COMODULES OVER WEAK MULTIPLIER BIALGEBRAS

GABRIELLA BÖHM

ABSTRACT. This is a sequel paper of [2] in which we study the comodules over a regular weak multiplier bialgebra over a field, with a full comultiplication. Replacing the usual notion of coassociative coaction over a (weak) bialgebra, a comodule is defined via a pair of compatible linear maps. Both the total algebra and the base (co)algebra of a regular weak multiplier bialgebra with a full comultiplication are shown to carry comodule structures. Kahng and Van Daele's integrals [9] are interpreted as comodule maps from the total to the base algebra. Generalizing the counitality of a comodule to the multiplier setting, we consider the particular class of so-called full comodules. They are shown to carry bi(co)module structures over the base (co)algebra and constitute a monoidal category via the (co)module tensor product over the base (co)algebra. If a regular weak multiplier bialgebra with a full comultiplication possesses an antipode, then finite dimensional full comodules are shown to possess duals in the monoidal category of full comodules. Hopf modules are introduced over regular weak multiplier bialgebras with a full comultiplication. Whenever there is an antipode, the Fundamental Theorem of Hopf Modules is proven. It asserts that the category of Hopf modules is equivalent to the category of firm modules over the base algebra.

INTRODUCTION

The categories of modules and comodules over a weak bialgebra (say, over a field), are well-studied in the literature. The most important features are the following. To any weak bialgebra A over a field, there is an associated separable Frobenius algebra R (known as the 'base (co)algebra'), see [11, 4, 1]. It is a subalgebra and a quotient coalgebra of A. Both the category M_A of A-modules and the category M^A of A-comodules are monoidal, admitting strict monoidal forgetful functors to the category $_RM_R$ of R-bimodules [5, 1]. If A is a weak Hopf algebra, then finite dimensional modules and comodules possess duals in the appropriate monoidal category [5], see also [14, 12].

The aim of this paper is to study analogous questions about weak multiplier bialgebras in [2]. These are generalizations of weak bialgebras in the spirit of [17, 18, 19]. That is, the algebra underlying a weak multiplier bialgebra A, is not required to possess a unit. Instead, the multiplication is assumed to be non-degenerate and surjective, so that the notion of multiplier algebra $\mathbb{M}(A)$ [7] is available. The comultiplication is a multiplicative map from A to the multiplier algebra of $A \otimes A$. It is subject to certain compatibility axioms in [2, Definition 2.1]. A central role is played by a canonical idempotent element E in the multiplier algebra of $A \otimes A$. Whenever the comultiplication is regular and full (in the sense discussed in [2, Definition 2.3 and Theorem 3.13], respectively), there is a

Date: Nov 2013.

²⁰¹⁰ Mathematics Subject Classification. 16T05, 16T10, 16D90, 18B40.

I am grateful to José Gómez-Torrecillas and Alfons Van Daele for highly enlightening discussions; and to the referee for several helpful comments. My research was supported by the Hungarian Scientific Research Fund OTKA, grant K108384.

co-separable co-Frobenius coalgebra R associated to A, see [2, Theorem 4.6] (it is a nonunital subalgebra of the multiplier algebra of A). As shown in [6], R possesses then a firm algebra structure and the monoidal category of R-bicomodules is isomorphic to the monoidal category $_RM_R$ of firm R-bimodules. As a crucial difference from usual weak bialgebras, a weak multiplier bialgebra A is not known to induce any monad or comonad. Still, whenever the comultiplication is regular and full, it was shown in [2, Section 5] that the category $M_{(A)}$, of non-degenerate A-modules with surjective action, is monoidal and it possesses a strict monoidal (forgetful) functor $U_{(A)}: M_{(A)} \to _RM_R$.

In this paper we continue the study started in [2] by analyzing the category of comodules. Comodules over a weak multiplier bialgebra A need to be defined in such a way that in particular A itself is a comodule via the comultiplication. Since in this particular case the comultiplication Δ lands in the multiplier algebra of $A \otimes A$ (rather than in $A \otimes A$), the existence of a coaction from any comodule V to $V \otimes A$ would be too much to expect. However, we can also not work with a 'multiplier valued' coaction, what would be the most naive generalization of the comultiplication Δ , as the notion of multiplier on $V \otimes A$ is not available for an arbitrary vector space V. Instead, one can observe that the comultiplication of a regular weak multiplier bialgebra A is uniquely encoded in the maps $T_1(a \otimes b) := \Delta(a)(1_{\mathbb{M}(A)} \otimes b)$ and $T_3(a \otimes b) := (1_{\mathbb{M}(A)} \otimes b)\Delta(a)$ from $A \otimes A$ to $A \otimes A$. In a similar spirit, in the same way as in [21, 10] in the non-weak case, we define a right A-comodule V in terms of a compatible pair of maps $V \otimes A \to V \otimes A$ (instead of a map $V \to V \otimes A$). As a generalized coassociativity condition, we require the same pentagonal equation in [21], but we supplement it with a normalization condition in terms of the canonical idempotent element E. The counitality condition is replaced by the property of 'fullness': the 'first leg' of either coaction map $V \otimes A \to V \otimes A$ is required to generate the vector space V.

Throughout the paper, we deal with regular weak multiplier bialgebras over a field. Their definition and some of the most important properties are recalled from [2] in Section 1, where also some new technical lemmata are proven. The axioms of weak multiplier bialgebra are presented in a somewhat new but equivalent form. In Section 2 we spell out the notions of comodule and its morphisms. We show that any regular weak multiplier bialgebra A is a right comodule over itself. Whenever in addition the comultiplication is right full, there is another distinguished right comodule on the base algebra R. In this case, in Section 3 we describe the integrals in [9] as comodule homomorphisms from A to R. Under the same assumption of right full comultiplication, in Section 4 we equip any full right A-comodule with a firm R-bimodule structure. In Section 5, the full subcategory of full right A-comodules is shown to be monoidal, in such a way that there is a strict monoidal forgetful functor to the category of firm *R*-bimodules. A similar result for multiplier Hopf algebras was obtained in [21]. In Section 6 we show that the dual vector space of any finite dimensional right A-comodule admits a left A-comodule structure. The dual of a finite dimensional full right comodule is proven to be an (obviously finite dimensional) full left comodule. If A possesses an antipode, then it is shown to induce a (full) right A-comodule structure on any (full) left A-comodule; in particular it induces a right comodule structure on the dual vector space of any finite dimensional right comodule. A finite dimensional full right comodule, and the corresponding full right comodule on its dual vector space are shown to be duals in the monoidal category of full right comodules. In Section 7 we introduce Hopf modules over a regular weak multiplier bialgebra A. They are both nondegenerate A-modules with a surjective action, and full A-comodules, in a compatible way. In particular, whenever the comultiplication is right full, A itself is an A-Hopf module. Assuming that the comultiplication is both left and right full and there exists an antipode, for any A-Hopf module V we discuss a distinguished subspace of the vector space of Hopf module homomorphisms $A \to V$. It plays the role of the space of A-coinvariants in V and it can be smaller indeed than the vector space of Hopf module homomorphisms $A \to V$. For example, the space of coinvariants of the A-Hopf module A is isomorphic to the base algebra of A (which is a non-unital subalgebra of the unital algebra of Hopf module endomorphisms of A). For a regular weak multiplier bialgebra A with left and right full comultiplication and possessing an antipode, we prove the Fundamental Theorem of Hopf Modules; that is, an equivalence of the category of A-Hopf modules and the category of firm modules over the base algebra of A.

1. Preliminaries

In this section we recall from [2] the definition and the basic properties of weak multiplier bialgebra. We also prove some new technical lemmata for use in the later sections.

1.1. Notation. Throughout the paper, for any vector space V, we denote the identity map $V \to V$ also by V. On elements $v \in V$, also the notation $v \mapsto 1v$ or $v \mapsto v1$ is used, meaning multiplication with the unit 1 of the base field. For any vector spaces Vand W, the space of linear maps $V \to W$ is denoted by Lin(V, W). For any subset I of a vector space V, $\langle v \mid v \in I \rangle$ denotes the subspace generated by I in V. For linear maps $f: V \to W$ and $g: W \to Z$, the composite is denoted by juxtaposition gf. The tensor product of vector spaces is denoted by \otimes . For vector spaces V and W, we denote by tw the flip map

$$\mathsf{tw}: V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto w \otimes v.$$

For any linear map $f: V \otimes W \to V' \otimes W'$, we consider the associated linear maps $f^{21} := \operatorname{\mathsf{tw}} f\operatorname{\mathsf{tw}} : W \otimes V \to W' \otimes V', f^{13} := (V' \otimes \operatorname{\mathsf{tw}})(f \otimes Z)(V \otimes \operatorname{\mathsf{tw}}) : V \otimes Z \otimes W \to V' \otimes Z \otimes W', f^{31} := (W' \otimes \operatorname{\mathsf{tw}})(f^{21} \otimes Z)(W \otimes \operatorname{\mathsf{tw}}) : W \otimes Z \otimes V \to W' \otimes Z \otimes V'$ and so on, for any vector space Z.

1.2. Multiplier algebra. Let A be an associative algebra over a field, with multiplication $\mu : A \otimes A \to A$, $a \otimes b \mapsto ab$, possibly possessing no unit. It is said to be *idempotent* if μ is surjective; that is, A is spanned by elements of the form ab for $a, b \in A$. It is termed non-degenerate if any of the conditions ($ab = 0 \forall b \in A$) and ($ba = 0 \forall b \in A$) implies a = 0. A multiplier [7] on an idempotent non-degenerate algebra A is a pair (λ, ρ) of linear maps $A \to A$ such that

(1.1)
$$a\lambda(b) = \varrho(a)b, \quad \forall a, b \in A$$

For any multiplier (λ, ϱ) on A, λ is a morphism of right A-modules and ϱ is a morphism of left A-modules. Multipliers on A constitute a unital associative algebra, the so-called *multiplier algebra* $\mathbb{M}(A)$ with multiplication $(\lambda, \varrho)(\lambda', \varrho') = (\lambda\lambda', \varrho'\varrho)$ and unit 1 = (A, A). The multiplication in $\mathbb{M}(A)$ will be denoted by μ too. Any element a of A can be regarded as a multiplier via the embedding

$$A \to \mathbb{M}(A), \qquad a \mapsto (a(-), (-)a).$$

This makes A a two-sided ideal in $\mathbb{M}(A)$. Indeed,

$$a(\lambda, \varrho) = \varrho(a)$$
 and $(\lambda, \varrho)a = \lambda(a)$.

The ideal A is dense in $\mathbb{M}(A)$ in the sense that for $\nu \in \mathbb{M}(A)$, any of the conditions $(\nu b = 0 \ \forall b \in A)$ and $(b\nu = 0 \ \forall b \in A)$ implies $\nu = 0$.

The opposite A^{op} of an idempotent non-degenerate algebra A is the same vector space A equipped with the multiplication $\mu^{op}: a \otimes b \mapsto ba$. It is again an idempotent nondegenerate algebra and $\mathbb{M}(A^{\mathsf{op}}) \cong \mathbb{M}(A)^{\mathsf{op}}$.

The tensor product $A \otimes B$ of idempotent non-degenerate algebras A and B is again an idempotent non-degenerate algebra via the factorwise multiplication. We extend the leg numbering notation of linear maps in Section 1.1 to multipliers on $A \otimes B$ by putting $(\lambda, \rho)^{ij} := (\lambda^{ij}, \rho^{ij}), \text{ for any labels } i, j.$

The following result, about the extension of maps to the multiplier algebra, is due to Van Daele and Wang [18, Proposition A.3]: Let A and B be idempotent non-degenerate algebras. Let $\phi: A \to \mathbb{M}(B)$ be a multiplicative linear map and e be an idempotent element of B (i.e. such that $e^2 = e$). If

$$\langle \phi(a)b \mid a \in A, b \in B \rangle = \langle eb \mid b \in B \rangle$$
 and $\langle b\phi(a) \mid a \in A, b \in B \rangle = \langle be \mid b \in B \rangle$

then there is a unique multiplicative map $\overline{\phi}: \mathbb{M}(A) \to \mathbb{M}(B)$ such that $\overline{\phi}(1) = e$ and $\overline{\phi}(a) = \phi(a)$ for any $a \in A$.

1.3. Weak multiplier bialgebra. We present here the definition of weak multiplier bialgebra in [2, Definition 2.1] in a slightly different, but equivalent form. Recall from Section 1.1 that for any vector space A, we denote both as $a \mapsto 1a$ and as $a \mapsto a1$ the action of the identity map A on elements $a \in A$, where 1 means the unit of the base field.

Definition 1.1. A weak multiplier bialgebra over a field k is given by an idempotent nondegenerate k-algebra A with multiplication $\mu: A \otimes A \to A$, and linear maps E_1, E_2, T_1, T_2 : $A \otimes A \to A \otimes A$ and a linear map $\epsilon : A \to k$ — called the *counit* — subject to the following axioms.

- (i) For all $a, b, c, d \in A$, $(a \otimes b)E_1(c \otimes d) = E_2(a \otimes b)(c \otimes d)$.
- (ii) $E_1^2 = E_1$, equivalently, $E_2^2 = E_2$. (iii) $(T_2 \otimes A)(A \otimes T_1) = (A \otimes T_1)(T_2 \otimes A)$.
- (iv) $(\epsilon \otimes A)T_1 = \mu = (A \otimes \epsilon)T_2.$
- (v) $(\mu \otimes A)T_1^{13}(A \otimes T_1) = T_1(\mu \otimes A)$, equivalently, $(A \otimes \mu)T_2^{13}(T_2 \otimes A) = T_2(A \otimes \mu)$.
- (vi) $\langle T_1(a \otimes c)(b \otimes 1) \mid a, b, c \in A \rangle = \langle E_1(b \otimes c) \mid b, c \in A \rangle$ and $\langle (1 \otimes c)T_2(b \otimes a) \mid a, b, c \in A \rangle = \langle E_2(b \otimes c) \mid b, c \in A \rangle.$
- (vii) $(E_1 \otimes A)(A \otimes T_1) = (A \otimes T_1)(E_1 \otimes A)$ (equivalently, $(A \otimes E_2)(T_2 \otimes A) = (T_2 \otimes A)(A \otimes E_2))$ and $(E_2 \otimes A)(A \otimes T_2) = (A \otimes T_2)E_2^{13}$ (equivalently, $(A \otimes E_1)(T_1 \otimes A) = (T_1 \otimes A)E_1^{13}).$ (viii) For all $a, b, c \in A$,

$$(\epsilon \otimes A)((1 \otimes a)E_1(b \otimes c)) = (\epsilon \otimes A)(T_1(a \otimes c)(b \otimes 1)) \text{ and } \\ (\epsilon \otimes A)(E_2(a \otimes b)(1 \otimes c)) = (\epsilon \otimes A)((1 \otimes b)T_2(a \otimes c)).$$

The weak multiplier bialgebra A is said to be *regular* if there exist further two maps $T_3, T_4: A \otimes A \to A \otimes A$ such that for all $a, b, c \in A$,

(ix)
$$(1 \otimes a)T_1(c \otimes b) = T_3(c \otimes a)(1 \otimes b)$$
 and $T_2(a \otimes c)(b \otimes 1) = (a \otimes 1)T_4(b \otimes c)$.

Throughout the paper, we shall often use the index notation $T_i(a \otimes b) =: a^i \otimes b^i$, for $i \in \{1, 2, 3, 4\}$, where implicit summation is understood.

It follows from the axioms in Definition 1.1 that

$$(\mu \otimes A)(A \otimes T_1) \stackrel{\text{(iv)}}{=} (A \otimes \epsilon \otimes A)(T_2 \otimes A)(A \otimes T_1)$$
$$\stackrel{\text{(iii)}}{=} (A \otimes \epsilon \otimes A)(A \otimes T_1)(T_2 \otimes A) \stackrel{\text{(iv)}}{=} (A \otimes \mu)(T_2 \otimes A).$$

That is, for all $a, b, c \in A$,

(1.2)
$$(a \otimes 1)T_1(c \otimes b) = T_2(a \otimes c)(1 \otimes b)$$

Theorem 1.2. The definition of (regular) weak multiplier bialgebra in Definition 1.1 above, is equivalent to that in [2, Definition 2.1 (and Definition 2.3)].

Proof. Axiom (i) in Definition 1.1 is equivalent to saying that there is a multiplier $E = (E_1, E_2)$ on $A \otimes A$ and axiom (ii) is equivalent to E being idempotent. Identity (1.2) is equivalent to the existence of a linear map $\Delta : A \to \mathbb{M}(A \otimes A)$ — the so-called *comultiplication* — defined by

$$\Delta(c)(a \otimes b) := T_1(c \otimes b)(a \otimes 1) \quad \text{and} \quad (a \otimes b)\Delta(c) := (1 \otimes b)T_2(a \otimes c)$$

(so that $T_1(c \otimes b) = \Delta(c)(1 \otimes b)$ and $T_2(a \otimes c) = (a \otimes 1)\Delta(c)$). By the non-degeneracy of the right A-module $A \otimes A$ (via multiplication in the second factor), and by the non-degeneracy of the left A-module $A \otimes A$ (via multiplication in the first factor), either equality in axiom (v) is equivalent to the multiplicativity of the map Δ . Axioms (iii), (iv), (vi) and (viii) are literally the same as in [2, Definition 2.1]. Let us investigate axiom (vii). Its first equality is equivalent to

$$\begin{array}{rcl} ((E_1 \otimes A)(A \otimes T_1)(a \otimes b \otimes c))(1 \otimes d \otimes 1) &=& ((A \otimes T_1)(E_1 \otimes A)(a \otimes b \otimes c))(1 \otimes d \otimes 1) \\ \Leftrightarrow & (E \otimes 1)(a \otimes \Delta(b)(d \otimes c)) &=& (A \otimes \Delta)(E(a \otimes b))(1 \otimes d \otimes c) \end{array}$$

holding true, for any $a, b, c, d \in A$. By (i), (ii), (1.2), (v) and (vi), $A \otimes \Delta$ extends to a unique multiplicative map $\overline{A \otimes \Delta} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ such that $(\overline{A \otimes \Delta})(1) = 1 \otimes E$. Hence for any $w \in \mathbb{M}(A \otimes A)$, $(\overline{A \otimes \Delta})(w) = (\overline{A \otimes \Delta})(w1) = (\overline{A \otimes \Delta})(w)(\overline{A \otimes \Delta})(1) = (\overline{A \otimes \Delta})(w)(1 \otimes E)$. In terms of this extended map, the first equality in (vii) is further equivalent to

$$(E \otimes 1)(a \otimes \Delta(b)(d \otimes c)) = (\overline{A \otimes \Delta})(E)(a \otimes \Delta(b)(d \otimes c))$$

holding true for any $a, b, c, d \in A$. By (vi), this is equivalent to the validity of

$$(E \otimes 1)(1 \otimes E)(a \otimes d \otimes c) = (\overline{A \otimes \Delta})(E)(1 \otimes E)(a \otimes d \otimes c) \quad \forall a, d, c \in A \qquad \Leftrightarrow (1.3) \qquad (E \otimes 1)(1 \otimes E) = (\overline{A \otimes \Delta})(E),$$

which is one of the axioms in [2, Definition 2.1] (v). One proves symmetrically the equivalence of the second equality in (vii) and of the other axiom in [2, Definition 2.1] (v).

If A is a regular weak multiplier bialgebra in the sense of [2, Definition 2.3], then $T_3(a \otimes b) := (1 \otimes b)\Delta(a)$ and $T_4(a \otimes b) := \Delta(b)(a \otimes 1)$ clearly obey (ix). In the opposite direction, we need to show that the maps T_3 and T_4 satisfying (ix) must be of this form. Using the first equality in (ix), it follows for any $a, b, c, d \in A$ that

$$(a \otimes 1)T_3(c \otimes b)(1 \otimes d) = (a \otimes b)T_1(c \otimes d) = (a \otimes b)\Delta(c)(1 \otimes d)$$

Using the non-degeneracy of A and simplifying by a and d, we conclude that T_3 is of the desired form. The claim about T_4 follows symmetrically by the second equality in (ix). \Box

If the algebra A in Definition 1.1 is in addition unital, then we obtain an equivalent definition of usual weak bialgebra in [4], in light of [2, Theorem 2.10]. On the other hand, if we add any of the equivalent conditions

- $E_1 = A \otimes A$
- $E_2 = A \otimes A$
- $\epsilon \mu = \epsilon \otimes \epsilon$

in Definition 1.1, then we obtain an equivalent definition of non-weak multiplier bialgebra over a field in [2, Theorem 2.11].

Note that if A is a regular weak multiplier bialgebra, then its comultiplication $\Delta : A \to \mathbb{M}(A \otimes A)$ can be defined equivalently by the prescriptions

$$\Delta(c)(a \otimes b) := T_1(c \otimes b)(a \otimes 1) \quad \text{and} \quad (a \otimes b)\Delta(c) := (a \otimes 1)T_3(c \otimes b).$$

Using the first identity in axiom (ix) in the first equality and (iv) in the second one, for all $a, b, c \in A$

$$((\epsilon \otimes A)T_3(b \otimes a))c = a((\epsilon \otimes A)T_1(b \otimes c)) = abc.$$

Symmetrically, by the second equality in (ix) and by (iv),

$$c((A \otimes \epsilon)T_4(b \otimes a)) = ((A \otimes \epsilon)T_2(c \otimes a))b = cab.$$

Thus by the non-degeneracy of A,

(1.4)
$$(\epsilon \otimes A)T_3 = \mu^{\mathsf{op}} = (A \otimes \epsilon)T_4$$

A symmetric reasoning shows that (1.4) is in fact an equivalent form of the counitality axiom (iv) in Definition 1.1. The axioms in Definition 1.1 imply the further identities

(1.5) $T_1(c \otimes b)(a \otimes 1) = \Delta(c)(a \otimes b) = T_4(a \otimes c)(1 \otimes b)$

(1.6)
$$(1 \otimes b)T_2(a \otimes c) = (a \otimes b)\Delta(c) = (a \otimes 1)T_3(c \otimes b)$$

(1.7)
$$T_3(c \otimes b)(a \otimes 1) = (1 \otimes b)\Delta(c)(a \otimes 1) = (1 \otimes b)T_4(a \otimes c)$$

for any regular weak multiplier bialgebra A and all $a, b, c \in A$. It is immediate from axioms (ii) and (vi) that $E_1T_1 = T_1$ and $E_2T_2 = T_2$ for any weak multiplier bialgebra; and by (1.5-1.6), also $E_2T_3 = T_3$ and $E_1T_4 = T_4$ in the regular case.

By [2, Proposition 2.4 and Proposition 2.6], for any weak multiplier bialgebra A there exist linear maps $\overline{\sqcap}^L, \overline{\sqcap}^R : A \to \mathbb{M}(A)$ defined by the prescriptions

(1.8)
$$\overline{\sqcap}^{L}(a)b := (\epsilon \otimes A)T_{2}(a \otimes b) \quad \text{and} \quad b\overline{\sqcap}^{L}(a) := (\epsilon \otimes A)E_{2}(a \otimes b)$$
$$\overline{\sqcap}^{R}(a)b := (A \otimes \epsilon)E_{1}(b \otimes a) \quad \text{and} \quad b\overline{\sqcap}^{R}(a) := (A \otimes \epsilon)T_{1}(b \otimes a).$$

They obey similar properties to the analogous maps in a usual, unital weak bialgebra in [4], see [2]. If A is regular, then there are further two similar linear maps $\sqcap^L, \sqcap^R : A \to \mathbb{M}(A)$ defined as in (3.1), (3.2) and (3.3) in [2], by the prescriptions

(1.9)
$$\sqcap^{L}(a)b := (\epsilon \otimes A)E_{1}(a \otimes b) \quad \text{and} \quad b \sqcap^{L}(a) := (\epsilon \otimes A)T_{4}(a \otimes b)$$
$$\sqcap^{R}(a)b := (A \otimes \epsilon)T_{3}(b \otimes a) \quad \text{and} \quad b \sqcap^{R}(a) := (A \otimes \epsilon)E_{2}(b \otimes a).$$

Lemma 1.3. For any weak multiplier bialgebra A and any $a, b \in A$, the following identities hold.

