_0 = -1$. The corresponding minimum is $-(1/4)N(N - 1)(N + 3)$, and it is attained at

$$\left(\frac{N - 1}{2}, 0, 0, \frac{N + 1}{2}\right). \quad (68)$$

Cases $c=0$ or $d=0$. Similarly, for either $c = 0$ or $d = 0$, the minimal value of I_N^o is $-(1/4)N(N - 1)(N + 3)$, which is attained at the five vertices (63), (66), and (68).

In conclusion, the lowest value of $I_N^{e/o}$ for both even and odd N is the one given in Eq. (45) and is realized by five elements of \mathbb{T}_N : (54), (56), and (58) for even N , and (63), (66), and (68) for odd N . This also implies that the Bell inequality $I_N^{e/o}$ is tangent to $\mathbb{P}_{2,S}$ on five vertices. Since these are linearly independent, the Bell inequality indeed represents a facet of \mathbb{P}_S^2 .

A bipartite reduction of the Dicke states

The bipartite subsystem of the Dicke state $|D_N^k\rangle$ can be determined analytically (see e.g. Ref. [27]). For $k = \lceil N/2 \rceil$, it has the following form

$$\rho_N^{\lceil N/2 \rceil} = \frac{1}{N(N-1)} \begin{pmatrix} p_N & 0 & 0 & 0 \\ 0 & q_N & q_N & 0 \\ 0 & q_N & q_N & 0 \\ 0 & 0 & 0 & r_N \end{pmatrix}, \quad (69)$$

where $p_N = (\lfloor N/2 \rfloor - 1)\lfloor N/2 \rfloor$, $q_N = \lfloor N/2 \rfloor \lceil N/2 \rceil$, and $r_N = (\lceil N/2 \rceil - 1)\lceil N/2 \rceil$.