(1)
$$(\overline{\sqcap}^R \otimes A) E_1(a \otimes b) = E(\overline{\sqcap}^R(a) \otimes b).$$

(2) $(A \otimes \overline{\sqcap}^L) E_2(a \otimes b) = (a \otimes \overline{\sqcap}^L(b)) E.$

If A is regular, then also the following hold for any $a, b \in A$.

- (3) $(\sqcap^R \otimes A)E_2(a \otimes b) = (\sqcap^R(a) \otimes b)E.$
- (4) $(A \otimes \sqcap^L) E_1(a \otimes b) = E(a \otimes \sqcap^L(b)).$

Proof. We only prove part (1), all other parts are proven analogously. For any $b, c \in A$ introduce the notation $T_1(b \otimes c) = b^1 \otimes c^1$ where implicit summation is understood. Then for any $a \in A$,

$$(\overline{\sqcap}^R \otimes A)(E(a \otimes bc)) = \overline{\sqcap}^R(\overline{\sqcap}^R(b^1)a) \otimes c^1 = \overline{\sqcap}^R(b^1)\overline{\sqcap}^R(a) \otimes c^1 = E(\overline{\sqcap}^R(a) \otimes bc).$$

From this we conclude by the idempotency of A. The first and the last equalities follow by identity (2.3) in [2] and the middle one follows by [2, Lemma 3.4].

Lemma 1.4. For any regular weak multiplier bialgebra A and any $a, b \in A$, the following identities hold.

- (1) $(A \otimes \sqcap^L) T_1(a \otimes b) = E(a \overline{\sqcap}^R(b) \otimes 1).$ (2) $(\overline{\sqcap}^R \otimes \sqcap^L) T_1(a \otimes b) = E(\overline{\sqcap}^R(ab) \otimes 1).$

Proof. (1) For any $a, b, c \in A$,

$$((A \otimes \sqcap^L)T_1(a \otimes b))(c \otimes 1) = (A \otimes \sqcap^L)(\Delta(a)(c \otimes b)) = (A \otimes \sqcap^L)(\Delta(a)(c \otimes \overline{\sqcap}^R(b)))$$
$$= (A \otimes \sqcap^L)T_4(c \otimes a\overline{\sqcap}^R(b)) = E(a\overline{\sqcap}^R(b)c \otimes 1),$$

from which we conclude by the non-degeneracy of A. In the second equality we used that by (1.9) and [2, Lemma 3.1], for any $a, b, c \in A$,

$$\sqcap^{L}(a\overline{\sqcap}^{R}(b))c = (\epsilon \otimes A)(E(a\overline{\sqcap}^{R}(b) \otimes c)) = (\epsilon \otimes A)(E(ab \otimes c)) = \sqcap^{L}(ab)c,$$

hence $\sqcap^L(ab) = \sqcap^L(a\overline{\sqcap}^R(b))$. The third equality follows by [2, Lemma 3.3] and the multiplicativity of the comultiplication. The last equality follows by (3.4) in [2].

(2) For any $a, b \in A$,

$$(\overline{\sqcap}^R \otimes \sqcap^L) T_1(a \otimes b) \stackrel{(1)}{=} (\overline{\sqcap}^R \otimes A) (E(a\overline{\sqcap}^R(b) \otimes 1)) = E(\overline{\sqcap}^R(a\overline{\sqcap}^R(b)) \otimes 1) = E(\overline{\sqcap}^R(ab) \otimes 1),$$

where the second equality follows by Lemma 1.3 (1) and the last equality follows by [2, Lemma 3.2].

By Lemma 3.4 and identities (3.8) in [2], the ranges of the maps (1.8) and (1.9) are (nonunital) subalgebras of $\mathbb{M}(A)$. In particularly nice cases they carry even more structure: Let A be a regular weak multiplier bialgebra over a field k. As in [2, Theorem 3.13], we say that its comultiplication is *right full* if

$$\langle (A \otimes \omega)T_1(a \otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k) \rangle = A$$

equivalently,

$$\langle (A \otimes \omega)T_3(a \otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A, k) \rangle = A.$$

These conditions hold if and only if the ranges of the maps $\overline{\sqcap}^R$ and \sqcap^R coincide. Let us denote this coinciding range by R. It carries a coalgebra structure as follows. By [2, Proposition 4.3 (1)], there is a multiplier F on $A \otimes A$ defined by the prescriptions (1.10)

$$F(a \otimes bc) := ((\overline{\sqcap}^R \otimes A) \mathsf{tw} T_4(c \otimes b))(a \otimes 1) \quad \text{and} \quad (ab \otimes c)F := (1 \otimes c)((A \otimes \sqcap^R) T_2(a \otimes b)).$$

By [2, Theorem 4.4], it induces a coassociative comultiplication

(1.11)
$$\delta: R \to R \otimes R, \qquad \sqcap^R(a) \mapsto (\sqcap^R(a) \otimes 1)F = F(1 \otimes \sqcap^R(a))$$

with the counit

(1.12)
$$\varepsilon: R \to k, \qquad \sqcap^R(a) \mapsto \epsilon(a).$$

The comultiplication δ is an *R*-bimodule map and it provides a section of the multiplication $R \otimes R \to R$. That is to say, R is a co-separable co-Frobenius coalgebra, see [2, Theorem 4.6]. The algebra R is not unital but it has (idempotent) local units. There is a unique automorphism, the so-called Nakayama automorphism ϑ of R, for which $\varepsilon(rs) = \varepsilon(\vartheta(s)r)$, for any $s, r \in R$.

As a particular consequence of the above properties, R is a firm subalgebra of $\mathbb{M}(A)$, meaning that the multiplication $R \otimes R \to R$ projects to an isomorphism $R \otimes_R R \to R$ (where \otimes_R stands for the R-module tensor product). There are evident notions of (associative but not unital) left and right R-modules, and hence of R-bimodules. A (say, right) R-module M is said to be firm if the action $M \otimes R \to M$ projects to an isomorphism $M \otimes_R R \to M$. Since R is a coseparable coalgebra, it follows by [6, Proposition 2.17] that the category of firm modules over the firm algebra R is isomorphic to the category of comodules over the coalgebra R. A bimodule is said to be firm if it is firm as a left and as a right R-module. Firm R-bimodules constitute a monoidal category $_RM_R$ (which is isomorphic to the category of R-bicomodules). The monoidal product is the module tensor product \otimes_R over the firm algebra R or, what is isomorphic to it, the comodule tensor product over the coalgebra R. It is a linear retract of the tensor product of vector spaces, see [6, Proposition 2.17]. The monoidal unit is R (with (co)actions provided by the (co)multiplication).

One defines symmetrically the *left full* property of the comultiplication of a regular weak multiplier bialgebra. It implies analogous properties of the coinciding image L of the maps $\overline{\sqcap}^L$ and $\sqcap^L : A \to \mathbb{M}(A)$, see [2, Theorem 3.13].

If A is a regular weak multiplier bialgebra with a right full comultiplication, then by [2, Lemma 4.8] there are anti-multiplicative maps

$$\begin{aligned} \tau : \sqcap^R(A) &= \overline{\sqcap}^R(A) \to \overline{\sqcap}^L(A), \qquad \quad \sqcap^R(a) \mapsto \overline{\sqcap}^L(a), \\ \overline{\tau} : \sqcap^R(A) &= \overline{\sqcap}^R(A) \to \sqcap^L(A), \qquad \quad \overline{\sqcap}^R(a) \mapsto \sqcap^L(a). \end{aligned}$$

Symmetrically, if the comultiplication is left full then there are anti-multiplicative maps

$$\sigma: \sqcap^{L}(A) = \overline{\sqcap}^{L}(A) \to \sqcap^{R}(A), \qquad \overline{\sqcap}^{L}(a) \mapsto \sqcap^{R}(a), \\ \overline{\sigma}: \sqcap^{L}(A) = \overline{\sqcap}^{L}(A) \to \overline{\sqcap}^{R}(A), \qquad \sqcap^{L}(a) \mapsto \overline{\sqcap}^{R}(a).$$

If the comultiplication is both right and left full, then $\tau = \sigma^{-1}$ and $\overline{\tau} = \overline{\sigma}^{-1}$; by [2, Proposition 4.9] they are anti-coalgebra maps; and the Nakayama automorphism of R is equal to $\sigma\overline{\sigma}^{-1}$ and the Nakayama automorphism of L is equal to $\overline{\sigma}^{-1}\sigma$.

Lemma 1.5. For any regular weak multiplier bialgebra A, the following hold.

- (1) The vector space A is spanned by elements of the form $a\overline{\sqcap}^{L}(b)$, for $a, b \in A$.
- (2) The vector space A is spanned by elements of the form $\overline{\sqcap}^R(b)a$, for $a, b \in A$.
- (3) The vector space A is spanned by elements of the form $\sqcap^{L}(b)a$, for $a, b \in A$.
- (4) The vector space A is spanned by elements of the form $a \sqcap^R (b)$, for $a, b \in A$.

If the comultiplication of A is right full, then also the following hold.

- (5) The vector space A is spanned by elements of the form $a\overline{\sqcap}^R(b)$, for $a, b \in A$.
- (6) The vector space A is spanned by elements of the form $\sqcap^R(b)a$, for $a, b \in A$.

If the comultiplication of A is left full, then also the following hold.

- (7) The vector space A is spanned by elements of the form $\overline{\sqcap}^L(b)a$, for $a, b \in A$.
- (8) The vector space A is spanned by elements of the form $a \sqcap^L (b)$, for $a, b \in A$.

Proof. Assertions (1)-(4) follow by the idempotency of A and [2, Lemma 3.7]. Assertions (5)-(8) follow from assertions (1) to (4) noting that by [2, Theorem 3.13] the ranges of \sqcap^R and $\overline{\sqcap}^R$ coincide whenever the comultiplication is right full; and the ranges of \sqcap^L and $\overline{\sqcap}^L$ coincide whenever the comultiplication is left full. \square

1.4. The antipode. For a regular weak multiplier bialgebra A, consider the idempotent map

 $(1.13) \quad G_1: A \otimes A \to A \otimes A, \qquad a \otimes bc \mapsto (a \otimes 1)F(1 \otimes bc) = (a \otimes 1)((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_4(c \otimes b))$

in (6.5) of [2] and its symmetric counterpart

(1.14) $G_2: A \otimes A \to A \otimes A, \quad ab \otimes c \mapsto ((A \otimes \overline{\sqcap}^L) \mathsf{tw} T_3(b \otimes a))(1 \otimes c)$

in (6.11) of [2]. By [2, Proposition 6.3] and its symmetric counterpart, they obey $E_iT_i = T_iG_i$, for $i \in \{1, 2\}$. By [2, Theorem 6.8], the maps T_i are 'weakly invertible' — in the sense that there are linear maps $R_1, R_2 : A \otimes A \to A \otimes A$ obeying $R_iT_i = G_i, T_iR_i = E_i$ and $R_iT_iR_i = R_i$ for $i \in \{1, 2\}$ — if and only if there is a linear map $S : A \to \mathbb{M}(A)$ — the so-called *antipode* — obeying the equalities in part (2) of [2, Theorem 6.8]. The resulting structure is in generality between arbitrary, and regular weak multiplier Hopf algebra in [19]. The conditions on S in part (2) of [2, Theorem 6.8] imply

(1.15)
$$\mu(S \otimes A)T_1 = \mu(\sqcap^R \otimes A), \qquad \mu(A \otimes S)T_2 = \mu(A \otimes \sqcap^L), \\ \mu(S \otimes A)E_1 = \mu(S \otimes A), \qquad \mu(A \otimes S)E_2 = \mu(A \otimes S),$$

see (6.14) in [2], but do not seem be equivalent to them. The antipode is anti-multiplicative by [2, Theorem 6.12], see also [19, Proposition 3.5], and anti-comultiplicative (in the multiplier sense) by [2, Corollary 6.16].

Lemma 1.6. Let A be a regular weak multiplier bialgebra possessing an antipode S. If the comultiplication is right full, then for any element $a \in A$ the following are equivalent.

(a)
$$a = 0$$
.

(b) aS(b) = 0 for all $b \in A$.

If the comultiplication is left full, then for any element $a \in A$ the following are equivalent.

(a')
$$a = 0$$
.
(b') $S(b)a = 0$ for all $b \in A$.

Proof. The implication (a) \Rightarrow (b) is trivial. If (b) holds then for all $b, c \in A$, using the implicit summation index notation $T_1(b \otimes c) = b^1 \otimes c^1$,

(1.16)
$$0 = aS(b^1)c^1 = a \sqcap^R (b)c,$$

where the last equality follows by (6.14) in [2]. By Lemma 1.5 (6), any element of A can be written as a linear combination of elements of the form $\sqcap^R(b)c$. Thus we conclude by the non-degeneracy of A from (1.16) that (a) holds. The equivalence (a') \Leftrightarrow (b') follows symmetrically.

Lemma 1.7. Consider a regular weak multiplier bialgebra A possessing an antipode S. For any $b, d \in A$, introduce the implicit summation index notation $T_2(d \otimes b) = d^2 \otimes b^2$. For any $a, b, c, d \in A$,

$$T_1(aS(b) \otimes S(d)c) = T_1(a \otimes S(d^2)c)(S(b^2) \otimes 1).$$

Proof. Using the multiplicativity of Δ and hence of its extension $\overline{\Delta} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$ in the second equality; the anti-comultiplicativity of S (cf. [2, Corollary 6.16]) in the third equality; and the anti-multiplicativity of S (cf. [2, Theorem 6.12]) in the fourth equality,

$$T_1(aS(b) \otimes S(d)c) = \Delta(aS(b))(1 \otimes S(d)c) = \Delta(a)\Delta S(b)(1 \otimes S(d)c)$$

= $\Delta(a)(\overline{S \otimes S})(\Delta(b)^{21})(1 \otimes S(d)c) = \Delta(a)(S(b^2) \otimes S(d^2))(1 \otimes c)$
= $T_1(a \otimes S(d^2)c)(S(b^2) \otimes 1).$

Lemma 1.8. For a regular weak multiplier bialgebra A with a left and right full comultiplication and possessing an antipode S, and for any elements $a, b \in A$, the following identities hold.

- (1) $T_3 \mathsf{tw} R_1(a \otimes b) = (S(a) \otimes 1) \Delta(b).$
- (2) $T_4 \mathsf{tw} R_2(a \otimes b) = \Delta(a)(1 \otimes S(b)).$

Proof. We only prove part (1), part (2) follows symmetrically. For any $a, c \in A$, introduce the notation $T_2(c \otimes a) = c^2 \otimes a^2$ where implicit summation is understood. For any $b, d \in A$,

$$\begin{aligned} (S(d) \otimes c)(T_3 \mathsf{tw} R_1(a \otimes b)) &= (S(d) \otimes 1)(T_3 \mathsf{tw}((c \otimes 1)R_1(a \otimes b))) \\ &= (S(d) \otimes 1)(T_3 \mathsf{tw}(((A \otimes S)T_2(c \otimes a))(1 \otimes b))) \\ &= (S(d) \otimes c^2) \Delta(S(a^2)b) \\ &= (S(d) \otimes c^2)((\overline{S \otimes S}) \Delta(a^2)^{21}) \Delta(b) \\ &= (\mathsf{tw}(\mu \otimes A)(A \otimes S \otimes S)(A \otimes T_1)(T_2 \otimes A)(c \otimes a \otimes d)) \Delta(b) \\ &= (\mathsf{tw}(\mu \otimes A)(A \otimes S \otimes S)(T_2 \otimes A)(A \otimes T_1)(c \otimes a \otimes d)) \Delta(b) \\ &= (\mathsf{tw}(\mu \otimes A)(A \otimes \sqcap^L \otimes S)(A \otimes T_1)(c \otimes a \otimes d)) \Delta(b) \\ &= (S\mu(\sqcap^L \otimes A)T_1(a \otimes d) \otimes c) \Delta(b) \\ &= (S(ad) \otimes c) \Delta(b) \\ &= (S(ad) \otimes c) \Delta(b). \end{aligned}$$

Simplifying by $S(d) \otimes c$ (cf. Lemma 1.6), we conclude the claim. In the first equality we used the left A-module property $T_3(b \otimes ca) = (1 \otimes c)T_3(b \otimes a)$. The second equality follows applying identity (6.3) in [2] to R_1 . The fourth equality follows by the multiplicativity of Δ and the anti-comultiplicativity of S (in the multiplier sense, cf. [2, Corollary 6.16]) and the fifth and the last equalities follow by the anti-multiplicativity of S (see [2, Theorem 6.12]). In the sixth equality we made use of the coassociativity axiom (iii) of weak multiplier bialgebra in Definition 1.1. The seventh equality follows by an identity in (6.14) in [2] and the eighth one follows by [2, Lemma 3.9 and Lemma 6.14]. The penultimate equality is a consequence of [2, Lemma 3.7 (3)].

2. Comodules and their morphisms

Before we can formulate the definition of *comodule* over a regular weak multiplier bialgebra, some preparation is needed. Let k be a field, V a k-vector space and A a (not necessarily unital) algebra over k. Then we can regard the tensor product vector space $V \otimes A$ as an $A \cong k \otimes A$ -bimodule via the left and right actions

$$(1 \otimes a)(v \otimes b) = v \otimes ab$$
 and $(v \otimes b)(1 \otimes a) = v \otimes ab$,

where 1 stands for the unit element of the base field.

Proposition 2.1. Let A be a regular weak multiplier bialgebra, let V be a vector space and let λ and ρ be linear maps $V \otimes A \to V \otimes A$ such that

(2.1)
$$(1 \otimes a)\lambda(v \otimes b) = \varrho(v \otimes a)(1 \otimes b)$$
 for all $v \in V$ and $a, b \in A$.

Then the following statements hold.

(1) The map λ is a morphism of right A-modules and ϱ is a morphism of left A-modules.

(2.a) $(V \otimes E_1)(\lambda \otimes A)\lambda^{13} = (\lambda \otimes A)\lambda^{13}$. (2.b) $(\rho \otimes A)\rho^{13}(V \otimes E_2) = (\rho \otimes A)\rho^{13}.$ (3) The following assertions are equivalent. (3.a) $(\lambda \otimes A)\lambda^{13}(V \otimes E_1) = (\lambda \otimes A)\lambda^{13}.$ (3.b) $(V \otimes E_2)(\varrho \otimes A)\varrho^{13} = (\varrho \otimes A)\varrho^{13}.$ (4) The following assertions are equivalent. (4.a) The equalities in part (3) hold and $(\lambda \otimes A)\lambda^{13}(V \otimes T_1) = (V \otimes T_1)(\lambda \otimes A).$ (2.2)(4.b) The equalities in part (3) hold and $(\lambda \otimes A)\lambda^{13}(V \otimes T_4) = (V \otimes T_4)\lambda^{13}.$ (2.3)(4.c) The equalities in part (3) hold and $(\lambda \otimes A)(V \otimes T_3)\rho^{13} = (V \otimes T_3)(\lambda \otimes A).$ (2.4)(4.d) The equalities in part (2) hold and $(\varrho \otimes A)\varrho^{13}(V \otimes T_3) = (V \otimes T_3)(\varrho \otimes A).$ (2.5)(4.e) The equalities in part (2) hold and $(\rho \otimes A)\rho^{13}(V \otimes T_2) = (V \otimes T_2)\rho^{13}.$ (2.6)(4.f) The equalities in part (2) hold and

(2) The following assertions are equivalent.

(2.7)
$$(\varrho \otimes A)(V \otimes T_1)\lambda^{13} = (V \otimes T_1)(\varrho \otimes A).$$

Proof. (1) Applying twice (2.1),

$$(1 \otimes a)\lambda(v \otimes bc) = \varrho(v \otimes a)(1 \otimes bc) = (1 \otimes a)\lambda(v \otimes b)(1 \otimes c),$$

for any $v \in V$ and $a, b, c \in A$. Hence by non-degeneracy of the left A-module $V \otimes A$ it follows that λ is a right A-module map. It is proven symmetrically that ρ is a left A-module map.

(2) Again, applying twice (2.1),

$$(2.8) \qquad ((\varrho \otimes A)\varrho^{13}(v \otimes a \otimes b))(1 \otimes c \otimes d) \\ = (1 \otimes a \otimes 1)(\lambda \otimes A)[\varrho^{13}(v \otimes c \otimes b)(1 \otimes 1 \otimes d)] \\ = (1 \otimes a \otimes b)((\lambda \otimes A)\lambda^{13}(v \otimes c \otimes d)),$$

for any $v \in V$ and $a, b, c, d \in A$. With this identity at hand,

$$((\varrho \otimes A)\varrho^{13}(V \otimes E_2)(v \otimes a \otimes b))(1 \otimes c \otimes d) = ((\varrho \otimes A)\varrho^{13}(v \otimes (a \otimes b)E))(1 \otimes c \otimes d)$$

$$\stackrel{(2.8)}{=} (1 \otimes (a \otimes b)E)((\lambda \otimes A)\lambda^{13}(v \otimes c \otimes d))$$

$$= (1 \otimes a \otimes b)((V \otimes E_1)(\lambda \otimes A)\lambda^{13}(v \otimes c \otimes d)).$$

Hence the implication $(2.a) \Rightarrow (2.b)$ follows by non-degeneracy of the right $A \otimes A$ -module $V \otimes A \otimes A$ and $(2.b) \Rightarrow (2.a)$ follows by non-degeneracy of the left $A \otimes A$ -module $V \otimes A \otimes A$.

(3) is proven analogously to (2).

 $(4.a) \Leftrightarrow (4.b)$ The equality in (2.2) is equivalent to

$$((\lambda \otimes A)\lambda^{13}(V \otimes T_1)(v \otimes a \otimes b))(1 \otimes c \otimes 1) = ((V \otimes T_1)(\lambda \otimes A)(v \otimes a \otimes b))(1 \otimes c \otimes 1),$$

for all $v \in V$ and $a, b, c \in A$. Using (1.5) and the right A-module map property of λ from part (1), the displayed equality is equivalent to

$$((\lambda \otimes A)\lambda^{13}(V \otimes T_4)(v \otimes c \otimes a))(1 \otimes 1 \otimes b) = ((V \otimes T_4)\lambda^{13}(v \otimes c \otimes a))(1 \otimes 1 \otimes b).$$

Its validity for all $v \in V$ and $a, b, c \in A$ is equivalent to (2.3). The equivalence (4.d) \Leftrightarrow (4.e) is proven symmetrically.

 $(4.a) \Leftrightarrow (4.c)$ The equality in (2.2) is equivalent also to

$$(1 \otimes 1 \otimes c)((\lambda \otimes A)\lambda^{13}(V \otimes T_1)(v \otimes a \otimes b)) = (1 \otimes 1 \otimes c)((V \otimes T_1)(\lambda \otimes A)(v \otimes a \otimes b)),$$

for all $v \in V$ and $a, b, c \in A$. Combining the first equality in axiom (ix) in Definition 1.1 with (2.1), the displayed equality is equivalent to

$$((\lambda \otimes A)(V \otimes T_3)\varrho^{13}(v \otimes a \otimes c))(1 \otimes 1 \otimes b) = ((V \otimes T_3)(\lambda \otimes A)(v \otimes a \otimes c))(1 \otimes 1 \otimes b).$$

Its validity for all $v \in V$ and $a, b, c \in A$ is equivalent to (2.4). The equivalence (4.d) \Leftrightarrow (4.f) is proven symmetrically.

 $(4.a) \Rightarrow (4.d)$ First we show that (4.a) implies (2.a). By axiom (vi) of weak multiplier bialgebra in Definition 1.1, for any elements $a, b \in A$ there exist finitely many elements $p^i, q^i, r^i \in A$ such that $E(a \otimes b) = \sum_i \Delta(p^i)(q^i \otimes r^i)$. In terms of these elements and for any $v \in V$, omitting for brevity the summation symbols,

$$(2.9) \quad (\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b) \stackrel{(3.a)}{=} \quad (\lambda \otimes A)\lambda^{13}(v \otimes E(a \otimes b)) \\ = \quad (\lambda \otimes A)\lambda^{13}(v \otimes T_1(p^i \otimes r^i)(q^i \otimes 1)) \\ \stackrel{\text{part } (1)}{=} \quad ((\lambda \otimes A)\lambda^{13}(V \otimes T_1)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1) \\ \stackrel{(2.2)}{=} \quad ((V \otimes T_1)(\lambda \otimes A)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1).$$

Using this identity in the first and the last equalities, the right A-module map property of E_1 in the second equality and $E_1T_1 = T_1$ in the third one,

$$\begin{aligned} (V \otimes E_1)(\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b) &= (V \otimes E_1)[((V \otimes T_1)(\lambda \otimes A)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1)] \\ &= ((V \otimes E_1)(V \otimes T_1)(\lambda \otimes A)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1) \\ &= ((V \otimes T_1)(\lambda \otimes A)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1) \\ &= (\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b). \end{aligned}$$

Finally — using the same notation as above and the implicit summation index notation $\varrho(v \otimes a) = v^{\varrho} \otimes a^{\varrho}, \lambda(v \otimes p) = v^{\lambda} \otimes p^{\lambda}$ — we show that (4.a) implies (2.5). Take any $v \in V$ and $a, b, c, d \in A$. Then

$$\begin{aligned} ((V \otimes T_3)(\varrho \otimes A)(v \otimes c \otimes d))(1 \otimes a \otimes b) \\ &= ((V \otimes E_2)(V \otimes T_3)(\varrho \otimes A)(v \otimes c \otimes d))(1 \otimes a \otimes b) \\ &= ((V \otimes T_3)(\varrho \otimes A)(v \otimes c \otimes d))(1 \otimes E(a \otimes b)) \\ &= v^{\varrho} \otimes (1 \otimes d)\Delta(c^{\varrho}p^i)(q^i \otimes r^i) \\ &= v^{\lambda} \otimes (1 \otimes d)\Delta(cp^{i\lambda})(q^i \otimes r^i) \\ &= (1 \otimes T_3(c \otimes d))((V \otimes T_1)(\lambda \otimes A)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1). \end{aligned}$$

In the first equality we used $E_2T_3 = T_3$ and in the second equality we used axiom (i) of weak multiplier bialgebra in Definition 1.1. In the third and the last equalities we used

the multiplicativity of Δ and in the penultimate one we applied (2.1). On the other hand,

$$\begin{aligned} ((\varrho \otimes A)\varrho^{13}(V \otimes T_3) & (v \otimes c \otimes d))(1 \otimes a \otimes b) \\ \stackrel{(3.b)}{=} & ((V \otimes E_2)(\varrho \otimes A)\varrho^{13}(V \otimes T_3)(v \otimes c \otimes d))(1 \otimes a \otimes b) \\ \stackrel{(i)}{=} & ((\varrho \otimes A)\varrho^{13}(V \otimes T_3)(v \otimes c \otimes d))(1 \otimes E(a \otimes b)) \\ & = & ((\varrho \otimes A)\varrho^{13}(V \otimes T_3)(v \otimes c \otimes d))(1 \otimes T_1(p^i \otimes r^i)(q^i \otimes 1)) \\ \stackrel{(2.8)}{=} & (1 \otimes T_3(c \otimes d))((\lambda \otimes A)\lambda^{13}(V \otimes T_1)(v \otimes p^i \otimes r^i))(1 \otimes q^i \otimes 1). \end{aligned}$$

The expressions we obtained are equal by (2.2) so that (2.5) holds by the non-degeneracy of the right $A \otimes A$ -module $V \otimes A \otimes A$. The converse implication (4.d) \Rightarrow (4.a) follows symmetrically.

We define a right comodule over a regular weak multiplier bialgebra A as a triple (V, λ, ϱ) satisfying (2.1) and the equivalent conditions in Proposition 2.1 (4). That is, we propose the following.

Definition 2.2. A right comodule over a regular weak multiplier bialgebra A is a triple consisting of a vector space V and linear maps $\lambda, \varrho : V \otimes A \to V \otimes A$ satisfying

(2.10)
$$(1 \otimes a)\lambda(v \otimes b) = \varrho(v \otimes a)(1 \otimes b), \quad \forall v \in V, \ a, b \in A$$

together with

(2.11)
$$(\lambda \otimes A)\lambda^{13}(V \otimes E_1) = (\lambda \otimes A)\lambda^{13}$$

(2.12)
$$(\lambda \otimes A)\lambda^{13}(V \otimes T_1) = (V \otimes T_1)(\lambda \otimes A).$$

Equivalently, λ and ρ satisfy (2.10) and

(2.13)
$$(\varrho \otimes A)\varrho^{13}(V \otimes E_2) = (\varrho \otimes A)\varrho^{13}$$

(2.14)
$$(\varrho \otimes A)\varrho^{13}(V \otimes T_3) = (V \otimes T_3)(\varrho \otimes A).$$

Remark 2.3. If A is a usual weak bialgebra as in [4, 11] possessing an algebraic unit 1, and V is a vector space, then a pair of linear maps $\lambda, \varrho : V \otimes A \to V \otimes A$ satisfying (2.10) is equivalent to a linear map

$$\tau: V \to V \otimes A, \qquad v \mapsto \lambda(v \otimes 1) \stackrel{(2.10)}{=} \varrho(v \otimes 1).$$

For this, the implicit summation index notation $\tau(v) = v^0 \otimes v^1$ is widely used in the literature. In this situation the normalization condition (2.11) translates to

$$((\tau \otimes A)\tau(v))(1 \otimes \Delta(1)) = (\tau \otimes A)\tau(v), \qquad \forall v \in V$$

and the coassociativity condition (2.12) translates to

$$((\tau \otimes A)\tau(v))(1 \otimes \Delta(1)) = (V \otimes \Delta)\tau(v), \quad \forall v \in V.$$

These conditions together are equivalent to the usual coassociativity condition $(\tau \otimes A)\tau = (V \otimes \Delta)\tau$.

Comodules over coalgebras (so in particular over (weak) bialgebras) are usually required to be counital as well. Later we will require an additional condition replacing it; see Definition 4.2 and Remark 4.6.

Example 2.4. For any regular weak multiplier bialgebra A, there is a right A-comodule (A, T_1, T_3) .

Proof. Condition (2.10) holds by the first equality in axiom (ix) in Definition 1.1. For any $b, c \in A$, introduce the notation $T_1(b \otimes c) =: b^1 \otimes c^1$, where implicit summation is understood. Then for any $a, b, c, d \in A$,

$$(d \otimes 1 \otimes 1)((T_1 \otimes A)T_1^{13}(A \otimes T_1)(a \otimes b \otimes c))$$

= $((T_2 \otimes A)(A \otimes T_1)(d \otimes a \otimes c^1))(1 \otimes b^1 \otimes 1)$
= $((A \otimes T_1)(T_2 \otimes A)(d \otimes a \otimes c^1))(1 \otimes b^1 \otimes 1)$
= $(A \otimes T_1)(T_2(d \otimes a)(1 \otimes b) \otimes c)$
= $(d \otimes 1 \otimes 1)((A \otimes T_1)(T_1 \otimes A)(a \otimes b \otimes c)).$

In the first and the last equalities we used (1.2), in the second equality we used the coassociativity axiom (iii), and in the penultimate equality we used axiom (v) of weak multiplier bialgebra in Definition 1.1. By non-degeneracy of the left A-module $A \otimes A \otimes A$, this proves the coassociativity condition (2.12). It remains to check the normalization condition (2.11). To this end, note that for any $a, b, d \in A$,

$$(2.15) T_1(a \otimes \overline{\sqcap}^R(b)d) = \Delta(a)(1 \otimes \overline{\sqcap}^R(b)d) = \Delta(a\overline{\sqcap}^R(b))(1 \otimes d) = T_1(a\overline{\sqcap}^R(b) \otimes d)$$

and

$$(2.16) \quad T_1(a \otimes d)(\overline{\sqcap}^R(b) \otimes 1) = \Delta(a)(\overline{\sqcap}^R(b) \otimes d) = \Delta(a)(1 \otimes \sqcap^L(b)d) = T_1(a \otimes \sqcap^L(b)d),$$

where the second equalities follow by Lemma 3.3 and Lemma 3.9 in [2], respectively. Combining these equalities, for any $a, b, c, d \in A$

(2.17)
$$(T_1 \otimes A)T_1^{13}(a \otimes \overline{\sqcap}^R(b)d \otimes c) = (T_1 \otimes A)T_1^{13}(a \otimes d \otimes \sqcap^L(b)c).$$

With this identity at hand, and using again the notation $T_1(b \otimes c) =: b^1 \otimes c^1$ (with implicit summation understood),

$$(T_1 \otimes A)T_1^{13}(A \otimes E_1)(a \otimes d \otimes bc) = (T_1 \otimes A)T_1^{13}(a \otimes \overline{\sqcap}^R(b^1)d \otimes c^1)$$
$$\stackrel{(2.17)}{=} (T_1 \otimes A)T_1^{13}(a \otimes d \otimes \sqcap^L(b^1)c^1)$$
$$= (T_1 \otimes A)T_1^{13}(a \otimes d \otimes bc).$$

The first equality follows by identity (2.3) in [2], and the last equality follows by Lemma 3.7 (3) in [2]. Using that the algebra A is idempotent, this proves the normalization condition (2.11).

Example 2.5. For any regular weak multiplier bialgebra A with a right full comultiplication, denote by R the coinciding range of the maps $\sqcap^R : A \to \mathbb{M}(A)$ and $\overline{\sqcap}^R : A \to \mathbb{M}(A)$ (cf. [2, Theorem 3.13]). Then there is a right A-comodule $(R, r \otimes a \mapsto E(1 \otimes ra), r \otimes a \mapsto (1 \otimes ar)E)$.

Proof. Both maps $\lambda : r \otimes a \mapsto E(1 \otimes ra)$ and $\varrho : r \otimes a \mapsto (1 \otimes ar)E$ have their range in $R \otimes A$ by identities (2.3) and (3.4) in [2], respectively. By [2, Lemma 3.3], we can write equivalently

$$\lambda(r \otimes a) = (1 \otimes r)E(1 \otimes a)$$
 and $\varrho(r \otimes a) = (1 \otimes a)E(1 \otimes r)$

so the compatibility condition (2.10) evidently holds. For any $r \in R$ and $a, b \in A$,

$$(2.18) \quad (\lambda \otimes A)\lambda^{13}(r \otimes a \otimes b) = (\lambda \otimes A)(E^{13}(1 \otimes a \otimes rb)) = (E \otimes 1)(1 \otimes E)(1 \otimes a \otimes rb).$$

Thus

$$(R \otimes T_1)(\lambda \otimes A)(r \otimes a \otimes b) = (R \otimes T_1)((E \otimes 1)(1 \otimes ra \otimes b))$$

$$= (E \otimes 1)(1 \otimes E)(1 \otimes T_1(ra \otimes b))$$

$$= (E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes r)(1 \otimes T_1(a \otimes b))$$

$$\stackrel{(2.18)}{=} (\lambda \otimes A)\lambda^{13}(R \otimes T_1)(r \otimes a \otimes b).$$

The second equality follows by axiom (vii) of weak multiplier bialgebra in Definition 1.1 and $E_1T_1 = T_1$. The third equality follows by [2, Lemma 3.3]. This proves the coassociativity condition (2.12). Finally,

$$\begin{aligned} (\lambda \otimes A)\lambda^{13}(R \otimes E_1)(r \otimes a \otimes b) &\stackrel{(2.18)}{=} & (E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes r)(1 \otimes E(a \otimes b)) \\ &= & (E \otimes 1)(1 \otimes E)(1 \otimes a \otimes rb) \\ \stackrel{(2.18)}{=} & (\lambda \otimes A)\lambda^{13}(r \otimes a \otimes b). \end{aligned}$$

The second equality follows by [2, Lemma 3.3] and $E^2 = E$. This proves the normalization condition (2.11).

In the following proposition the compatibility of comodules with the counit is studied.

Proposition 2.6. Let A be a regular weak multiplier bialgebra and let (V, λ, ϱ) be a right A-comodule. Then the following equalities hold.

- (1) $(V \otimes \epsilon \otimes A)(\lambda \otimes A)\lambda^{13} = \lambda(V \otimes \mu(\sqcap^L \otimes A)).$
- (2) $(V \otimes \epsilon \otimes A)\lambda^{13}(\lambda \otimes A) = \lambda(V \otimes \mu(\overline{\sqcap}^R \otimes A)).$
- (3) $(V \otimes A \otimes \epsilon)(\varrho \otimes A)\varrho^{13} = \varrho(V \otimes \mu(A \otimes \sqcap^R)).$
- (4) $(V \otimes A \otimes \epsilon) \varrho^{13}(\varrho \otimes A) = \varrho(V \otimes \mu(A \otimes \overline{\sqcap}^L)).$

Proof. By axiom (vi) of weak multiplier bialgebra in Definition 1.1, for any $a, b \in A$ there exist finitely many elements $p^i, q^i, r^i \in A$ such that $E(a \otimes b) = \sum_i \Delta(p^i)(q^i \otimes r^i) = \sum_i T_1(p^i \otimes r^i)(q^i \otimes 1)$. In terms of these elements, omitting the summation symbols for brevity, for any $v \in V$ the equality (2.9) holds. Equivalently, by (1.5),

(2.19)
$$(\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b) = ((V \otimes T_4)\lambda^{13}(v \otimes q^i \otimes p^i))(1 \otimes 1 \otimes r^i).$$

The identities (2.19) and (2.9) are used to prove parts (1) and (2), respectively.

(1) Using the above elements $p^i, q^i, r^i \in A$ associated to $a, b \in A$, for any $v \in V$

$$(V \otimes \epsilon \otimes A)(\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b) \stackrel{(2.19)}{=} ((V \otimes \epsilon \otimes A)(V \otimes T_4)\lambda^{13}(v \otimes q^i \otimes p^i))(1 \otimes r^i)$$
$$\stackrel{(1.9)}{=} \lambda(v \otimes p^i)(1 \otimes \sqcap^L(q^i)r^i)$$
$$= \lambda(v \otimes p^i \sqcap^L(q^i)r^i) = \lambda(v \otimes \sqcap^L(a)b).$$

The penultimate equality follows by Proposition 2.1 (1) and the last one follows by

$$p^{i} \sqcap^{L} (q^{i}) r^{i} \stackrel{(1.9)}{=} (\epsilon \otimes A) (\Delta(p^{i})(q^{i} \otimes r^{i})) = (\epsilon \otimes A) (E(a \otimes b)) \stackrel{(1.9)}{=} \sqcap^{L} (a) b$$

(2) Using the same notation as above,

$$(V \otimes A \otimes \epsilon)(\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b) \stackrel{(2.9)}{=} ((V \otimes A \otimes \epsilon)(V \otimes T_1)(\lambda(v \otimes p^i) \otimes r^i))(1 \otimes q^i)$$
$$\stackrel{(1.8)}{=} \lambda(v \otimes p^i)(1 \otimes \overline{\sqcap}^R(r^i)q^i)$$
$$= \lambda(v \otimes p^i \overline{\sqcap}^R(r^i)q^i) = \lambda(v \otimes \overline{\sqcap}^R(b)a).$$

The penultimate equality follows by Proposition 2.1 (1) and the last equality follows by

$$p^{i}\overline{\sqcap}^{R}(r^{i})q^{i} \stackrel{(1.8)}{=} (A \otimes \epsilon)(\Delta(p^{i})(q^{i} \otimes r^{i})) = (A \otimes \epsilon)(E(a \otimes b)) \stackrel{(1.8)}{=} \overline{\sqcap}^{R}(b)a.$$

The remaining assertions follow symmetrically.

Lemma 2.7. For any regular weak multiplier bialgebra A, any right A-comodule (V, λ, ϱ) and any $v \in V$ and $a, b \in A$, the following statements hold.

- (1) $(V \otimes A \otimes \epsilon)\lambda^{13}(\rho \otimes A)(v \otimes a \otimes b) = \rho(v \otimes a)(1 \otimes \sqcap^L(b)).$
- (2) $(V \otimes A \otimes \epsilon) \varrho^{13}(\lambda \otimes A)(v \otimes a \otimes b) = (1 \otimes \overline{\sqcap}^L(b))\lambda(v \otimes a).$

Proof. We only prove part (1), part (2) is proven symmetrically. For any $v \in V$ and $a, b, c \in A$,

$$((V \otimes A \otimes \epsilon)\lambda^{13}(\varrho \otimes A)(v \otimes a \otimes b))(1 \otimes c)$$

$$\stackrel{(2.10)}{=} (1 \otimes a)((V \otimes A \otimes \epsilon)\lambda^{13}(\lambda \otimes A)(v \otimes c \otimes b))$$

$$= (1 \otimes a)\lambda(v \otimes \sqcap^{L}(b)c) \stackrel{(2.10)}{=} \varrho(v \otimes a)(1 \otimes \sqcap^{L}(b)c),$$

from which we conclude by the non-degeneracy of the right A-module $V \otimes A$. The second equality follows by Proposition 2.6 (1).

In order to define morphisms between comodules, we need the following lemma. Since for any vector space V, the left and right modules $V \otimes A$ over a weak multiplier bialgebra A are non-degenerate, it follows immediately from (2.10).

Lemma 2.8. Let A be a regular weak multiplier bialgebra and let (V, λ, ϱ) and (V', λ', ϱ') be right A-comodules. For any linear map $f : V \to V'$, the following assertions are equivalent.

(a)
$$\lambda'(f \otimes A) = (f \otimes A)\lambda$$
.
(b) $\varrho'(f \otimes A) = (f \otimes A)\varrho$.

We define the morphisms between right comodules over a regular weak multiplier bialgebra as linear maps satisfying the equivalent conditions in Lemma 2.8. That is, we put the following.

Definition 2.9. A morphism of right comodules over a regular weak multiplier bialgebra A is a linear map $f: V \to V'$ such that any (hence both) of the following diagrams commutes.

$V \otimes A \xrightarrow{\lambda}$	$\rightarrow V \otimes A$	$V \otimes A \xrightarrow{\varrho}$	$\rightarrow V \otimes A$
$f \otimes A$	$f \otimes A$	$f \otimes A$	$f \otimes A$
$V' \otimes A - \frac{1}{\lambda'}$	$\rightarrow V' \otimes A$	$V' \otimes A - \underline{\varrho'}$	$\rightarrow V' \otimes A$

Left comodules are defined and treated symmetrically. One of several equivalent definitions is the following.

Definition 2.10. A *left comodule* over a regular weak multiplier bialgebra A is a triple consisting of a vector space V and linear maps $\lambda, \varrho : A \otimes V \to A \otimes V$ obeying the following conditions. Compatibility, in the sense that

(2.20)
$$(a \otimes 1)\lambda(b \otimes v) = \varrho(a \otimes v)(b \otimes 1)$$
 for any $v \in V$ and $a, b \in A$;

normalization in the sense

(2.21)
$$(A \otimes \lambda)\lambda^{13}(E_1 \otimes V) = (A \otimes \lambda)\lambda^{13}$$

and coassociativity in the sense

(2.22)
$$(A \otimes \lambda)\lambda^{13}(T_1 \otimes V) = (T_1 \otimes V)\lambda^{13}.$$

3. INTEGRALS ON WEAK MULTIPLIER BIALGEBRAS

Integrals on unital (weak) bialgebras can be interpreted as comodule maps from the regular comodule to the comodule defined on the base object. Our aim in this section is to give a similar interpretation of integrals on regular weak multiplier bialgebras with a right full comultiplication (when the comodule on the base object is available). A very general notion of *integral* is studied in [9]. Applying it to a regular weak multiplier bialgebra with right full comultiplication, it yields the notion in forthcoming Proposition 3.2.

Consider a regular weak multiplier bialgebra A over a field k. By the first axiom in (ix) in Definition 1.1, any linear map $\psi: A \to k$ determines a linear map $(-) \leftarrow \psi: A \to \mathbb{M}(A)$ by the prescriptions

$$(a \leftarrow \psi)b = (\psi \otimes A)T_1(a \otimes b)$$
 and $b(a \leftarrow \psi) = (\psi \otimes A)T_3(a \otimes b),$

see [20]. For this map $(-) \leftarrow \psi$, the following holds.

Lemma 3.1. Let A be a regular weak multiplier bialgebra over a field k. For any linear map $\psi : A \to k$ and for any $a, b \in A$,

$$((-) \leftarrow \psi \otimes A)T_3(a \otimes b) = (1 \otimes b)\overline{\Delta}(a \leftarrow \psi).$$

Proof. By axiom (vi) of weak multiplier bialgebra in Definition 1.1, for any fixed elements $b, c \in A$ there exist finitely many elements $p^i, q^i, r^i \in A$ such that $(c \otimes b)E = \sum_i (p^i \otimes q^i)\Delta(r^i) = \sum_i (p^i \otimes 1)T_3(r^i \otimes q^i)$. In terms of these elements (omitting the summation symbols for brevity), for any $a \in A$,

$$\begin{aligned} (c \otimes 1)[((-) \leftarrow \psi \otimes A)T_3(a \otimes b)] &= (\psi \otimes A \otimes A)(T_3 \otimes A)T_3^{13}(a \otimes c \otimes b) \\ &= (\psi \otimes A \otimes A)(T_3 \otimes A)T_3^{13}(A \otimes E_2)(a \otimes c \otimes b) \\ &= (\psi \otimes A \otimes A)(T_3 \otimes A)T_3^{13}(a \otimes (p^i \otimes 1)T_3(r^i \otimes q^i)) \\ &= (p^i \otimes 1)[(\psi \otimes A \otimes A)(T_3 \otimes A)T_3^{13}(A \otimes T_3)(a \otimes r^i \otimes q^i)] \\ &= (p^i \otimes 1)[(\psi \otimes A \otimes A)(A \otimes T_3)(T_3 \otimes A)(a \otimes r^i \otimes q^i)] \\ &= (p^i \otimes 1)T_3(r^i(a \leftarrow \psi) \otimes q^i) = (p^i \otimes 1)T_3(r^i \otimes q^i)\overline{\Delta}(a \leftarrow \psi) \\ &= (c \otimes b)E\overline{\Delta}(a \leftarrow \psi) = (c \otimes b)\overline{\Delta}(a \leftarrow \psi), \end{aligned}$$

from which we conclude by simplifying with c. The second equality follows by part (2.b), and the fifth equality follows by part (4.d) of Proposition 2.1, applied to the A-comodule (A, T_1, T_3) in Example 2.4.

Proposition 3.2. For a regular weak multiplier bialgebra A over a field k with right full comultiplication; and for a linear map $\psi : A \to k$, the following assertions are equivalent.

- (a) $(\psi \otimes A)T_1 = (\psi \otimes A)G_1$ (where G_1 is the idempotent map in (1.13)).
- (b) $(\psi \otimes A)T_3 = (\psi \otimes A)[(-)F]$ (where $F \in \mathbb{M}(A \otimes A)$ is as in (1.10)).
- (c) $\overline{\Delta}(a \leftarrow \psi) = (1 \otimes (a \leftarrow \psi))E$, for all $a \in A$.
- (d) $a \leftarrow \psi \in \sqcap^R(A)$, for all $a \in A$.

If these equivalent assertions hold then we term ψ a right integral on A.

Proof. By the form of G_1 in (6.1) of [2] (cf. (1.13)) and by the non-degeneracy of $A \otimes A$ as a left and right A-module via multiplication in the second factor, both assertions (a) and (b) are equivalent to

$$(\psi \otimes A)[(1 \otimes c)\Delta(a)(1 \otimes b)] = (\psi \otimes A)[(a \otimes c)F(1 \otimes b)] \qquad \forall a, b, c \in A,$$

hence they are equivalent also to each other.

Assertion (d) implies (c) by (3.7) in [2].

If (a) holds then $a \leftarrow \psi = (\psi \otimes A)[(a \otimes 1)F]$ is an element of $\sqcap^R(A)$ for any $a \in A$ by [2, Proposition 4.3 (1)] (cf. (1.10)) and the idempotency of A, proving that (d) holds. The implication (c) \Rightarrow (b) can be found in [9, Proposition 2.7]: For any $a, b, c \in A$,

$$\begin{aligned} (\psi \otimes A \otimes A)(T_3 \otimes A)T_3^{13}(a \otimes c \otimes b) &= (c \otimes 1)[((-) \leftarrow \psi \otimes A)T_3(a \otimes b)] \\ &= (c \otimes b)\overline{\Delta}(a \leftarrow \psi) \\ &\stackrel{(c)}{=} (c \otimes b(a \leftarrow \psi))E \\ &= (\psi \otimes A \otimes A)[T_3^{13}(a \otimes c \otimes b)(1 \otimes E)] \\ &= (\psi \otimes A \otimes A)[T_3^{13}(a \otimes c \otimes b)](F \otimes 1)]. \end{aligned}$$

From this we conclude by using that the comultiplication is right full; that is, any element of A is a linear combination of elements of the form $(A \otimes \omega)T_3(a \otimes b)$, for $a, b \in A$ and $\omega \in \text{Lin}(A, k)$. The second equality follows by Lemma 3.1 and the last one follows since for any $a, p, q, r \in A$,

$$E^{13}(1 \otimes E)(r \otimes pq \otimes 1) = E^{13}[r \otimes (A \otimes \sqcap^L)T_4(q \otimes p)]$$

= $E^{13}[((\overline{\sqcap^R} \otimes A)\mathsf{tw}T_4(q \otimes p))(r \otimes 1) \otimes 1]$
= $E^{13}(F \otimes 1)(r \otimes pq \otimes 1),$

— where in the first equality we applied (3.4) in [2], in the second equality we used [2, Lemma 3.9], and in the last one we made use of [2, Proposition 4.3 (1)] (cf. (1.10)) — hence by $T_3 = E_2 T_3$, $T_3^{13}(a \otimes c \otimes b)(1 \otimes E) = T_3^{13}(a \otimes c \otimes b)(F \otimes 1)$, for any $a, b, c \in A$. \Box

The aim of this section is to prove an isomorphism between the vector space of right integrals on A; and the space of homomorphisms from the comodule A in Example 2.4 to the comodule R in Example 2.5 — assuming that the comultiplication of A is right full.

Let A be a regular weak multiplier bialgebra over a field k with a right full comultiplication. Recall that, by Theorem 3.13 and Theorem 4.6 in [2], the coinciding range of the maps \sqcap^R and $\overline{\sqcap}^R : A \to \mathbb{M}(A)$ — the so-called base algebra R — is a co-separable co-Frobenius coalgebra over k and it is a firm k-subalgebra of $\mathbb{M}(A)$. Applying this fact, we can relate the k-dual and the R-dual of any firm right R-module.

Proposition 3.3. Let A be a regular weak multiplier bialgebra over a field k with right full comultiplication, and let R be the coinciding range of the maps \sqcap^R and $\overline{\sqcap}^R : A \to \mathbb{M}(A)$. For any firm right R-module M, the vector space $\operatorname{Lin}(M, k)$ of linear maps $M \to k$ is isomorphic to the vector space $\operatorname{Hom}_R(M, R)$ of R-module maps $M \to R$.

Proof. In terms of the comultiplication $\delta : R \to R \otimes R$ in [2, Proposition 4.3 (3-4)] (cf. (1.11)) and the counit $\varepsilon : R \to k$ in [2, Proposition 4.1] (cf. (1.12)), consider the maps

(3.1) $\operatorname{Hom}_R(M, R) \to \operatorname{Lin}(M, k), \qquad \Psi \mapsto \varepsilon \Psi \quad \text{and}$

(3.2) $\operatorname{Lin}(M,k) \to \operatorname{Hom}_R(M,R), \qquad \psi \mapsto [mr \mapsto (\psi \otimes R)((m \otimes 1)\delta(r))].$

Since M is a firm right R-module by assumption, any element of M can be written as a linear combination of elements of the form mr for $m \in M$ and $r \in R$. So it is enough to define the image of ψ in (3.2) on such elements. In order to see that it is a well-defined linear map, we need to show that it maps a zero element to zero. So assume that for some elements $m^i \in M$ and $r^i \in R$, $\sum_i m^i r^i = 0$. Then omitting the summation symbol for brevity, and using in the second equality that δ is a morphism of R-bimodules, it follows for any $s \in R$ that

$$0 = (\psi \otimes R)[(m^i r^i \otimes 1)\delta(s)] = (\psi \otimes R)[(m^i \otimes 1)\delta(r^i)(1 \otimes s)] = (\psi \otimes R)[(m^i \otimes 1)\delta(r^i)]s.$$

Since the multiplication in R is non-degenerate by [2, Theorem 4.6 (2)], this proves that the image of ψ in (3.2) is a well-defined linear map. It is a homomorphism of right R-modules since δ is so. We claim that the maps (3.1) and (3.2) establish the stated isomorphism.

Take a linear map $\psi: M \to k$ and apply to it the constructions in (3.2) and in (3.1). For any $m \in M$ and $r \in R$, the resulting map takes $mr \in M$ to

$$\varepsilon(\psi \otimes R)[(m \otimes 1)\delta(r)] = \psi[m((R \otimes \varepsilon)\delta(r))] = \psi(mr).$$

We conclude by the surjectivity of the *R*-action on *M* that we re-obtained ψ . In the last equality we used that by [2, Theorem 4.4], ε is the counit of the comultiplication δ .

Take next an *R*-module map $\Psi: M \to R$ and apply to it the constructions in (3.1) and in (3.2). For any $m \in M$ and $r \in R$, the resulting *R*-module map takes $mr \in M$ to

$$\begin{aligned} (\varepsilon \Psi \otimes R)[(m \otimes 1)\delta(r)] &= (\varepsilon \otimes R)[(\Psi(m) \otimes 1)\delta(r)] \\ &= (\varepsilon \otimes R)\delta(\Psi(m)r) = \Psi(m)r = \Psi(mr). \end{aligned}$$

So by surjectivity of the *R*-action on *M* we re-obtained Ψ . In the first and the last equalities we used the right *R*-module map property of Ψ , the second and the third equalities follow by left *R*-linearity and the counitality of δ , respectively.

Proposition 3.3 provides in particular a vector space isomorphism between Lin(A, k)and $Hom_R(A, R)$, for any regular weak multiplier bialgebra A over a field k with a right full comultiplication, and its base algebra R. In this particular case, using the multiplier F on $A \otimes A$ in [2, Proposition 4.3 (1)] (cf. (1.10)), the map in (3.2) can be rewritten in the equivalent form

$$\operatorname{Lin}(A,k) \to \operatorname{Hom}_R(A,R), \quad \psi \mapsto (\psi \otimes R)[(-\otimes 1)F],$$

see [2, Proposition 4.3] (or (1.11)).

Proposition 3.4. Let A be a regular weak multiplier bialgebra with right full comultiplication. Any comodule homomorphism Ψ from the comodule A in Example 2.4 to the comodule R in Example 2.5, is a homomorphism of right R-modules. Moreover, for the counit ε of R in [2, Proposition 4.1] (cf. (1.12)), the composite $\varepsilon \Psi$ is a right integral on A.

Proof. First we show that Ψ is a homomorphism of right *R*-modules. Since Ψ is a comodule map, it renders commutative the diagram

$$\begin{array}{c|c} A \otimes A \xrightarrow{\Psi \otimes A} R \otimes A \\ T_1 \bigvee & \bigvee r \otimes a \mapsto (1 \otimes r) E(1 \otimes a) \\ A \otimes A \xrightarrow{\Psi \otimes A} R \otimes A. \end{array}$$

That is, $(\Psi \otimes A)T_1(a \otimes b) = (1 \otimes \Psi(a))E(1 \otimes b)$ for any $a, b \in A$. Applying $\varepsilon \otimes A$ to both sides of this equality, we conclude that

(3.3)
$$(\varepsilon \Psi \otimes A)T_1(a \otimes b) = \Psi(a)((\varepsilon \otimes A)(E(1 \otimes b))) = \Psi(a)b.$$

The last equality follows by the idempotency of A and

$$(\varepsilon \otimes A)(E(1 \otimes bc)) = (\varepsilon \overline{\sqcap}^R \otimes A)T_1(b \otimes c) = (\epsilon \otimes A)T_1(b \otimes c) = bc,$$

for any $b, c \in A$, where we applied identity (2.3) in [2], Lemma 4.5 (3) in [2] and the counitality axiom (iv) of weak multiplier bialgebra in Definition 1.1. Therefore,

$$\Psi(a\overline{\sqcap}^R(c))b \stackrel{(3.3)}{=} (\varepsilon\Psi\otimes A)T_1(a\overline{\sqcap}^R(c)\otimes b) \stackrel{(2.15)}{=} (\varepsilon\Psi\otimes A)T_1(a\otimes\overline{\sqcap}^R(c)b) \stackrel{(3.3)}{=} \Psi(a)\overline{\sqcap}^R(c)b$$

for any $a, b, c \in A$, so that Ψ is a right *R*-module map.

Applying (3.3) in the second equality, it follows for any $a, b \in A$ that

$$(a \leftarrow \varepsilon \Psi)b = (\varepsilon \Psi \otimes A)T_1(a \otimes b) = \Psi(a)b$$

so that $a \leftarrow \varepsilon \Psi = \Psi(a)$ is an element of R. Thus $\varepsilon \Psi$ is a right integral on A by Proposition 3.2.

Proposition 3.5. For a regular weak multiplier bialgebra A with right full comultiplication, consider the multiplier F on $A \otimes A$ in [2, Proposition 4.3 (1)] (cf. (1.10)). For any right integral ψ on A, the map

$$(3.4) A \to R, a \mapsto (\psi \otimes R)((a \otimes 1)F)$$

is a homomorphism from the comodule in Example 2.4 to the comodule in Example 2.5.

Proof. By Lemma 1.5 (4), the right *R*-action on *A* is surjective. Since *R* has local units by [2, Theorem 4.6 (2)], this proves that *A* is a firm right *R*-module and thus (3.4) — being a particular instance of (3.2) — is a well-defined (right *R*-module) map by (the proof of) Proposition 3.3.

In light of the form of the structure map ρ in Example 2.5, (3.4) is a comodule map provided that for all $a, b, c \in A$,

$$[(\psi \otimes A)((-\otimes c)F) \otimes A]T_3(a \otimes b) = [c \otimes (\psi \otimes A)((a \otimes b)F)]E.$$

By Proposition 3.2 (b), this is equivalent to

$$[(\psi \otimes A)T_3(-\otimes c) \otimes A]T_3(a \otimes b) = [c \otimes (\psi \otimes A)T_3(a \otimes b)]E,$$

hence to

$$[c((-) \leftarrow \psi) \otimes A]T_3(a \otimes b) = [c \otimes b(a \leftarrow \psi)]E$$

This equality holds by Proposition 3.2 (c) and Lemma 3.1.

Combining Proposition 3.3 with Proposition 3.4 and Proposition 3.5, we proved the following.

Theorem 3.6. Consider a regular weak multiplier bialgebra A with right full comultiplication. The constructions in Proposition 3.4 and in Proposition 3.5 establish a vector space isomorphism between the right integrals on A on one hand, and the homomorphisms from the comodule in Example 2.4 to the comodule in Example 2.5 on the other hand.

20

In this section we consider a particular class of comodules over a regular weak multiplier bialgebra A that we call *full* comodules. Assuming that also the comultiplication of A is right full, full right comodules are shown to carry natural bimodule structures over the base algebra R of A. This results in a faithful functor from the full subcategory of full right A-comodules to the category of firm R-bimodules.

The definition of full comodule will be based on the following lemma.

Lemma 4.1. For a regular weak multiplier bialgebra A over a field k, and a right Acomodule (V, λ, ϱ) , the following statements are equivalent.

- (a) $V = \langle (V \otimes \omega) \lambda(v \otimes a) \mid v \in V, a \in A, \omega \in Lin(A, k) \rangle.$
- (b) $V = \langle (V \otimes \omega) \varrho(v \otimes a) \mid v \in V, a \in A, \omega \in Lin(A, k) \rangle.$

Proof. We only prove (a) \Rightarrow (b), the converse implication follows symmetrically. The idea of the proof is the same as in [18, Lemma 1.11]. Take any linear decomposition of V of the form

$$\langle (V \otimes \omega) \varrho(v \otimes a) \mid v \in V, \ a \in A, \ \omega \in \mathsf{Lin}(A, k) \rangle \oplus W$$

and any linear functional $\varphi: V \to k$ vanishing on $\langle (V \otimes \omega) \varrho(v \otimes a) \mid v \in V, a \in A, \omega \in Lin(A, k) \rangle$. Then

$$0 = \varphi(V \otimes \omega)\varrho(v \otimes a) = \omega(\varphi \otimes A)\varrho(v \otimes a),$$

for all $v \in V$, $a \in A$, and $\omega \in Lin(A, k)$. Hence $0 = (\varphi \otimes A)\varrho(v \otimes a)$ for all $v \in V$ and $a \in A$ and therefore

$$0 = ((\varphi \otimes A)\varrho(v \otimes a))b \stackrel{(2.10)}{=} a((\varphi \otimes A)\lambda(v \otimes b))$$

for all $v \in V$ and $a, b \in A$. Thus by the non-degeneracy of A, $0 = (\varphi \otimes A)\lambda(v \otimes b)$ for all $v \in V$ and $b \in A$ and so

$$0 = \omega(\varphi \otimes A)\lambda(v \otimes b) = \varphi(V \otimes \omega)\lambda(v \otimes b)$$

for all $v \in V$, $b \in A$, and $\omega \in Lin(A, k)$. If (a) holds then this implies $\varphi = 0$ and hence $W = \emptyset$, proving that also (b) holds.

Definition 4.2. A right comodule (V, λ, ϱ) for a regular weak multiplier bialgebra A over a field k is said to be *full* if the equivalent conditions in Lemma 4.1 hold. That is, V is equal to any (hence both) of the subspaces

$$\langle (V \otimes \omega)\lambda(v \otimes a) \mid v \in V, \ a \in A, \ \omega \in \mathsf{Lin}(A, k) \rangle \quad \text{and} \\ \langle (V \otimes \omega)\varrho(v \otimes a) \mid v \in V, \ a \in A, \ \omega \in \mathsf{Lin}(A, k) \rangle.$$

For a regular weak multiplier bialgebra A, we denote by $M^{(A)}$ the full subcategory in the category of right A-comodules whose objects are the full comodules.

As discussed in Remark 2.3, for a usual, unital weak bialgebra A as in [4, 11], a right Acomodule in the sense of Definition 2.2 is given by a coassociative coaction $\tau : V \to V \otimes A$. Its fullness in the sense of Definition 4.2 is in fact equivalent to counitality in the usual sense: $(V \otimes \epsilon)\tau = V$. This will be proven in Remark 4.6.

Example 4.3. Consider a regular weak multiplier bialgebra A. The comultiplication of A is right full if and only if the A-comodule in Example 2.4 is full.

Example 4.4. For a regular weak multiplier bialgebra A with right full comultiplication, the A-comodule in Example 2.5 is full.

Proof. By equality (3.2) in [2], any element of R can be written as $(R \otimes \epsilon)((1 \otimes a)E)$ for an appropriate element a of A. By Lemma 1.5 (4), any element of A can be written as a linear combination of elements of the form ar for $a \in A$ and $r \in R$. Hence

$$R \subseteq \langle (R \otimes \epsilon)((1 \otimes a)E) \mid a \in A \rangle$$

$$\subseteq \langle (R \otimes \omega)((1 \otimes ar)E) = (R \otimes \omega)\varrho(r \otimes a) \mid r \in R, \ a \in A, \ \omega \in \mathsf{Lin}(A,k) \rangle \subseteq R,$$

so that the comodule in Example 2.5 is full.

Theorem 4.5. For any regular weak multiplier bialgebra A with right full comultiplication, there is a faithful functor $M^{(A)} \to {}_RM_R$.

Proof. First we equip any full right A-comodule V with the structure of R-bimodule. As right and left actions, we put

$$v\overline{\sqcap}^R(a) := (V \otimes \epsilon)\lambda(v \otimes a) \quad \text{and} \quad \sqcap^R(a)v := (V \otimes \epsilon)\varrho(v \otimes a).$$

In order to see that the right action is well defined, we need to check that whenever $\overline{\sqcap}^R(a) = 0$, also $(V \otimes \epsilon)\lambda(v \otimes a) = 0$ for any $v \in V$. So let us assume that $\overline{\sqcap}^R(a) = 0$ for some $a \in A$. Then also $\sqcap^L(a) = 0$ by [2, Lemma 4.8 (4)], so that for any $v \in V$ and $b \in A$,

$$0 = \lambda(v \otimes \sqcap^{L}(a)b) = (V \otimes \epsilon \otimes A)(\lambda \otimes A)\lambda^{13}(v \otimes a \otimes b),$$

where the equality follows by Proposition 2.6 (1). Applying any $\omega \in \text{Lin}(A, k)$ to the second factor, we obtain

$$0 = (V \otimes \epsilon)\lambda((V \otimes \omega)\lambda(v \otimes b) \otimes a),$$

for any $v \in V$, $b \in A$ and $\omega \in \text{Lin}(A, k)$. By the assumption that V is a full comodule, this implies $(V \otimes \epsilon)\lambda(v \otimes a) = 0$ for all $v \in V$. Thus the right *R*-action on V is well-defined. It is proven symmetrically that the left action is well-defined.

The right action is associative by

$$(v\overline{\sqcap}^R(a))\overline{\sqcap}^R(b) = (V \otimes \epsilon \otimes \epsilon)(\lambda \otimes A)\lambda^{13}(v \otimes b \otimes a) = (V \otimes \epsilon)\lambda(v \otimes \sqcap^L(b)a)$$

= $v\overline{\sqcap}^R(\sqcap^L(b)a) = v(\overline{\sqcap}^R(a)\overline{\sqcap}^R(b)),$

for any $v \in V$ and $a, b \in A$. The second equality follows by Proposition 2.6 (1) and the last one follows by [2, Lemma 3.12]. Associativity of the left action is proven symmetrically.

In order to see that the left and right actions commute, compute for any $v \in V$ and $a, b \in A$

$$(\sqcap^{R}(a)v)\overline{\sqcap}^{R}(b) = (V \otimes \epsilon \otimes \epsilon)\lambda^{13}(\varrho \otimes A)(v \otimes a \otimes b)$$

= $(V \otimes \epsilon)(\varrho(v \otimes a)(1 \otimes \sqcap^{L}(b)))$
= $(V \otimes \epsilon)(\varrho(v \otimes a)(1 \otimes b)).$

The second equality follows by Lemma 2.7 (1) and the last one follows by identity (3.5) in [2]. Symmetrically, compute

$$\sqcap^{R}(a)(v \overline{\sqcap}^{R}(b)) = (V \otimes \epsilon \otimes \epsilon) \varrho^{13}(\lambda \otimes A)(v \otimes b \otimes a)$$

= $(V \otimes \epsilon)((1 \otimes \overline{\sqcap}^{L}(a))\lambda(v \otimes b))$
= $(V \otimes \epsilon)((1 \otimes a)\lambda(v \otimes b)).$

The second equality follows by Lemma 2.7 (2) and the last one follows by Lemma 3.1 in [2]. These expressions are equal by (2.10), proving that V is an R-bimodule.

Since R has local units by [2, Theorem 4.6 (2)], an R-module is firm if and only if the R-action on it is surjective. So we only need to show that the above R-actions are

surjective. In the case of the right action it can be seen by the following considerations; for the left action symmetric reasoning applies. By Lemma 1.5 (3), A is spanned by elements of the form $\sqcap^L(b)a$ for $a, b \in A$. Since V is a full comodule by assumption, we conclude that it is spanned by elements of the form

$$(V \otimes \omega)\lambda(v \otimes \sqcap^{L}(b)a) = (V \otimes \epsilon)\lambda((V \otimes \omega)\lambda(v \otimes a) \otimes b),$$

for $v \in V$, $a, b \in A$ and $\omega \in \text{Lin}(A, k)$, where we applied Proposition 2.6 (1). Hence V is spanned by elements of the form $(V \otimes \epsilon)\lambda(v \otimes b) = v\overline{\sqcap}^R(b)$, for $v \in V$ and $b \in A$.

The above construction gives the object map of the stated functor $M^{(A)} \to {}_R M_R$. On the morphisms it acts as the identity map. Any morphism f in $M^{(A)}$ is indeed a morphism of right R-modules by

$$f(v\overline{\sqcap}^R(a)) = f(V \otimes \epsilon)\lambda(v \otimes a) = (V' \otimes \epsilon)(f \otimes A)\lambda(v \otimes a)$$

= $(V' \otimes \epsilon)\lambda'(f(v) \otimes a) = f(v)\overline{\sqcap}^R(a)$

for any $v \in V$ and $a \in A$; and it is a morphism of left *R*-modules by a symmetric reasoning.

Remark 4.6. Consider a usual weak bialgebra A as in [4, 11] possessing an algebraic unit 1. Let (V, λ, ρ) be a right A-comodule in the sense of Definition 2.2. As discussed in Remark 2.3, this means the existence of a coassociative coaction

$$\tau := \lambda(-\otimes 1) = \varrho(-\otimes 1) : V \to V \otimes A, \qquad v \mapsto v^0 \otimes v^1.$$

Here we claim that τ is counital — i.e. $(V \otimes \epsilon)\tau = V$ — if and only if V is a full comodule in the sense of Definition 4.2.

Assume first that τ is counital. Then

$$V = \langle v = (V \otimes \epsilon)\tau(v) = (V \otimes \epsilon)\lambda(v \otimes 1) \mid v \in V \rangle$$

$$\subseteq \langle (V \otimes \omega)\lambda(v \otimes a) \mid v \in V, \ a \in A, \ \omega \in \mathsf{Lin}(A,k) \rangle \subseteq V$$

so that V is a full comodule.

Conversely, assume that the comodule (V, λ, ϱ) is full. Then the *R*-actions on *V* are surjective by Theorem 4.5. Because in this case *R* is a unital algebra, this implies that the *R*-actions on *V* are unital as well. Then for any $v \in V$,

$$v = v \overline{\sqcap}^R(1) = (V \otimes \epsilon) \lambda(v \otimes 1) = (V \otimes \epsilon) \tau(v),$$

so that τ is counital.

Example 4.7. Consider a regular weak multiplier bialgebra A with right full comultiplication. Applying the functor in Theorem 4.5 to the full comodule in Example 4.3, we obtain the R-actions

$$a \otimes \overline{\sqcap}^R(b) \mapsto (A \otimes \epsilon) T_1(a \otimes b) = a \overline{\sqcap}^R(b)$$
$$\sqcap^R(b) \otimes a \mapsto (A \otimes \epsilon) T_3(a \otimes b) = \sqcap^R(b)a,$$

for $a, b \in A$. In the last equalities we used the identities (2.2) and (3.3) in [2] (see (1.8) and (1.9)), respectively.

Example 4.8. Consider a regular weak multiplier bialgebra A with right full comultiplication. Applying the functor in Theorem 4.5 to the full comodule in Example 4.4, we obtain the R-actions

$$r \otimes \overline{\sqcap}^R(a) \mapsto (R \otimes \epsilon)(E(1 \otimes ra)) = \overline{\sqcap}^R(ra) = r\overline{\sqcap}^R(a)$$
$$\sqcap^R(a) \otimes r \mapsto (R \otimes \epsilon)((1 \otimes ar)E) = \sqcap^R(ar) = \sqcap^R(a)r,$$

for any $r \in R$ and $a \in A$. In the first line we used identity (2.2) and Lemma 3.4 in [2], and in the second line we used identities (3.2) and (3.8) in [2].

In terms of the R-actions in Theorem 4.5, we can reformulate Proposition 2.6 in the following form.

Lemma 4.9. For a regular weak multiplier bialgebra A with right full comultiplication, and for a full right A-comodule (V, λ, ϱ) , the following statements hold for any $v \in V$ and $a, b \in A$.

(1) $\lambda(v \otimes \sqcap^{L}(a)b) = \lambda(v \otimes b)(\overline{\sqcap}^{R}(a) \otimes 1).$ (2) $\lambda(v \otimes \overline{\sqcap}^{R}(a)b) = \lambda(v\overline{\sqcap}^{R}(a) \otimes b).$ (3) $\varrho(v \otimes a \sqcap^{R}(b)) = \varrho(\sqcap^{R}(b)v \otimes a).$ (4) $\varrho(v \otimes a\overline{\sqcap}^{L}(b)) = (\sqcap^{R}(b) \otimes 1)\varrho(v \otimes a).$ (5) $\varrho(v \otimes b)(1 \otimes \sqcap^{L}(a)) = \varrho(v \otimes b)(\overline{\sqcap}^{R}(a) \otimes 1).$ (6) $\varrho(v \otimes b)(1 \otimes \overline{\sqcap}^{R}(a)) = \varrho(v\overline{\sqcap}^{R}(a) \otimes b).$ (7) $(1 \otimes \sqcap^{R}(a))\lambda(v \otimes b) = \lambda(\sqcap^{R}(a)v \otimes b).$ (8) $(1 \otimes \overline{\sqcap^{L}}(a))\lambda(v \otimes b) = (\sqcap^{R}(a) \otimes 1)\lambda(v \otimes b).$

Proof. Assertions (1)-(4) are evident reformulations of the statements in Proposition 2.6. The remaining parts are obtained from them applying (2.10). For example, (5) follows by

$$\varrho(v \otimes b)(1 \otimes \sqcap^{L}(a)c) \stackrel{(2.10)}{=} (1 \otimes b)\lambda(v \otimes \sqcap^{L}(a)c) \\
\stackrel{(1)}{=} (1 \otimes b)\lambda(v \otimes c)(\overline{\sqcap}^{R}(a) \otimes 1) \\
\stackrel{(2.10)}{=} \varrho(v \otimes b)(\overline{\sqcap}^{R}(a) \otimes c)$$

— for any $v \in V$ and $a, b, c \in A$ — and non-degeneracy of the right A-module $V \otimes A$. \Box

In the next section we shall need the following compatibilities between full comodules and the idempotent E. Recall that for a regular weak multiplier bialgebra A with right full comultiplication, by identities (2.3) and (3.4) in [2], both $E(1 \otimes a)$ and $(1 \otimes a)E$ are elements of $R \otimes A$, for any $a \in A$. Hence the maps $E_1, E_2 : A \otimes A \to A \otimes A$ allow for the generalizations

$$E_1: V \otimes A \to V \otimes A, \qquad v \otimes a \mapsto ((-)v \otimes A)[E(1 \otimes a)] \equiv E(v \otimes a)$$
$$E_2: V \otimes A \to V \otimes A, \qquad v \otimes a \mapsto (v(-) \otimes A)[(1 \otimes a)E] \equiv (v \otimes a)E$$

for any full right A-comodule V.

Lemma 4.10. For a regular weak multiplier bialgebra A with right full comultiplication, and for a full right A-comodule (V, λ, ϱ) , the following statements hold.

(1)
$$E_1 \lambda = \lambda$$
.
(2) $(V \otimes E_1)(\lambda \otimes A) = (\lambda \otimes A)E_1^{13}$.
(3) $E_2 \varrho = \varrho$.
(4) $(V \otimes E_2)(\varrho \otimes A) = (\varrho \otimes A)E_2^{13}$.

Proof. (1) For any $v \in V$ and $a, b, c \in A$,

$$(1 \otimes bc)(E_1\lambda(v \otimes a)) \stackrel{(!)}{=} E_2(1 \otimes bc)\lambda(v \otimes a)$$

= $((\sqcap^R \otimes A)T_3(c \otimes b))\lambda(v \otimes a)$
= $(V \otimes \mu^{\mathsf{op}}(\overline{\sqcap^L} \otimes A)T_3(c \otimes b))\lambda(v \otimes a)$
= $(1 \otimes bc)\lambda(v \otimes a),$

from which we conclude by the non-degeneracy of the left A-module $V \otimes A$. In the second equality we used identity (3.4) in [2], in the third equality we used Lemma 4.9 (8) and in the last equality we used [2, Lemma 3.7 (1)].

(2) For any $v \in V$ and $a, b, c \in A$,

$$(V \otimes E_1)(\lambda \otimes A)(v \otimes a \otimes bc) = (1 \otimes (\overline{\sqcap}^R \otimes A)T_1(b \otimes c))(\lambda(v \otimes a) \otimes 1)$$

= $(\lambda \otimes A)((\overline{\sqcap}^R \otimes A)T_1(b \otimes c))^{13}(v \otimes a \otimes 1)$
= $(\lambda \otimes A)E_1^{13}(v \otimes a \otimes bc),$

from which we conclude by the idempotency of A. In the first and the last equalities we used identity (2.3) in [2] and in the second equality we used Lemma 4.9 (7).

The remaining assertions follow symmetrically.

The following technical lemmata will be applied in Section 7.

Lemma 4.11. Consider a regular weak multiplier bialgebra A with right full comultiplication, and a full right A-comodule (V, λ, ϱ) . For any $v \in V$ and $a \in A$, introduce the index notation $\varrho(v \otimes a) =: v^{\varrho} \otimes a^{\varrho}$ and $\lambda(v \otimes a) =: v^{\lambda} \otimes a^{\lambda}$ (where in both cases implicit summation is understood). Then for any $v \in V$ and $a \in A$, the following identities hold.

(1) $v^{\varrho} \sqcap^R (a^{\varrho}) = \sqcap^R (a)v.$

(2)
$$\overline{\sqcap}^R(a^\lambda)v^\lambda = v\overline{\sqcap}^R(a)$$

Proof. (1) For any $a, b \in A$ and any $v \in V$,

$$v \sqcap^R (ab) = v \sqcap^R (\sqcap^R (a)b) = (V \otimes \epsilon)[(v \otimes 1)((\sqcap^R \otimes A)T_3(b \otimes a))] = (V \otimes \epsilon)E_2(v \otimes ab).$$

The first equality follows by identity (3.6), the second one follows by (3.3), and the last one follows by (3.4) in [2]. By the idempotency of A, this proves $v \sqcap^R (a) = (V \otimes \epsilon)E_2(v \otimes a)$. With this identity at hand,

$$v^{\varrho} \sqcap^{R} (a^{\varrho}) = (V \otimes \epsilon) E_{2} \varrho(v \otimes a) = (V \otimes \epsilon) \varrho(v \otimes a) = \sqcap^{R} (a) v.$$

In the second equality we used Lemma 4.10 (3). Assertion (2) follows symmetrically. \Box

Lemma 4.12. Let A be a regular weak multiplier bialgebra with right full comultiplication and let (V, λ, ϱ) be a full right A-comodule. For any $v \in V$, $a, b \in A$ and $r \in R$, the following identities hold.

- (1) $(V \otimes \sqcap^R) \varrho(vr \otimes b) = (\sqcap^R(b)v \otimes 1)\delta(r)$ (where δ denotes the comultiplication on the coalgebra R in [2, Theorem 4.4]; cf. (1.11)).
- (2) $(1 \otimes b)((V \otimes \sqcap^{L})\varrho(v \otimes a)) = ((\sqcap^{R} \otimes A)T_{4}(a \otimes b))(v \otimes 1).$

Proof. (1) For any $v \in V$, $r \in R$ and $a, b, c \in A$, denote $\varrho(vr \otimes b) = (vr)^{\varrho} \otimes b^{\varrho}$ and $T_2(c \otimes a) = c^2 \otimes a^2$, where in both cases implicit summation is understood. Then

$$(1 \otimes c)((V \otimes \sqcap^{R})\varrho(vr \otimes b))(1 \otimes a) = (vr)^{\varrho} \otimes (A \otimes \epsilon)((c \otimes 1)T_{3}(a \otimes b^{\varrho}))$$

$$= (vr)^{\varrho} \otimes (A \otimes \epsilon)((1 \otimes b^{\varrho})T_{2}(c \otimes a))$$

$$= (vr)^{\varrho} \otimes c^{2}\epsilon(b^{\varrho}\overline{\sqcap^{R}}(a^{2}))$$

$$= (V \otimes \epsilon)\varrho(vr\overline{\sqcap^{R}}(a^{2}) \otimes b) \otimes c^{2}$$

$$= (1 \otimes c)(\sqcap^{R}(b)v \otimes 1)\delta(r)(1 \otimes a),$$

from which we conclude simplifying by a and c. The first equality follows by (3.3) in [2] (cf. (1.9)). In the second equality we applied (1.6). The third equality holds by [2, Lemma

3.1] and the fourth one does by Lemma 4.9 (6). In the last equality we used that by [2, Proposition 4.3 (1)] (cf. (1.10)) and by the second equality in axiom (ix) in Definition 1.1, the multiplier F in [2, Proposition 4.3 (1)] satisfies

$$(1 \otimes c)F(1 \otimes ab) = (1 \otimes c)((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_4(b \otimes a)) = ((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_2(c \otimes a))(1 \otimes b).$$

Hence $\delta(r) = (r \otimes 1)F$ satisfies $r \overline{\sqcap}^R(a^2) \otimes c^2 = (1 \otimes c)\delta(r)(1 \otimes a)$. (2) For any $v \in V$ and $a, b, c \in A$,

$$(1 \otimes c \otimes 1)((V \otimes T_4)(\varrho \otimes A)(v \otimes a \otimes b)) = (1 \otimes T_2(c \otimes b))(\varrho(v \otimes a) \otimes 1)$$

$$= (\varrho \otimes A)(v \otimes T_2(c \otimes b)(a \otimes 1))$$

$$= (\varrho \otimes A)(v \otimes (c \otimes 1)T_4(a \otimes b))$$

$$= (1 \otimes c \otimes 1)((\varrho \otimes A)(V \otimes T_4)(v \otimes a \otimes b)).$$

In the first and the third equalities we used the second equality in axiom (ix) in Definition 1.1. In the second and the last equalities we used that ρ is a left A-module map, cf. Proposition 2.1 (1). Simplifying by c, we obtain

(4.1)
$$(V \otimes T_4)(\varrho \otimes A) = (\varrho \otimes A)(V \otimes T_4).$$

Using this identity in the second equality and (3.3) in [2] (cf. (1.9)) in the first one, it follows for any $v \in V$ and $a, b \in A$ that

$$(1 \otimes b)((V \otimes \sqcap^{L})\varrho(v \otimes a)) = (V \otimes \epsilon \otimes A)(V \otimes T_{4})(\varrho \otimes A)(v \otimes a \otimes b)$$

= $(V \otimes \epsilon \otimes A)(\varrho \otimes A)(V \otimes T_{4})(v \otimes a \otimes b)$
= $((\sqcap^{R} \otimes A)T_{4}(a \otimes b))(v \otimes 1).$

		_
		ч
_	_	_

5. The monoidal category of full comodules

For a unital (weak) bialgebra, both the category of modules and the category of comodules are known to be monoidal with respect to the monoidal product provided by the module tensor product over the base algebra. For a regular (weak) multiplier bialgebra A with a full comultiplication, it was shown in [2, Section 5] that the category of idempotent non-degenerate A-modules is monoidal with respect to the same monoidal product provided by the module tensor product over the base algebra. The aim of this section is to prove a similar result about A-comodules: We show that the category $M^{(A)}$ of full A-comodules is monoidal such that the functor in Theorem 4.5 is strict monoidal.

Lemma 5.1. For a regular weak multiplier bialgebra A and any right A-comodules $(V, \lambda_V, \varrho_V)$ and $(W, \lambda_W, \varrho_W)$, there is an A-comodule

$$(V \otimes W, \lambda_V^{13}(V \otimes \lambda_W), (V \otimes \varrho_W) \varrho_V^{13}).$$

Proof. Since (2.10) holds both for V and W, it clearly holds for the stated comodule $V \otimes W$. Since the normalization condition (2.11) holds for W, the top left region in

commutes. Postcomposing both paths around this diagram by λ_V^{13} , we conclude that the normalization condition (2.11) holds for the stated comodule $V \otimes W$. Finally, since the coassociativity condition (2.12) holds both for V and W,

commutes proving that (2.12) holds also for the stated comodule $V \otimes W$.

The comodule in Lemma 5.1 is not yet the appropriate monoidal product. With the aim of lifting the monoidal structure of
$${}_{R}M_{R}$$
 to $M^{(A)}$, the candidate monoidal unit in $M^{(A)}$ is the full A-comodule R in Example 4.4. The candidate monoidal product of full A-comodules V and W is their R-module tensor product $V \otimes_{R} W$.

Proposition 5.2. Let A be a regular weak multiplier bialgebra with right full comultiplication. For any full right A-comodules $(V, \lambda_V, \varrho_V)$ and $(W, \lambda_W, \varrho_W)$, $V \otimes_R W$ is isomorphic to the range of the idempotent map

$$G_1: V \otimes W \to V \otimes W, \qquad \sqcap^R(a) v \otimes w \overline{\sqcap}^R(b) \mapsto v^\varrho \otimes w^\lambda \epsilon(a^\varrho b^\lambda),$$

where the implicit summation index notation $\varrho_V(v \otimes a) =: v^{\varrho} \otimes a^{\varrho}$ and $\lambda_W(w \otimes b) =: w^{\lambda} \otimes b^{\lambda}$ is used.

Proof. By [2, Theorem 4.6 (1)] R is a coseparable coalgebra. Thus by [6, Proposition 2.17], for any firm R-bimodules V and W, $V \otimes_R W$ is isomorphic to the range of the idempotent map

(5.1)
$$V \otimes W \to V \otimes W, \quad vr \otimes w \mapsto (v \otimes 1)\delta(r)(1 \otimes w),$$

where δ denotes the comultiplication in the coalgebra R (cf. [2, Theorem 4.4] and (1.11)) and we use that V is spanned by elements of the form vr in terms of $v \in V$ and $r \in R$.

In view of Theorem 4.5 and the idempotency of A, the tensor product vector space $V \otimes W$ is spanned by elements of the form $\Box^R(a)v \Box^R(cd) \otimes w \overline{\Box}^R(b)$, for $v \in V$, $w \in W$

and $a, b, c, d \in A$. Evaluate (5.1) on this element. Introducing the implicit summation index notation $T_2(c \otimes d) = c^2 \otimes d^2$ and applying (4.1) in [2], we obtain

$$\begin{split} \sqcap^{R}(a)v \sqcap^{R}(c^{2}) \otimes \sqcap^{R}(d^{2})w\overline{\sqcap}^{R}(b) \\ &= (V \otimes \epsilon)\varrho_{V}(v \sqcap^{R}(c^{2}) \otimes a) \otimes (W \otimes \epsilon)\lambda_{W}(\sqcap^{R}(d^{2})w \otimes b) \\ &= (V \otimes \epsilon)(\varrho_{V}(v \otimes a)(1 \otimes \sqcap^{R}(c^{2}))) \otimes (W \otimes \epsilon)((1 \otimes \sqcap^{R}(d^{2}))\lambda_{W}(w \otimes b)) \\ &= (V \otimes \epsilon)(\varrho_{V}(v \otimes a)(1 \otimes \sqcap^{R}(c^{2}))) \otimes (W \otimes \epsilon)((1 \otimes d^{2})\lambda_{W}(w \otimes b)) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho}(\sqcap^{R} \otimes \epsilon)(T_{2}(c \otimes d)(1 \otimes \eqsim^{\lambda}))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho}(\sqcap^{R} \otimes \epsilon)T_{2}(c \otimes d(1 \otimes \sqcap^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho}(\sqcap^{R} \otimes \epsilon)T_{2}(c \otimes d(1 \otimes \sqcap^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho} \cap \sqcap^{R}(cd(1 \otimes \bowtie^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho} \cap \sqcap^{R}(cd(1 \otimes \bowtie^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho} \cap \sqcap^{R}(cd(1 \otimes \bowtie^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho} \cap \sqcap^{R}(cd(1 \otimes \bowtie^{R}(b^{\lambda})))) \\ &= v^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho} \cap \sqcap^{R}(cd(1 \otimes \bowtie^{R}(b^{\lambda})))) \\ &= (v \cap \shortparallel^{R}(cd))^{\varrho} \otimes w^{\lambda}\epsilon(a^{\varrho}b^{\lambda}). \end{split}$$

This proves that the idempotent map (5.1) is equal to the stated map G_1 . In the first equality we used the R-actions in Theorem 4.5. In the second and the last equalities we used parts (6)-(7) and (6) of Lemma 4.9, respectively. The third equality follows by one of the identities in (3.5) in [2]. The fifth and the penultimate equalities follow by [2, Lemma 3.1]. In the sixth equality we used that by [2, Lemma 3.3], $T_2(c \otimes d)(1 \otimes \overline{\sqcap}^R(b)) = T_2(c \otimes d)(1 \otimes \overline{\sqcap}^R(b))$ $d\overline{\sqcap}^{R}(b)$). The seventh equality follows by the counitality axiom (iv) of weak multiplier bialgebra in Definition 1.1 and the eighth one follows by [2, Lemma3.11].

Example 5.3. It is immediate from the proof of Proposition 5.2 that for the full Acomodule A in Example 4.3, $G_1: A \otimes A \to A \otimes A$ is equal to the idempotent map in (1.13). More generally, for any full right A-comodule V, we can write $G_1: V \otimes A \to V \otimes A$ as $v \otimes a \mapsto (v(-) \otimes A)[F(1 \otimes a)]$ (which is meaningful by [2, Proposition 4.3 (1)]; cf. (1.10)).

In order to see that the *R*-module tensor product of any full right comodules is again a full right comodule, we need some compatibilities between the coactions and the map G_1 in Proposition 5.2.

Lemma 5.4. Consider a regular weak multiplier bialgebra A with right full comultiplication. For any full right A-comodules V and W, the map G_1 in Proposition 5.2 satisfies the following identities.

(1)
$$\lambda_V G_1 = \lambda_V$$
.

(2)
$$(V \otimes \lambda_W)(G_1 \otimes A) = G_1^{13}(V \otimes \lambda_W)$$

(3)
$$\lambda_V^{13}(V \otimes \lambda_W)(G_1 \otimes A) = \lambda_V^{13}(V \otimes \lambda_W).$$

(4)
$$(G_1 \otimes A)\lambda_V^{13} = \lambda_V^{13}(V \otimes E_1)$$

- (5) $(G_1 \otimes A)\lambda_V^{13}(V \otimes \lambda_W) = \lambda_V^{13}(V \otimes \lambda_W).$
- (6) $\varrho_V G_1^{21} = \varrho_V.$

- $(6) \quad \varrho_V G_1 \varrho_V.$ $(7) \quad \varrho_V^{13}(G_1 \otimes A) = (V \otimes G_1^{21})\varrho_V^{13}.$ $(8) \quad (V \otimes \varrho_W)\varrho_V^{13}(G_1 \otimes A) = (V \otimes \varrho_W)\varrho_V^{13}.$ $(9) \quad (G_1 \otimes A)(V \otimes \varrho_W) = (V \otimes \varrho_W)E_2^{13}.$ $(10) \quad (G_1 \otimes A)(V \otimes \varrho_W)\varrho_V^{13} = (V \otimes \varrho_W)\varrho_V^{13}.$

Proof. (1) For any $v \in V$, $r \in R$ and $a \in A$,

$$\lambda_V G_1(v \otimes ra) = \lambda_V((v \otimes 1)\delta(r)(1 \otimes a)) = \lambda_V(v \otimes \mu\delta(r)a) = \lambda_V(v \otimes ra).$$

The second equality follows by Lemma 4.9 (2) and in the last equality we used that the comultiplication δ of the coseparable coalgebra R is a section of the multiplication μ , see [2, Proposition 4.3 (3)]. Since A is spanned by elements of the form ra, for $r \in R$ and $a \in A$ (cf. Lemma 1.5), this proves the claim.

(2) Let us use the implicit summation index notation $T_2(a \otimes b) =: a^2 \otimes b^2$ and $\lambda_W(w \otimes c) =: w^{\lambda} \otimes c^{\lambda}$ for any $a, b, c \in A$ and $w \in W$. For any $v \in V$,

$$(V \otimes \lambda_W)(G_1 \otimes A)(v \otimes \sqcap^R(ab)w \otimes c)$$

= $(V \otimes \lambda_W)(v \sqcap^R(a^2) \otimes \sqcap^R(b^2)w \otimes c)$
= $v \sqcap^R(a^2) \otimes w^{\lambda} \otimes \sqcap^R(b^2)c^{\lambda}$
= $G_1^{13}(v \otimes w^{\lambda} \otimes \sqcap^R(ab)c^{\lambda})$
= $G_1^{13}(V \otimes \lambda_W)(v \otimes \sqcap^R(ab)w \otimes c).$

The first and the third equalities follow by (4.1) in [2] and in the second and the last equalities we applied Lemma 4.9 (7). Since R is idempotent and the left R-action on W is surjective by Theorem 4.5, this proves the claim.

- (3) follows immediately from (1) and (2).
- (4) For any $v \in V$, $w \in W$ and $a, b, c \in A$,

$$(G_1 \otimes A)\lambda_V^{13}(v \otimes \overline{\sqcap}^R(ab)w \otimes c) = v^{\lambda}\overline{\sqcap}^R(b^1) \otimes \overline{\sqcap}^R(a^1)w \otimes c^{\lambda}$$
$$= \lambda_V^{13}(v \otimes \overline{\sqcap}^R(a^1)w \otimes \sqcap^L(b^1)c)$$
$$= \lambda_V^{13}(V \otimes E_1)(v \otimes \overline{\sqcap}^R(ab)w \otimes c).$$

where we used the notation $\lambda_V(v \otimes c) = v^{\lambda} \otimes c^{\lambda}$ and $T_1(a \otimes b) = a^1 \otimes b^1$. The first equality follows by [2, Lemma 4.5 (2)]. The second one follows by Lemma 4.9 (1) and the last one does by Lemma 1.4 (2). Since *R* is idempotent and the left *R*-action on *W* is surjective by Theorem 4.5, this proves the claim.

(5) is immediate by (4) and Lemma 4.10 (1). The remaining assertions are proven symmetrically.

Since the module tensor product is a coequalizer (of linear maps), we study next comodule structures on such coequalizers.

Lemma 5.5. Consider a regular weak multiplier bialgebra A. For some right A-comodules (V, λ, ϱ) and (V', λ', ϱ') , comodule maps $f, g: V \to V'$, a vector space P and a linear map $\pi: V' \to P$, assume that

(5.2)
$$V \xrightarrow{f} V' \xrightarrow{\pi} P$$

is a coequalizer of linear maps. Then P carries a unique right A-comodule structure such that π is a comodule map.

Proof. Since f and g are comodule maps, the diagrams

serially commute (meaning that they commute with either simultaneous choice of the upper or lower ones of the parallel arrows). Since (5.2) is a coequalizer by assumption, also the top rows are coequalizers. Hence by their universality, there exist unique morphisms λ_P and ϱ_P making the respective diagrams commutative. Let us see that they define a comodule $(P, \lambda_P, \varrho_P)$. For any $v' \in V'$ and $a, b \in A$,

$$(1 \otimes a)\lambda_P(\pi(v') \otimes b) = (\pi \otimes A)[(1 \otimes a)\lambda'(v' \otimes b)] = (\pi \otimes A)[\varrho'(v' \otimes a)(1 \otimes b)] = \varrho_P(\pi(v') \otimes a)(1 \otimes b).$$

In the second equality we used that (2.10) holds for V'. Since $\pi \otimes A$ is surjective, this proves that (2.10) holds for P. Since also $\pi \otimes A \otimes A$ is surjective, the normalization condition (2.11) on P follows by

$$\begin{aligned} (\lambda_P \otimes A)\lambda_P^{13}(P \otimes E_1)(\pi \otimes A \otimes A) &= (\pi \otimes A \otimes A)(\lambda' \otimes A)\lambda'^{13}(V' \otimes E_1) \\ &= (\pi \otimes A \otimes A)(\lambda' \otimes A)\lambda'^{13} \\ &= (\lambda_P \otimes A)\lambda_P^{13}(\pi \otimes A \otimes A), \end{aligned}$$

where in the second equality we used that (2.11) holds for V'. Similarly, the coassociativity condition (2.12) follows by

$$\begin{aligned} (\lambda_P \otimes A)\lambda_P^{13}(P \otimes T_1)(\pi \otimes A \otimes A) &= (\pi \otimes A \otimes A)(\lambda' \otimes A)\lambda'^{13}(V' \otimes T_1) \\ &= (\pi \otimes A \otimes A)(V' \otimes T_1)(\lambda' \otimes A) \\ &= (P \otimes T_1)(\lambda_P \otimes A)(\pi \otimes A \otimes A), \end{aligned}$$

where in the second equality we used that (2.12) holds for V'.

Proposition 5.6. Consider a regular weak multiplier bialgebra A with right full comultiplication. For any full right A-comodules V and W, there is a unique right A-comodule structure on $V \otimes_R W$ such that the canonical epimorphism $\pi_{V,W} : V \otimes W \to V \otimes_R W$ is a morphism of comodules from the comodule in Lemma 5.1.

Proof. By Proposition 5.2, $\pi_{V,W}$ is split by the *R*-bimodule map

$$V \otimes_R W \cong \operatorname{Im}(G_1) \to V \otimes W, \qquad v \otimes_R w \mapsto G_1(v \otimes w).$$

Since G_1 is idempotent,

$$V \otimes W \xrightarrow[V \otimes W]{G_1} V \otimes W \xrightarrow[V \otimes W]{\pi_{V,W}} V \otimes_R W$$

is a (split) coequalizer of linear maps. Moreover, by Lemma 5.4 (3) and (5), G_1 is a comodule map $V \otimes W \to V \otimes W$ for the comodule $V \otimes W$ in Lemma 5.1. Then we conclude by Lemma 5.5.

Our next aim is to prove that the product of full comodules in Proposition 5.6 is a full comodule again. This starts with the following.

Lemma 5.7. Let A be a regular weak multiplier bialgebra with right full comultiplication. For any full right A-comodules V and W, consider the product A-comodule $(V \otimes_R W, \lambda_{V \otimes_R W}, \varrho_{V \otimes_R W})$ in Proposition 5.6. For all $v \otimes_R w \in V \otimes_R W$ and $a \in A$, the following assertions hold.

- (1) $(V \otimes_R W \otimes \epsilon) \varrho_{V \otimes_R W}(v \otimes_R w \otimes a) = \sqcap^R(a) v \otimes_R w.$
- (2) $(V \otimes_R W \otimes \epsilon) \lambda_{V \otimes_R W} (v \otimes_R w \otimes a) = v \otimes_R w \overline{\sqcap}^R(a).$

Proof. (1) The canonical epimorphism $\pi_{V,W} : V \otimes W \to V \otimes_R W$ is a morphism of comodules by Proposition 5.6. Hence for any $v \in V$, $w \in W$, $a \in A$ and $v \otimes_R w := \pi_{V,W}(v \otimes w)$,

$$(V \otimes_R W \otimes \epsilon) \varrho_{V \otimes_R W}(v \otimes_R w \otimes a)$$

= $\pi_{V,W}(V \otimes W \otimes \epsilon)(V \otimes \varrho_W) \varrho_V^{13}(v \otimes w \otimes a)$
= $v^{\varrho} \otimes_R \sqcap^R (a^{\varrho}) w = v^{\varrho} \sqcap^R (a^{\varrho}) \otimes_R w = \sqcap^R (a) v \otimes_R w,$

where the implicit summation index notation $\rho_V(v \otimes a) =: v^{\rho} \otimes a^{\rho}$ is used and the last equality holds by Lemma 4.11 (1). Part (2) is proven symmetrically.

Proposition 5.8. Consider a regular weak multiplier bialgebra A with right full comutiplication. For any full right A-comodules V and W, the product A-comodule $(V \otimes_R W, \lambda_{V \otimes_R W}, \varrho_{V \otimes_R W})$ in Proposition 5.6 is full.

Proof. Since V is a full A-comodule, its left R-action is surjective by Theorem 4.5. So we conclude by Lemma 5.7 (1) that

$$V \otimes_{R} W \subseteq \langle (V \otimes_{R} W \otimes \epsilon) \varrho_{V \otimes_{R} W} (v \otimes_{R} w \otimes a) \mid v \otimes_{R} w \in V \otimes_{R} W, \ a \in A \rangle$$

$$\subseteq \langle (V \otimes_{R} W \otimes \omega) \varrho_{V \otimes_{R} W} (v \otimes_{R} w \otimes a) \mid v \otimes_{R} w \in V \otimes_{R} W, \ a \in A, \ \omega \in \mathsf{Lin}(A, k) \rangle$$

$$\subseteq V \otimes_{R} W$$

so that $V \otimes_R W$ is a full A-comodule.

It follows immediately from Lemma 5.7 that the left and right *R*-actions on the full *A*-comodule $V \otimes_R W$ in Proposition 5.8 are

$$\sqcap^{R}(a)(v \otimes_{R} w) = \sqcap^{R}(a)v \otimes_{R} w \quad \text{and} \quad (v \otimes_{R} w)\overline{\sqcap}^{R}(a) = v \otimes_{R} w\overline{\sqcap}^{R}(a).$$

Remark 5.9. Consider right comodules $(V, \lambda_V, \varrho_V)$, (P, λ, ϱ) and (P', λ', ϱ') over a regular weak multiplier bialgebra A. Note that for a surjective homomorphism $\pi : V \to P$ of comodules and a linear map $f : P \to P'$, $f\pi$ is a morphism of comodules if and only if f is so. Indeed, the composite of comodule maps π and f is evidently a morphism of comodules and the converse follows by

$$\lambda'(f \otimes A)(\pi \otimes A) = (f \otimes A)(\pi \otimes A)\lambda_V = (f \otimes A)\lambda(\pi \otimes A)$$

and the surjectivity of $\pi \otimes A$.

We are ready to prove the main result of this section.

Theorem 5.10. For a regular weak multiplier bialgebra A with right full comultiplication, the category $M^{(A)}$ of full right A-comodules is monoidal and the functor $M^{(A)} \rightarrow {}_{R}M_{R}$ in Theorem 4.5 is strict monoidal.

Proof. The tensor product $f \otimes g : V \otimes W \to V' \otimes W'$ of any A-comodule maps $f : V \to V'$ and $g : W \to W'$ is evidently an A-comodule map between the comodules as in Lemma 5.1. Since $\pi_{V,W}$ and $\pi_{V',W'}$ are surjective comodule maps by Proposition 5.6, also $\pi_{V',W'}(f \otimes g) = (f \otimes_R g)\pi_{V,W}$ is a comodule map. Thus $f \otimes_R g : V \otimes_R W \to V' \otimes_R W'$ is a comodule map by Remark 5.9.

It remains to show that the unitors and the associator of ${}_{R}M_{R}$ — if evaluated at objects of $M^{(A)}$ — are morphisms of A-comodules. For any full right A-comodule V the composition of the canonical epimorphism $\pi_{V,R}: V \otimes R \to V \otimes_{R} R$ with the right unitor $\mathbf{r}_{V}: V \otimes_{R} R \to V$ yields the right R-action $\nu: V \otimes R \to V$ on V in Theorem 4.5. Let us show that $\nu = \mathbf{r}_{V}\pi_{V,R}$ is a morphism of A-comodules. This is proven by the following

computation for any $v \in V$ and $a, b, c \in A$. (We use again the implicit summation index notation $T_1(b \otimes c) = b^1 \otimes c^1$.)

$$\begin{aligned} (\nu \otimes A)\lambda_V^{13}(V \otimes \lambda_R)(v \otimes \overline{\sqcap}^R(a) \otimes bc) \\ &= (\nu \otimes A)\lambda_V^{13}(v \otimes (1 \otimes \overline{\sqcap}^R(a))E(1 \otimes bc)) = \lambda_V(v \otimes \overline{\sqcap}^R(a)c^1)(\overline{\sqcap}^R(b^1) \otimes 1) \\ &= \lambda_V(v \otimes \sqcap^L(b^1)\overline{\sqcap}^R(a)c^1) = \lambda_V(v \otimes \overline{\sqcap}^R(a) \sqcap^L(b^1)c^1) \\ &= \lambda_V(v \otimes \overline{\sqcap}^R(a)bc) = \lambda_V(\nu \otimes A)(v \otimes \overline{\sqcap}^R(a) \otimes bc). \end{aligned}$$

The second equality follows by identity (2.3) in [2]. The third and the last equalities hold by parts (1) and (2) of Lemma 4.9, respectively. In the fourth equality we used (3.9) in [2] and in the fifth one we applied Lemma 3.7 (3) in [2]. We know from Proposition 5.6 that $\pi_{V,R}$ is a surjective morphism of A-comodules; hence \mathbf{r}_V is a comodule map by Remark 5.9. The left unitor is treated symmetrically.

For any full A-comodules V, W and Z, the associator $\mathbf{a}_{V,W,Z} : (V \otimes_R W) \otimes_R Z \to V \otimes_R (W \otimes_R Z)$ satisfies by construction

(5.3)
$$\mathbf{a}_{V,W,Z}\pi_{V\otimes_R W,Z}(\pi_{V,W}\otimes Z) = \pi_{V,W\otimes_R Z}(V\otimes \pi_{W,Z})$$

(where we omit explicitly denoting the associator in the monoidal category of vector spaces). By Proposition 5.6 both $\pi_{V\otimes_R W,Z}(\pi_{V,W}\otimes Z)$ and $\pi_{V,W\otimes_R Z}(V\otimes \pi_{W,Z})$ are surjective morphisms of A-comodules. Hence $\mathbf{a}_{V,W,Z}$ is a morphism of A-comodules by Remark 5.9.

6. ANTIPODE AND DUAL COMODULES

In this section we deal with a regular weak multiplier bialgebra with a left and right full comultiplication. Assuming that it possesses an antipode (in the sense of [2, Theorem 6.8]), we show that finite dimensional full right A-comodules possess duals in $M^{(A)}$.

To begin with, let A be a regular weak multiplier bialgebra over a field k. As a first step, we equip the k-dual $V^* := \text{Lin}(V, k)$ of any finite dimensional right A-comodule V with the structure of a left A-comodule.

Proposition 6.1. Let A be a regular weak multiplier bialgebra over a field k and let (V, λ, ϱ) be a finite dimensional right A-comodule. Using a dual basis basis $\{a_i\}$ in A and $\{\alpha_i\}$ in $A^* := \text{Lin}(A, k)$, the dual vector space $V^* := \text{Lin}(V, k)$ carries a left A-comodule structure

$$\lambda^* : A \otimes V^* \to A \otimes V^*, \qquad b \otimes \varphi \mapsto \sum_i a_i \otimes (\varphi \otimes \alpha_i) \lambda(-\otimes b)$$
$$\varrho^* : A \otimes V^* \to A \otimes V^*, \qquad b \otimes \varphi \mapsto \sum_i a_i \otimes (\varphi \otimes \alpha_i) \varrho(-\otimes b).$$

Proof. For any given elements $v \in V$ and $b \in A$, there are only finitely many values of the index *i* such that $(V \otimes \alpha_i)\lambda(v \otimes b) \neq 0$. Thus since *V* is a finite dimensional vector space, for any given element $b \in A$ there are only finitely many indices *i* such that $(V \otimes \alpha_i)\lambda(-\otimes b)$ is a non-zero map $V \to V$. Symmetrically, there are only finitely many indices *i* such that $(V \otimes \alpha_i)\varrho(-\otimes b) \neq 0$. This proves that the sums defining the maps λ^* and ϱ^* contain only finitely many non-zero terms; hence they are meaningful.

Introduce the implicit summation index notation $\lambda(v \otimes a) = v^{\lambda} \otimes a^{\lambda}$, $\varrho(v \otimes a) = v^{\varrho} \otimes a^{\varrho}$, $\lambda^*(a \otimes \varphi) = a^{\lambda^*} \otimes \varphi^{\lambda^*}$ and $\varrho^*(a \otimes \varphi) = a^{\varrho^*} \otimes \varphi^{\varrho^*}$ for any $a \in A$, $v \in V$ and $\varphi \in V^*$. By construction,

(6.1)
$$a^{\lambda^*} \varphi^{\lambda^*}(v) = \varphi(v^{\lambda}) a^{\lambda}$$
 and $a^{\varrho^*} \varphi^{\varrho^*}(v) = \varphi(v^{\varrho}) a^{\varrho}$,

hence

$$ab^{\lambda^*}\varphi^{\lambda^*}(v) = \varphi(v^{\lambda})ab^{\lambda} = \varphi(v^{\varrho})a^{\varrho}b = a^{\varrho^*}\varphi^{\varrho^*}(v)b.$$

In the second equality we used the compatibility condition (2.10) between λ and ρ . This proves the compatibility condition (2.20), that is,

 $(a \otimes 1)\lambda^*(b \otimes \varphi) = \rho^*(a \otimes \varphi)(b \otimes 1) \qquad \forall a, b \in A, \ \varphi \in V^*.$

In view of (6.1), the normalization condition

$$(A \otimes \lambda^*)\lambda^{*13}(E_1 \otimes V^*) = (A \otimes \lambda^*)\lambda^{*13}$$

on λ^* (see (2.21)) is immediate from the normalization condition (2.11) on λ and the coassociativity condition

$$(A \otimes \lambda^*)\lambda^{*13}(T_1 \otimes V^*) = (T_1 \otimes V^*)\lambda^{*13}$$

on λ^* (see (2.22)) is immediate from the coassociativity condition (2.12) on λ .

Proposition 6.2. Let A be a regular weak multiplier bialgebra with a right full comultiplication. For a finite dimensional full right A-comodule V, the (obviously finite dimensional) left A-comodule V^* in Proposition 6.1 is also full.

Proof. By Theorem 4.5, V is a firm right R-module so in particular the action $V \otimes R \to V$ is surjective. By [2, Theorem 4.6 (2)] R has local units; so for any finite set of elements in R there is a common right unit. Thus by the finite dimensionality of V there is an element $e \in A$ such that $v \overline{\sqcap}^R(e) = v$ for all $v \in V$. Then for any $v \in V$ and $\varphi \in V^*$,

$$\epsilon(e^{\lambda^*})\varphi^{\lambda^*}(v) \stackrel{(6.1)}{=} (\varphi \otimes \epsilon)\lambda(v \otimes e) = \varphi(v\overline{\sqcap}^R(e)) = \varphi(v),$$

so that $(\epsilon \otimes V^*)\lambda^*(e \otimes \varphi) = \varphi$. This proves

$$V^* \subseteq \langle (\epsilon \otimes V^*) \lambda^* (e \otimes \varphi) | \varphi \in V^* \rangle \subseteq \langle (\omega \otimes V^*) \lambda^* (a \otimes \varphi) | \varphi \in V^*, \ \omega \in A^*, \ a \in A \rangle \subseteq V^*$$
hence the fullness of V^* .

The notion of antipode $S: A \to \mathbb{M}(A)$ on a regular weak multiplier bialgebra A was introduced in [2, Section 6], see Section 1.4. In what follows, we study the bearing of the existence of an antipode on the relation between left and right comodules.

Theorem 6.3. Let A be a regular weak multiplier bialgebra with a left and right full comultiplication, possessing an antipode S. Any left A-comodule V (with structure maps $\lambda, \rho: A \otimes V \to A \otimes V$) is a right A-comodule as well with the structure maps

$$\begin{array}{lll} \lambda^{S}: & V \otimes A \to V \otimes A, & v \otimes S(b)a & \mapsto ((V \otimes S)\varrho^{21}(v \otimes b))(1 \otimes a)\\ \varrho^{S}: & V \otimes A \to V \otimes A, & v \otimes aS(b) & \mapsto (1 \otimes a)((V \otimes S)\lambda^{21}(v \otimes b)). \end{array}$$

This is the object map of a functor - acting on the morphisms as the identity map from the category of left A-comodules to the category of right A-comodules.

Proof. We need to show first that λ^S and ρ^S are well-defined linear maps. By [2, Proposition 6.13], any element of A is a linear combination of elements of the form S(b)a or, alternatively, a linear combination of elements of the form aS(b). So λ^S and ρ^S can be defined by giving their values on the elements above. For any $a, b, c \in A$ and $v \in V$,

$$(6.2) \quad ((V \otimes S)\lambda^{21}(v \otimes c))(1 \otimes S(b)a) = ((V \otimes S)\mathsf{tw}((b \otimes 1)\lambda(c \otimes v)))(1 \otimes a)$$
$$\stackrel{(2.20)}{=} ((V \otimes S)\mathsf{tw}(\varrho(b \otimes v)(c \otimes 1)))(1 \otimes a)$$
$$= (1 \otimes S(c))((V \otimes S)\varrho^{21}(v \otimes b))(1 \otimes a).$$

The first and the last equalities follow by the anti-multiplicativity of S, see [2, Theorem 6.12]. So if for some elements $v^i \in V$ and $a^i, b^i \in A$ the (finite) sum $\sum_i v^i \otimes S(b^i)a^i$ is equal to zero, then we conclude from (6.2) by Lemma 1.6 that $\sum_i ((V \otimes S) \varrho^{21} (v^i \otimes b^i))(1 \otimes a^i) = 0$; hence λ^S is a well-defined linear map. A symmetric reasoning applies to ϱ^S .

Multiplying both sides of (6.2) on the left by $1 \otimes d$ for any $d \in A$, we obtain

$$\varrho^{S}(v \otimes dS(c))(1 \otimes S(b)a) = (1 \otimes dS(c))\lambda^{S}(v \otimes S(b)a).$$

In light of [2, Proposition 6.13], this proves the compatibility condition (2.10) between λ^{S} and ϱ^{S} .

By Proposition 2.1, (V, λ^S, ρ^S) is a right A-comodule provided that it satisfies the conditions in Proposition 2.1 (4.f). First we prove that the identity in Proposition 2.1 (2.a) holds. For any $v \in V$ and $a, b, c, d \in A$,

$$\begin{aligned} (V \otimes E_1)(\lambda^S \otimes A)\lambda^{S13}(v \otimes S(b)a \otimes S(d)c) \\ &= v^{\varrho\varrho'} \otimes E(S(b^{\varrho'})a \otimes S(d^{\varrho})c) \\ &= v^{\varrho\varrho'} \otimes (S \otimes S)((b^{\varrho'} \otimes d^{\varrho})E^{21})(a \otimes c) \\ &= v^{\varrho\varrho'} \otimes S(b^{\varrho'})a \otimes S(d^{\varrho})c \\ &= (\lambda^S \otimes A)\lambda^{S13}(v \otimes S(b)a \otimes S(d)c). \end{aligned}$$

where we used the implicit summation index notation $\rho(b \otimes v) = b^{\rho} \otimes v^{\rho} = b^{\rho'} \otimes v^{\rho'}$. In the second equality we used that by [2, Theorem 6.12 and Proposition 6.15], $E_1(S \otimes S) = (S \otimes S)E_2^{21}$. The third equality holds since $(E_2 \otimes V)(A \otimes \rho)\rho^{13} = (A \otimes \rho)\rho^{13}$ by a symmetric form of Proposition 2.1 on left *A*-comodules. By [2, Proposition 6.13], this proves the normalization condition in the form $(V \otimes E_1)(\lambda^S \otimes A)\lambda^{S13} = (\lambda^S \otimes A)\lambda^{S13}$.

It remains to prove the coassociativity condition (2.7). For any $v \in V$ and $a, b, c, d \in A$, introduce the implicit summation index notation $\rho(d \otimes v) = d^{\rho} \otimes v^{\rho}$, $\lambda(b \otimes v) = b^{\lambda} \otimes v^{\lambda}$ and $T_2(d \otimes b) = d^2 \otimes b^2$. Then

$$\begin{aligned} (\varrho^{S} \otimes A)(V \otimes T_{1})\lambda^{S13}(v \otimes aS(b) \otimes S(d)c) &= (\varrho^{S} \otimes A)(V \otimes T_{1})(v^{\varrho} \otimes aS(b) \otimes S(d^{\varrho})c) \\ &= (\varrho^{S} \otimes A)(v^{\varrho} \otimes T_{1}(a \otimes S(d^{\varrho 2})c)(S(b^{2}) \otimes 1)) \\ &= v^{\varrho\lambda} \otimes T_{1}(a \otimes S(d^{\varrho 2})c)(S(b^{2\lambda}) \otimes 1) \\ &= v^{\lambda} \otimes T_{1}(a \otimes S(d^{2})c)(S(b^{\lambda 2}) \otimes 1) \\ &= (V \otimes T_{1})(v^{\lambda} \otimes aS(b^{\lambda}) \otimes S(d)c) \\ &= (V \otimes T_{1})(\varrho^{S} \otimes A)(v \otimes aS(b) \otimes S(d)c). \end{aligned}$$

From this we conclude by [2, Proposition 6.13] that the coassociativity condition (2.7) holds. The second and the penultimate equalities hold by Lemma 1.7 and the fourth equality holds since $(A \otimes \lambda)(T_2 \otimes V)\varrho^{13} = (T_2 \otimes V)(A \otimes \lambda)$ by a symmetric form of Proposition 2.1 on left A-comodules.

Left A-comodule maps are evidently maps between the induced right A-comodules above. $\hfill \Box$

Clearly, there is a symmetric functor from the category of right comodules to the category of left comodules: It takes a right A-comodule (V, λ, ϱ) to the left A-comodule $(V, \lambda^S, \varrho^S)$, where for any $v \in V$ and $a, b \in A$,

(6.3)
$$\lambda^{S}(S(b)a \otimes v) = ((S \otimes V)\varrho^{21}(b \otimes v))(a \otimes 1)$$
$$\varrho^{S}(aS(b) \otimes v) = (a \otimes 1)((S \otimes V)\lambda^{21}(b \otimes v)).$$

It acts on the morphisms as the identity map.

Let A be a regular weak multiplier bialgebra with left and right full comultiplication and base algebra R. Consider the linear maps

$$E_1: V \otimes A \to V \otimes A, \qquad v \otimes a \mapsto ((-)v \otimes A)[E(1 \otimes a)] \equiv E(v \otimes a)$$

$$G_1: V \otimes A \to V \otimes A, \qquad v \otimes a \mapsto (v(-) \otimes A)[F(1 \otimes a)] \equiv (v \otimes 1)F(1 \otimes a)$$

as before (they are meaningful since for any $a \in A$, both $E(1 \otimes a)$ and $F(1 \otimes a)$ are elements of $R \otimes A$, see (2.3) and Proposition 4.3 (1) in [2]).

Lemma 6.4. Let A be a regular weak multiplier bialgebra with left and right full comultiplication possessing an antipode S. For any right A-comodule (V, λ, ϱ) , the linear map $\lambda^S : V \otimes A \to V \otimes A$ in (6.3) obeys the following identities.

(1) $\lambda^{S21} E_1 = \lambda^{S21}$. (2) $\lambda\lambda^{S21} = E_1$. (3) $\lambda^{S21}\lambda = G_1$

(3)
$$\lambda \quad \lambda = G_1$$

Proof. From (6.3) we have

$$(6.4) \qquad \lambda^{S21}: V \otimes A \to V \otimes A, \qquad v \otimes S(b)a \mapsto ((V \otimes S)\varrho(v \otimes b))(1 \otimes a).$$

Since (V, λ^S, ρ^S) in (6.3) is a left A-comodule, it follows by the compatibility axiom (2.20) that for any $v \in V$ and $a, b, c \in A$,

$$(1 \otimes bS(c))\lambda^{S21}(v \otimes a) = \varrho^{S21}(v \otimes bS(c)))(1 \otimes a) = (1 \otimes b)((V \otimes S)\lambda(v \otimes c))(1 \otimes a).$$

Thus simplifying by b,

(6.5)
$$(1 \otimes S(c))\lambda^{S^{21}}(v \otimes a) = ((V \otimes S)\lambda(v \otimes c))(1 \otimes a) \qquad \forall v \in V, \ a, c \in A.$$

(1) Using (6.5) in the first and the last equalities, Lemma 4.10 (2) in the second equality, and an identity in (6.14) of [2] (cf. (1.15)) in the third one, we obtain

$$(1 \otimes S(c))(\lambda^{S21}E_1(v \otimes a)) = (V \otimes \mu(S \otimes A))(\lambda \otimes A)E_1^{13}(v \otimes c \otimes a)$$

= $(V \otimes \mu(S \otimes A)E_1)(\lambda \otimes A)(v \otimes c \otimes a)$
= $(V \otimes \mu(S \otimes A))(\lambda \otimes A)(v \otimes c \otimes a)$
= $(1 \otimes S(c))\lambda^{S21}(v \otimes a).$

In light of Lemma 1.6, this proves assertion (1).

(2) For any $v \in V$ and $a, b, c \in A$,

$$(1 \otimes c) \quad (\lambda \lambda^{S21}(v \otimes S(b)a))$$

$$\stackrel{(6.4)}{=} \quad (1 \otimes c)\lambda((V \otimes S)\varrho(v \otimes b)(1 \otimes a))$$

$$= \quad ((V \otimes \mu(A \otimes S))(\varrho \otimes A)\varrho^{13}(v \otimes c \otimes b))(1 \otimes a)$$

$$= \quad ((V \otimes \mu(A \otimes S)T_2)\varrho^{13}(V \otimes R_2)(v \otimes c \otimes b))(1 \otimes a)$$

$$= \quad ((V \otimes \epsilon \otimes A)(\varrho \otimes A)(V \otimes T_4 \mathsf{tw}R_2)(v \otimes c \otimes b))(1 \otimes a)$$

$$= \quad (V \otimes \epsilon \otimes A)(\varrho \otimes A)(v \otimes T_1(c \otimes S(b)a))$$

$$= \quad ((\sqcap^R \otimes A)T_1(c \otimes S(b)a))(v \otimes 1)$$

$$= \quad (1 \otimes c)E(v \otimes S(b)a).$$

From this we conclude simplifying by c and using that A is spanned by elements of the form S(b)a, cf. [2, Proposition 6.13]. The second equality follows by (2.10) and the third

equality follows by

$$(\varrho \otimes A)\varrho^{13} \stackrel{(2.13)}{=} (\varrho \otimes A)\varrho^{13} (V \otimes E_2) = (\varrho \otimes A)\varrho^{13} (V \otimes T_2 R_2) \stackrel{(2.6)}{=} (V \otimes T_2)\varrho^{13} (V \otimes R_2).$$

In the fourth equality we used that by an identity in (6.14) in [2] (cf. (1.15)); by (3.3) in

In the fourth equality we used that by an identity in (6.14) in [2] (cf. (1.15)); by (3.3) in [2] (cf. (1.9)); and by (4.1),

$$(V \otimes \mu(A \otimes S)T_2)\varrho^{13} = (V \otimes \mu(A \otimes \sqcap^L))\varrho^{13}$$

= $(V \otimes \epsilon \otimes A)(V \otimes T_4 \mathsf{tw})\varrho^{13}$
= $(V \otimes \epsilon \otimes A)(\varrho \otimes A)(V \otimes T_4 \mathsf{tw}).$

The fifth equality is a consequence of Lemma 1.8(2) and the last equality holds since by the first axiom in (ix) in Definition 1.1 and by (3.4) in [2],

$$(1 \otimes d)((\sqcap^R \otimes A)T_1(c \otimes a)) = ((\sqcap^R \otimes A)T_3(c \otimes d))(1 \otimes a) = (1 \otimes dc)E(1 \otimes a),$$

for all $a, c, d \in A$, so that $(\sqcap^R \otimes A)T_1(c \otimes a) = (1 \otimes c)E(1 \otimes a)$. (3) The proof is similar to part (2). For any $v \in V$ and $a, b, c \in A$,

$$(1 \otimes bS(c))(\lambda^{S21}\lambda(v \otimes a))$$

$$= (1 \otimes b)((V \otimes \mu(S \otimes A))(\lambda \otimes A)\lambda^{13}(v \otimes c \otimes a))$$

$$= (1 \otimes b)((V \otimes \mu(S \otimes A)T_1)(\lambda \otimes A)(V \otimes R_1)(v \otimes c \otimes a))$$

$$= (1 \otimes b)((V \otimes A \otimes \epsilon)(V \otimes T_3)\lambda^{13}(V \otimes \mathsf{tw}R_1)(v \otimes c \otimes a))$$

$$= (1 \otimes b)((V \otimes A \otimes \epsilon)\lambda^{13}(V \otimes T_3\mathsf{tw}R_1)(v \otimes c \otimes a))$$

$$= (V \otimes A \otimes \epsilon)\lambda^{13}(v \otimes T_2(bS(c) \otimes a))$$

$$= (v \otimes 1)((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_2(bS(c) \otimes a))$$

$$= (v \otimes bS(c))F(1 \otimes a).$$

From this we conclude simplifying by bS(c) (cf. [2, Proposition 6.13]). The first equality follows by (6.5). The second one holds since

$$(\lambda \otimes A)\lambda^{13} \stackrel{(2.11)}{=} (\lambda \otimes A)\lambda^{13} (V \otimes E_1) = (\lambda \otimes A)\lambda^{13} (V \otimes T_1 R_1) \stackrel{(2.12)}{=} (V \otimes T_1)(\lambda \otimes A) (V \otimes R_1).$$

In the third equality we used that by an identity in (6.14) in [2] (cf. (1.15)) and by (3.3) in [2] (cf. (1.9)),

$$V \otimes \mu(S \otimes A)T_1 = V \otimes \mu(\sqcap^R \otimes A) = V \otimes (A \otimes \epsilon)T_3 \mathsf{tw}.$$

In the fourth equality we used that since λ is a right A-module map (cf. Proposition 2.1 (1)) and T_3 is a left A-module map, it follows for any $v \in V$ and $a, b, c \in A$ that

$$(V \otimes T_3)\lambda^{13}(v \otimes a \otimes bc) = \lambda(v \otimes b)^{13}(1 \otimes T_3(a \otimes c)) = \lambda^{13}(V \otimes T_3)(v \otimes a \otimes bc).$$

Hence by the idempotency of A, $(V \otimes T_3)\lambda^{13} = \lambda^{13}(V \otimes T_3)$. The fifth equality follows by Lemma 1.8 (1) and the last one holds since by the second axiom in (ix) in Definition 1.1 and by [2, Proposition 4.3 (1)] (cf. (1.10)),

$$((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_2(b \otimes a))(1 \otimes c) = (\overline{\sqcap}^R \otimes A)\mathsf{tw}(T_2(b \otimes a)(c \otimes 1))$$

$$= (\overline{\sqcap}^R \otimes A)\mathsf{tw}((b \otimes 1)T_4(c \otimes a))$$

$$= (1 \otimes b)((\overline{\sqcap}^R \otimes A)\mathsf{tw}T_4(c \otimes a)) = (1 \otimes b)F(1 \otimes ac),$$

for all $a, b, c \in A$; hence $(\overline{\sqcap}^R \otimes A) \mathsf{tw} T_2(b \otimes a) = (1 \otimes b) F(1 \otimes a)$.

Proposition 6.5. Let A be a regular weak multiplier bialgebra with left and right full comultiplication, possessing an antipode S. For a full left A-comodule (V, λ, ϱ) , also the right A-comodule $(V, \lambda^S, \varrho^S)$ in Theorem 6.3 is full.

Proof. By Lemma 6.11, (3.5) and Lemma 6.10 in [2],

(6.6)
$$\epsilon(aS(b)) = \epsilon \mu(A \otimes \sqcap^L) R_2(a \otimes b) = \epsilon \mu R_2(a \otimes b) = \epsilon(a \sqcap^R (b)), \quad \forall a, b \in A.$$

By Lemma 3.12 and (3.8) in [2], $\Box^R(b \overline{\Box}^L(a)) = \Box^R(a) \Box^R(b) = \Box^R(a \Box^R(b))$ for any $a, b \in A$. Applying to this identity the counit ε of R and using [2, Proposition 4.1], we conclude that $\epsilon(b\overline{\Box}^L(a)) = \epsilon(a \Box^R(b))$. Combining this with (6.6), we obtain

(6.7)
$$\epsilon(aS(b)) = \epsilon(b\overline{\sqcap}^{L}(a)), \quad \forall a, b \in A.$$

Using this identity in the second equality, it follows for any $v \in V$ and $a, b \in A$ that

(6.8)
$$(V \otimes \epsilon)\varrho^S(v \otimes aS(b)) = \epsilon(aS(b^{\lambda}))v^{\lambda} = \epsilon(b^{\lambda}\overline{\sqcap}^L(a))v^{\lambda} = (\epsilon \otimes V)\lambda(b\overline{\sqcap}^L(a) \otimes v),$$

where the implicit summation index notation $\lambda(b \otimes v) = b^{\lambda} \otimes v^{\lambda}$ is used. In the third equality we used that λ is a morphism of right $\mathbb{M}(A)$ -modules by Proposition 2.1 (1) and the idempotency of A. The last expression in (6.8) is the right action of $\sqcap^{L}(b \sqcap^{L}(a))$ on vobtained from the left A-comodule structure of V (cf. a symmetric counterpart of Theorem 4.5 on left comodules). Hence by Lemma 1.5 (1) and a symmetric counterpart of Theorem 4.5 on left comodules, V is spanned by elements of the form in (6.8). Taking into account [2, Proposition 6.13], this proves that $(V, \lambda^S, \varrho^S)$ is a full right A-comodule.

Let A be a regular weak multiplier bialgebra with left and right full comultiplication, possessing an antipode S. Let V be a finite dimensional full right A-comodule. Combining the constructions in Proposition 6.1 and in Theorem 6.3, we obtain a right A-comodule structure on V^* with the structure maps

(6.9)
$$\begin{array}{cccc} \lambda^{*S} : & V^* \otimes A \to V^* \otimes A, & \varphi \otimes S(b)a & \mapsto & [v \mapsto \varphi(v^{\varrho})] \otimes S(b^{\varrho})a \\ \varrho^{*S} : & V^* \otimes A \to V^* \otimes A, & \varphi \otimes aS(b) & \mapsto & [v \mapsto \varphi(v^{\lambda})] \otimes aS(b^{\lambda}), \end{array}$$

where the implicit summation index notation $\varrho(v \otimes b) = v^{\varrho} \otimes b^{\varrho}$ and $\lambda(v \otimes b) = v^{\lambda} \otimes b^{\lambda}$ is used. This right A-comodule V^* is full by Proposition 6.2 and Proposition 6.5.

Proposition 6.6. Let A be a regular weak multiplier bialgebra with left and right full comultiplication, possessing an antipode S. Let (V, λ, ϱ) be a finite dimensional full right A-comodule and consider the A-comodule $(V^*, \lambda^{*S}, \varrho^{*S})$ in (6.9). In terms of the Nakayama automorphism ϑ of R (cf. [2, Theorem 4.6 (3)]), the corresponding R-actions in Theorem 4.5 on V^* come out as

$$r\varphi = \varphi(-\vartheta^{-1}(r))$$
 and $\varphi r = \varphi(r-).$

In particular, the isomorphism $V^* \cong \text{Hom}_R(V, R)$ in Proposition 3.3 is an isomorphism of R-bimodules.

Proof. By [2, Proposition 6.13], R is spanned by elements of the form $\overline{\sqcap}^R(S(b)a)$, for $a, b \in A$. So we only compute its right action on an arbitrary element $\varphi \in V^*$.

$$\begin{split} \varphi \overline{\sqcap}^R(S(b)a) &= (V^* \otimes \epsilon) \lambda^{*S}(\varphi \otimes S(b)a) = \varphi(-^{\varrho}) \epsilon(S(b^{\varrho})a) \\ &= \varphi(-^{\varrho}) \epsilon(\overline{\sqcap}^R(a)b^{\varrho}) = (\varphi \otimes \epsilon) \varrho(- \otimes \overline{\sqcap}^R(a)b) \\ &= \varphi(\sqcap^R(\overline{\sqcap}^R(a)b) -) = \varphi(\overline{\sqcap}^R(S(b)a) -). \end{split}$$

The fourth equality holds because ρ is a left A-module map (see Proposition 2.1 (1)), hence by the idempotency of A it is also a left R-module map. In the last equality we used that by Lemma 6.11, Lemma 3.11, and identity (2.2) in [2] and by the equality $T_1R_1 = E_1$,

$$\overline{\sqcap}^{R}(S(b)a) = \overline{\sqcap}^{R}\mu(\sqcap^{R}\otimes A)R_{1}(b\otimes a) = \sqcap^{R}\mu(A\otimes\overline{\sqcap}^{R})R_{1}(b\otimes a)$$
$$= \sqcap^{R}(A\otimes\epsilon)T_{1}R_{1}(b\otimes a) = \sqcap^{R}(A\otimes\epsilon)E_{1}(b\otimes a) = \sqcap^{R}(\overline{\sqcap}^{R}(a)b)$$

for any $a, b \in A$. Applying the counit ε of R to both sides, by [2, Proposition 4.1] and [2, Lemma 4.5 (3)] we obtain $\epsilon(S(b)a) = \epsilon(\overline{\sqcap}^R(a)b)$ justifying the third equality. For $\Phi \in \operatorname{Hom}_R(V, R), r \in R$, and the counit ε on R, $\varepsilon(\Phi r) = \varepsilon \Phi(r-)$ so the isomorphism in Proposition 3.3 is an isomorphism of right R-modules.

Symmetrically, by [2, Proposition 6.13], R is spanned by elements of the form $\sqcap^R(aS(b))$, for $a, b \in A$. So we only compute its left action on an arbitrary element $\varphi \in V^*$.

$$\Box^{R}(aS(b))\varphi = (V^{*} \otimes \epsilon)\varrho^{*S}(\varphi \otimes aS(b)) = \varphi(-^{\lambda})\epsilon(aS(b^{\lambda}))$$

= $\varphi(-^{\lambda})\epsilon(b^{\lambda}\overline{\sqcap}^{L}(a)) = (\varphi \otimes \epsilon)\lambda(-\otimes b\overline{\sqcap}^{L}(a))$
= $\varphi(-\overline{\sqcap}^{R}(b\overline{\sqcap}^{L}(a))) = \varphi(-\vartheta^{-1}\sqcap^{R}(aS(b))).$

In the third equality we applied (6.7) and in the last equality we used that by Proposition 4.9, Lemma 6.11, Lemma 3.11 and identity (2.1) in [2] and by the equality $T_2R_2 = E_2$,

$$\vartheta^{-1} \sqcap^R (aS(b)) = \overline{\sigma} \,\overline{\sqcap}^L \mu(A \otimes \sqcap^L) R_2(a \otimes b) = \overline{\sqcap}^R \mu(\overline{\sqcap}^L \otimes A) R_2(a \otimes b) = \overline{\sqcap}^R (\epsilon \otimes A) T_2 R_2(a \otimes b) = \overline{\sqcap}^R (\epsilon \otimes A) E_2(a \otimes b) = \overline{\sqcap}^R (b \overline{\sqcap}^L(a)).$$

For $\Phi \in \operatorname{Hom}_R(V, R)$, $r \in R$, and the counit ε on R,

$$\varepsilon(r\Phi) = \varepsilon(r\Phi(-)) = \varepsilon(\Phi(-)\vartheta^{-1}(r)) = \varepsilon\Phi(-\vartheta^{-1}(r))$$

so the isomorphism in Proposition 3.3 is an isomorphism of left R-modules.

We are ready to prove the main result of the section:

Theorem 6.7. Let A be a regular weak multiplier bialgebra over a field k with left and right full comultiplication, possessing an antipode S. Then for any finite dimensional full right A-comodule V, the right A-comodule $V^* := \text{Lin}(V, k)$ in (6.9) is the dual of V in the monoidal category $M^{(A)}$ of full right A-comodules in Theorem 5.10.

Proof. Let us choose a finite k-basis $\{v_i\}$ in V with dual basis $\{\varphi^i\}$ in V^* . The isomorphism in Proposition 3.3 takes $\{\varphi^i\}$ to the set $\{\Phi^i \in \mathsf{Hom}_R(V, R)\}$, satisfying

$$\sum_{i} v_i \Phi^i(wr) = \sum_{i} (\varphi_i \otimes V)((w \otimes v_i)\delta(r)) = w(\mu\delta(r)) = wr,$$

for any $w \in V$ and $r \in R$. Since the *R*-action on *V* is surjective by Theorem 4.5, this means that $\sum_i v_i \Phi^i(-) = V$ and hence $V \otimes_R V^* \cong V \otimes_R \operatorname{Hom}_R(V, R) \cong \operatorname{Hom}_R(V, V)$ via

$$(6.10) \qquad \kappa: V \otimes_R V^* \to \operatorname{Hom}_R(V, V), \qquad v \otimes_R \psi \mapsto [ws \mapsto (\psi \otimes V)((w \otimes v)\delta(s))].$$

We need to show that the evaluation map

$$\operatorname{Hom}_{R}(V,R)\otimes_{R}V\to R,\qquad\Psi\otimes_{R}w\mapsto\Psi(w)$$

and the coevaluation map

$$R \to V \otimes_R \operatorname{Hom}_R(V, R), \qquad r \mapsto \sum_i r v_i \otimes_R \Phi^i = \sum_i v_i \otimes_R \Phi^i r$$

are morphisms of right A-comodules. Equivalently, using the (R-bimodule) isomorphism $\operatorname{Hom}_R(V, R) \cong V^*$ in Proposition 3.3,

(6.11)
$$\operatorname{ev}: V^* \otimes_R V \to R, \qquad \psi \otimes wr \mapsto (\psi \otimes R)((w \otimes 1)\delta(r))$$

and

(6.12)
$$\operatorname{coev} : R \to V \otimes_R V^*, \qquad r \mapsto \sum_i r v_i \otimes_R \varphi^i = \sum_i v_i \otimes_R \varphi^i r$$

are morphisms of right A-comodules.

We know from Proposition 5.6 that the canonical epimorphism $\pi : V^* \otimes V \to V^* \otimes_R V$ is a morphism of A-comodules. Thus by Remark 5.9, ev is a morphism of A-comodules if and only if $ev\pi$ is so, that is,

(6.13)
$$(\operatorname{ev}\pi\otimes A)\lambda^{*S13}(V^*\otimes\lambda) = \lambda_R(\operatorname{ev}\pi\otimes A).$$

By the idempotency of A, by surjectivity of the R-actions on V (cf. Theorem 4.5) and Lemma 1.5 (3), the domain $V^* \otimes V \otimes A$ of these maps in (6.13) is spanned by elements of the form $\psi \otimes \sqcap^R(b)v \otimes \sqcap^L(cd)a$, for $v \in V$, $\psi \in V^*$ and $a, b, c, d \in A$. So it is enough to show that both sides of (6.13) are equal if evaluated on such elements. By Lemma 4.9 (7) and (1), $\psi \otimes \sqcap^R(b)v \otimes \sqcap^L(cd)a$ is taken by $V^* \otimes \lambda$ to

$$\psi \otimes v^{\lambda} \overline{\sqcap}^R(cd) \otimes \sqcap^R(b) a^{\lambda} = \psi \otimes v^{\lambda} \overline{\sqcap}^R(cd) \otimes S(b^1) a^{\lambda 1}$$

where the implicit summation index notation $\lambda(v \otimes a) = v^{\lambda} \otimes a^{\lambda}$ and $T_1(b \otimes a) = b^1 \otimes a^1$ is used and the equality follows by (6.14) in [2] (cf. (1.15)). Applying to this λ^{*S13} and using the implicit summation index notation $\varrho(-\otimes b) = (-)^{\varrho} \otimes b^{\varrho}$, we get

$$\psi(-^{\varrho}) \otimes v^{\lambda} \overline{\sqcap}^R(cd) \otimes S(b^{1\varrho}) a^{\lambda 1}.$$

In light of [2, Lemma 4.5 (2)], denoting $T_1(c \otimes d) = c^{1'} \otimes d^{1'}$, this is taken by $ev\pi \otimes A$ to

$$\begin{split} \psi((v^{\lambda}\overline{\sqcap}^{R}(d^{1'}))^{\varrho})\overline{\sqcap}^{R}(c^{1'})\otimes S(b^{1\varrho})a^{\lambda 1} \\ &= \psi(v^{\lambda\varrho})\overline{\sqcap}^{R}(c^{1'})\otimes S(b^{1\varrho}\overline{\sqcap}^{R}(d^{1'}))a^{\lambda 1} \\ &= \psi(v^{\lambda\varrho})\overline{\sqcap}^{R}(c^{1'})\otimes \sqcap^{L}(d^{1'})S(b^{1\varrho})a^{\lambda 1} \\ &= \psi(v^{\varrho})\overline{\sqcap}^{R}(c^{1'})\otimes \sqcap^{L}(d^{1'})S(b^{\varrho 1})a^{1} \\ &= \psi(v^{\varrho})\overline{\sqcap}^{R}(c^{1})\otimes \sqcap^{L}(d^{1})\sqcap^{R}(b^{\varrho})a \\ &= \psi(v^{\varrho})E(1\otimes \sqcap^{L}(cd)\sqcap^{R}(b^{\varrho})a) \\ &= E(1\otimes \sqcap^{L}(cd)\mathsf{ev}(\psi\otimes_{R}\sqcap^{R}(b)v)a) \\ &= E(1\otimes\mathsf{ev}(\psi\otimes_{R}\sqcap^{R}(b)v)\sqcap^{L}(cd)a) \\ &= \lambda_{R}(\mathsf{ev}\pi\otimes A)(\psi\otimes \sqcap^{R}(b)v\otimes \sqcap^{L}(cd)a). \end{split}$$

The first equality follows by Lemma 4.9 (6), the second one does by [2, Lemma 6.14], the third one does by (2.7), the fourth one does by (6.14) in [2] (cf. (1.15)) and the fifth one follows since by (2.3) in [2] and Lemma 1.3 (4),

$$\overline{\sqcap}^R(c^1) \otimes \sqcap^L(d^1) = (\mathbb{M}(A) \otimes \sqcap^L)(E(1 \otimes cd)) = E(1 \otimes \sqcap^L(cd)).$$

In the sixth equality we used that by Lemma 4.12 (1) and by surjectivity of the right *R*-action on *V*, $(\psi \otimes \sqcap^R) \varrho(v \otimes b) = \operatorname{ev}(\psi \otimes_R \sqcap^R(b)v)$, for any $v \in V$, $\psi \in V^*$ and $b \in A$. In the penultimate equality we made use of (3.9) in [2]. This proves that (6.13) holds hence ev is a morphism of *A*-comodules.

The coevaluation map $\operatorname{coev} : R \to V \otimes_R V^*$ factorizes through the canonical epimorphism $\pi : V \otimes V^* \to V \otimes_R V^*$ (which is a morphism of A-comodules by Proposition 5.6) via the map

$$\operatorname{coev}': R \to V \otimes V^*, \qquad r \mapsto \sum_i r v_i \otimes \varphi^i.$$

Hence coev is a morphism of A-comodules if and only if

(6.14)
$$(\kappa \,\pi \,\operatorname{coev}' \otimes A)\lambda_R = (\kappa \,\pi \otimes A)\lambda^{13}(V \otimes \lambda^{*S})(\operatorname{coev}' \otimes A),$$

where κ is the isomorphism in (6.10). By [2, Proposition 6.13], the domain $R \otimes A$ of these maps in (6.14) is spanned by elements of the form $r \otimes S(b)a$, for $a, b \in A$ and $r \in R$. So it is enough to prove that both sides of (6.14) are equal if evaluated on such elements. Applying $\operatorname{coev}' \otimes A$ to $r \otimes S(b)a$ and omitting the summation symbol for brevity, we obtain $rv_i \otimes \varphi^i \otimes S(b)a$. This is taken by $V \otimes \lambda^{*S}$ to $rv_i \otimes \varphi^i(-\varrho) \otimes S(b^\varrho)a$ (where the implicit summation index notation $\varrho(-\otimes b) = (-)^{\varrho} \otimes b^{\varrho}$ is used) and then by $(\pi \otimes A)\lambda^{13}$ to $(rv_i)^{\lambda} \otimes_R \varphi^i(-\varrho) \otimes (S(b^\varrho)a)^{\lambda}$ (where we introduced the implicit summation index notation $\lambda(v \otimes a) = v^{\lambda} \otimes a^{\lambda}$). Applying $\kappa \otimes A$ (cf. (6.10)), and introducing the implicit summation index notation $\delta(s) = s_1 \otimes s_2$ for the comultiplication δ of R and any $s \in R$, we get

$$[ws \mapsto \varphi^i((ws_1)^{\varrho})(rv_i)^{\lambda}s_2] \otimes (S(b^{\varrho})a)^{\lambda}$$

Now

$$(6.15) \varphi^{i}((ws_{1})^{\varrho})\lambda(rv_{i}\otimes S(b^{\varrho})a)(s_{2}\otimes 1) = \lambda(r(ws_{1})^{\varrho}\otimes S(b^{\varrho})a)(s_{2}\otimes 1)$$
$$= (1\otimes r)(\lambda\lambda^{S21}(ws_{1}\otimes S(b)a))(s_{2}\otimes 1)$$
$$= (1\otimes r)E(ws_{1}\otimes S(b)a))(s_{2}\otimes 1)$$
$$= E(ws\otimes rS(b)a)).$$

In the second equality we applied Lemma 4.9 (7) and in the penultimate equality we applied Lemma 6.4 (2) (the map λ^S appearing here is that in (6.3)). The last equality holds by identity (3.7) in [2] and the fact that the comultiplication of R splits its multiplication (i.e $s_1s_2 = s$).

On the other hand, applying λ_R to $r \otimes S(b)a$ we obtain $E(1 \otimes rS(b)a)$. Applying to this $\operatorname{coev} \otimes A = \pi \operatorname{coev}' \otimes A$ we get $(\pi \otimes A)[(v_i \otimes \varphi^i \otimes 1)(1 \otimes E(1 \otimes rS(b)a))]$ which is taken by the isomorphism $\kappa \otimes A$ (cf. (6.10)) to

(6.16)
$$[ws \mapsto v_i s_2] \otimes (\varphi^i \otimes A) E_1(ws_1 \otimes rS(b)a) = E(- \otimes rS(b)a).$$

The first expression is obtained applying Proposition 6.6 and the second one is obtained using again that $s_1s_2 = s$. Comparing (6.15) and (6.16), we conclude that (6.14) holds hence **coev** is a morphism of A-comodules too.

7. HOPF MODULES

This section is devoted to the study of Hopf modules over a regular weak multiplier bialgebra A. These are vector spaces carrying compatible (non-degenerate idempotent) module and (full) comodule structures. Whenever the comultiplication is left and right full and there exists an antipode, we prove the Fundamental Theorem of Hopf Modules. That is, an equivalence between the category of A-Hopf modules and the category of firm modules over the base algebra $\sqcap^L(A) = \overline{\sqcap}^L(A)$.

Definition 7.1. For a regular weak multiplier bialgebra A, a right-right Hopf module is a vector space V carrying the following structures.

• V is a non-degenerate right A-module with a surjective action $\cdot : V \otimes A \to V$,

• (V, λ, ϱ) is a full right A-comodule,

obeying the following equivalent conditions.

- (7.1) $\varrho(\cdot \otimes A) = (\cdot \otimes A)(V \otimes T_3)\varrho^{13}$
- (7.2) $\lambda(\cdot \otimes A) = (\cdot \otimes A)\lambda^{13}(V \otimes T_1).$

A morphism of Hopf modules is a linear map which is both a morphism of modules and a morphism of comodules. The category of A-Hopf modules will be denoted by $M_{(A)}^{(A)}$.

Conditions (7.1) and (7.2) are equivalent, indeed: any one of them asserts precisely that \cdot is a comodule map (with respect to the comodule structure in Lemma 5.1 on the domain).

Example 7.2. Consider a regular weak multiplier bialgebra A with right full comultiplication. Via the multiplication $\mu : A \otimes A \to A$ and the maps $T_1, T_3 : A \otimes A \to A \otimes A$, A is an A-Hopf module itself by the first condition in axiom (v) in Definition 1.1.

Remark 7.3. A usual, unital (weak) bialgebra A can be regarded as a monoid in the monoidal category of A-comodules. An A-Hopf module is then precisely a module over the monoid A in the category of A-comodules.

For a regular weak multiplier bialgebra A with right full comultiplication a similar interpretation is possible. It follows by Example 4.7 that the multiplication $\mu : A \otimes A \to A$ factorizes as the canonical epimorphism $\pi : A \otimes A \to A \otimes_R A$ composed with an associative R-bilinear multiplication $\mu_R : A \otimes_R A \to A$. By axiom (v) in Definition 1.1 μ is a morphism of A-comodules; hence so is μ_R by Remark 5.9. That is to say, via the multiplication μ_R , A is a (non-unital) monoid in $M^{(A)}$.

For any associative action $\cdot : V \otimes A \to V$, it follows by (1.13) and by [2, Lemma 3.7 (2)] that for any $v \in V$ and $a, b \in A$

(7.3)
$$G_1(v \otimes ab) = v \cdot \mu^{\mathsf{op}}(A \otimes \overline{\sqcap}^R) T_4(b \otimes a) = v \cdot ab$$

So by the idempotency of A and Proposition 5.2, the A-actions $\cdot : V \otimes A \to V$ are in a bijective correspondence with the A-actions $\star : V \otimes_R A \to V$ such that $\cdot = \star \pi$. Moreover, if V carries a right A-comodule structure (λ, ϱ) , then (7.1) (equivalently, (7.2)) asserts precisely that \cdot is a comodule map. Hence by Remark 5.9, any of (7.1) and (7.2) is equivalent to \star being a morphism of A-comodules; that is, to (V, \star) being a module over (A, μ_R) in $M^{(A)}$.

Proposition 7.4. Let A be a regular weak multiplier bialgebra with left and right full comultiplication. Denote by L the coinciding range of the maps \sqcap^L and $\overline{\sqcap}^L : A \to \mathbb{M}(A)$ and regard A as a left L-module via the multiplication in $\mathbb{M}(A)$. There is a functor $(-) \otimes_L A$ from the category M_L of firm right L-modules to the category $M_{(A)}^{(A)}$ of A-Hopf modules.

Proof. Since both A and L are subalgebras of the associative algebra $\mathbb{M}(A)$, the A-action provided by the multiplication of A is a morphism of left L-modules. The maps T_1 and T_3 are also left L-module maps by [2, Lemma 3.3]. These observations imply that the A-Hopf module in Example 7.2 induces an A-module structure $P \otimes_L \mu$ and an A-comodule structure $(P \otimes_L T_1, P \otimes_L T_3)$ on $P \otimes_L A$, for any firm right L-module P. By the surjectivity of μ , this A-action is surjective. Let us see that it is also non-degenerate. Denote by $\delta : L \to L \otimes L$ the comultiplication in the coseparable coalgebra L. If for some $p \in P$, $l \in L$ and $a \in A$ we have $pl \otimes_L ab = 0$ for all $b \in A$, then also $(p \otimes 1)\delta(l)(1 \otimes ab) \in P \otimes A$ is equal to

zero for all $b \in A$. By [8, Lemma 1.11], $P \otimes A$ is a non-degenerate right A-module hence $0 = (p \otimes 1)\delta(l)(1 \otimes a)$. Applying the epimorphism $P \otimes A \to P \otimes_L A$ and using that δ is a section of the multiplication in L, we conclude that $pl \otimes_L a = 0$ as needed. Since the comultiplication of A is right full by assumption, the comodule $P \otimes_L A$ is also full so that $P \otimes_L A$ is an A-Hopf module. For any right L-module map $f : P \to P'$, $f \otimes_L A$ is evidently a morphism of Hopf modules.

If A is a regular non-weak multiplier bialgebra (i.e. it obeys the equivalent conditions in [2, Theorem 2.11]) with left and right full comultiplication, then the functor in Proposition 7.4 reduces to the functor $(-) \otimes A$ from the category of vector spaces to the category $M_{(A)}^{(A)}$ of right-right A-Hopf modules, possessing a right adjoint $\operatorname{Hom}_{(A)}^{(A)}(A, -)$. An analogous result fails to hold if A is a weak multiplier bialgebra since then $\operatorname{Hom}_{(A)}^{(A)}(A, V)$ may not be a firm right L-module for an arbitrary Hopf module V. However, if the comultiplication is left and right full and there exists an antipode for A, then the functor $(-) \otimes_L A$ in Proposition 7.4 turns out to be even an equivalence. But its inverse is no longer of the form $\operatorname{Hom}_{(A)}^{(A)}(A, -)$ as we shall see below.

Proposition 7.5. Let A be a regular weak multiplier bialgebra with left and right full comultiplication possessing an antipode S. For any right-right A-Hopf module $(V, \cdot, \lambda, \varrho)$, there is a linear map

$$\varpi_V: V \to \operatorname{Hom}_{(A)}^{(A)}(A, V), \qquad v \mapsto \cdot \lambda^{S21}(v \otimes -),$$

where λ^S is the map from (6.3).

Proof. Since λ^{S21} is a morphism of right A-modules, so is $\varpi_V(v)$, for any $v \in V$. Let us see that $\varpi_V(v)$ is also a morphism of A-comodules. The proof of this is analogous to [21, Lemma 4.4] in the non-weak case. For any $v \in V$ and $a, b \in A$,

$$\begin{split} \lambda(\varpi_V(v) \otimes A)(a \otimes b) &= \lambda(\cdot \lambda^{S21} \otimes A)(v \otimes a \otimes b) \\ \stackrel{(7.2)}{=} & (\cdot \otimes A)\lambda^{13}(V \otimes T_1)(\lambda^{S21} \otimes A)(v \otimes a \otimes b) \\ \stackrel{(7.3)}{=} & (\cdot \otimes A)(G_1 \otimes A)\lambda^{13}(V \otimes T_1)(\lambda^{S21} \otimes A)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21}\lambda \otimes A)\lambda^{13}(V \otimes T_1)(\lambda^{S21} \otimes A)(v \otimes a \otimes b) \\ \stackrel{(2.12)}{=} & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(\lambda\lambda^{S21} \otimes A)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(E_1 \otimes A)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(E_1 \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes A)(V \otimes T_1)(v \otimes a \otimes b) \\ &= & (\cdot \otimes A)(\lambda^{S21} \otimes A)(V \otimes A)($$

The fourth, sixth and eighth equalities follow by parts (3), (2) and (1) of Lemma 6.4, respectively. In the seventh equality we used that by axiom (vii) in Definition 1.1, for any $v \in V$ and $a, b \in A$,

$$(V \otimes T_1)(E_1 \otimes A)(v \otimes a \otimes b) = ((-) \cdot v \otimes A \otimes A)(R \otimes T_1)[E(1 \otimes a) \otimes b]$$

= $((-) \cdot v \otimes A \otimes A)[(E \otimes 1)(1 \otimes T_1(a \otimes b))]$
= $(E_1 \otimes A)(V \otimes T_1)(v \otimes a \otimes b).$

Example 7.6. Let A be a regular weak multiplier bialgebra with left and right full comultiplication possessing an antipode S, and consider its right-right Hopf module A in Example 7.2. For any $a, b, c, d \in A$, it follows form (6.3) in [2], the anti-multiplicativity of S (cf. [2, Theorem 6.12]) and (1.6) that

$$(d \otimes 1)R_1(c \otimes S(b)a) = ((A \otimes S)T_2(d \otimes c))(1 \otimes S(b)a)$$

= $((A \otimes S)((1 \otimes b)T_2(d \otimes c)))(1 \otimes a)$
= $((A \otimes S)((d \otimes 1)T_3(c \otimes b)))(1 \otimes a)$
= $(d \otimes 1)((A \otimes S)T_3(c \otimes b))(1 \otimes a).$

Simplifying by d, we conclude that

$$T_1^{S21}(c \otimes S(b)a) \stackrel{(6.3)}{=} ((A \otimes S)T_3(c \otimes b))(1 \otimes a) = R_1(c \otimes S(b)a);$$

that is, (in view of [2, Proposition 6.13]) $T_1^{S21} = R_1$. The corresponding map in Proposition 7.5 comes out as

$$\varpi_A(a)(b) = \mu R_1(a \otimes b) = \sqcap^L(a)b$$

for any $a, b \in A$, where the last equality follows by [2, Lemma 6.10].

Proposition 7.7. Let A be a regular non-weak multiplier bialgebra (i.e. a regular weak multiplier bialgebra satisfying the equivalent assertions in [2, Theorem 2.11]) with left and right full comultiplication possessing an antipode S. For any right-right A-Hopf module $(V, \cdot, \lambda, \varrho)$, the map ϖ_V in Proposition 7.5 is surjective.

Proof. Under the hypotheses of the proposition, it follows by Lemma 6.4 that λ^{S21} is the inverse of λ . In particular, R_1 is the inverse of T_1 .

Take a morphism of Hopf modules $f : A \to V$. It is a right A-module map satisfying $\lambda(f \otimes A) = (f \otimes A)T_1$, equivalently, $(f \otimes A)R_1 = \lambda^{S21}(f \otimes A)$. Therefore for any $a, b \in A$,

$$\varpi_V(f(a))(b) = \cdot \lambda^{S^{21}}(f(a) \otimes b) = \cdot (f \otimes A)R_1(a \otimes b) = f\mu R_1(a \otimes b) = \epsilon(a)f(b).$$

where in the penultimate equality we used that f is a right A-module map and the last equality follows by [2, Lemma 6.10]. Thus if we choose $a \in A$ such that $\epsilon(a) = 1$, then $\varpi_V(f(a)) = f$.

Example 7.8. The equivalent assertions in [2, Theorem 2.11] are really needed to prove Proposition 7.7. If they do not hold, then the discussed map ϖ_V may not be surjective for all Hopf modules V. Consider e.g. an arbitrary groupoid C and the weak multiplier bialgebra A := kC in [2, Example 2.12], spanned by the morphisms in C. For its rightright Hopf module A = kC in Example 7.2, it follows by Example 7.6 that the range of ϖ_A is the range of the map $\sqcap^L : A \to \mathbb{M}(A)$; which is spanned by the identity morphisms in C. This is a proper non-unital subalgebra of $\operatorname{Hom}_{(A)}^{(A)}(A, A)$ (which contains also infinite linear combinations of the identity morphisms of C regarded as multipliers on A = kC).

Definition 7.9. Consider a regular weak multiplier bialgebra A with left and right full comultiplication possessing an antipode S. For any right-right A-Hopf module V, the coinvariant space is defined as the range $V^c := \varpi_V(V) \subseteq \operatorname{Hom}_{(A)}^{(A)}(A, V)$ of the map in Proposition 7.5.

Proposition 7.10. For any regular weak multiplier bialgebra A with left and right full comultiplication possessing an antipode S, there is a functor $(-)^c$ from the category $M_{(A)}^{(A)}$ of right-right A-Hopf modules to the category M_L of firm right modules over the base algebra $L := \sqcap^L(A) = \overline{\sqcap}^L(A)$.

Proof. Since L is a (non-unital) subalgebra of $\operatorname{Hom}_{(A)}^{(A)}(A, A)$, for any right-right A-Hopf module V the vector space $\operatorname{Hom}_{(A)}^{(A)}(A, V)$ is a right L-module via composition:

$$(fl)(a) := f(la), \quad \text{for } f \in \operatorname{Hom}_{(A)}^{(A)}(A, V), \ l \in L, \ a \in A.$$

Let us see that the subspace $V^c \subseteq \operatorname{Hom}_{(A)}^{(A)}(A, V)$ is closed under this action. Using [2, Lemma 6.14] in the third equality and Lemma 4.9 (3) in the penultimate one, for any $v \in V$ and $a, b, c \in A$ we get

$$(\varpi_{V}(v) \sqcap^{L}(c))(S(b)a) = \varpi_{V}(v)(\sqcap^{L}(c)S(b)a) = \cdot \lambda^{S21}(v \otimes \sqcap^{L}(c)S(b)a)$$
$$= \cdot \lambda^{S21}(v \otimes S(b\overline{\sqcap}^{R}(c))a) \stackrel{(6.4)}{=} \cdot [((V \otimes S)\varrho(v \otimes b\overline{\sqcap}^{R}(c)))(1 \otimes a)]$$
$$= \cdot [((V \otimes S)\varrho(\overline{\sqcap}^{R}(c)v \otimes b))(1 \otimes a)] = \varpi_{V}(\overline{\sqcap}^{R}(c)v)(S(b)a).$$

Thus regarding V as a left $R := \sqcap^R(A) = \overline{\sqcap}^R(A)$ -module as in Theorem 4.5, it follows by [2, Proposition 6.13] that

(7.4)
$$\varpi_V(v) \sqcap^L (c) = \varpi_V(\overline{\sqcap}^R(c)v), \quad \forall v \in V, \ c \in A$$

proving that V^c is an *L*-submodule of $\operatorname{Hom}_{(A)}^{(A)}(A, V)$. Moreover, since the left *R*-action on *V* is surjective by Theorem 4.5, (7.4) also implies that the right *L*-action on V^c is surjective. Since *L* has local units by a symmetric variant of [2, Theorem 4.6 (2)], this proves that V^c is a firm right *L*-module.

For a morphism of Hopf modules $f: V \to V'$, and for $v \in V$ and $a, b \in A$,

$$\begin{aligned} f\varpi_V(v)(S(b)a) &= f \cdot (((V \otimes S)\varrho(v \otimes b))(1 \otimes a)) \\ &= \cdot (((V' \otimes S)\varrho'(f(v) \otimes b))(1 \otimes a)) = \varpi_{V'}(f(v))(S(b)a). \end{aligned}$$

Hence (in view of [2, Proposition 6.13]) there is a map

$$f^c: V^c \to V'^c, \qquad \varpi_V(v) \mapsto f \varpi_V(v) = \varpi_{V'}(f(v)).$$

It is a morphism of L-modules since for any $v \in V$ and $c \in A$,

$$f(\varpi_V(v) \sqcap^L (c)) \stackrel{(7.4)}{=} f \varpi_V(\overline{\sqcap}^R(c)v) = \varpi_{V'}f(\overline{\sqcap}^R(c)v)$$
$$= \varpi_{V'}(\overline{\sqcap}^R(c)f(v)) \stackrel{(7.4)}{=} (\varpi_{V'}f(v)) \sqcap^L (c) = (f\varpi_V(v)) \sqcap^L (c).$$

In the third equality we used that f is a left R-module homomorphism by Theorem 4.5. \Box

Before we can prove the Fundamental Theorem of Hopf Modules, we need the following technical lemma.

Lemma 7.11. Consider a regular weak multiplier bialgebra A with left and right full comultiplication possessing an antipode S. For any right-right A-Hopf module $(V, \cdot, \lambda, \varrho)$, there is a linear map

$$\zeta^0: V \to V^c \otimes A, \qquad v \cdot a \mapsto (\varpi_V \otimes A) \lambda(v \otimes a),$$

where V^c is the coinvariant space in Definition 7.9 and ϖ_V is the map in Proposition 7.5.

Proof. We only need to prove that ζ^0 is a well-defined linear map; that is, it takes zero to zero. So assume that $v \cdot a = 0$. Then using the implicit summation index notation $T_1(a \otimes b) = a^1 \otimes b^1$ and $\lambda(v \otimes b) = v^\lambda \otimes b^\lambda$, it follows for any $b \in A$ that

(7.5)
$$0 = (\varpi_V \otimes A)\lambda(v \cdot a \otimes b) \stackrel{(7.2)}{=} \varpi_V(v^{\lambda} \cdot a^1) \otimes b^{1\lambda} = \varpi_V(v^{\lambda} \overline{\sqcap}^R(a^1)) \otimes b^{1\lambda}$$
$$= (\varpi_V \otimes A)\lambda(V \otimes \mu(\sqcap^L \otimes A)T_1)(v \otimes a \otimes b)$$
$$= (\varpi_V \otimes A)\lambda(v \otimes ab) = ((\varpi_V \otimes A)\lambda(v \otimes a))(1 \otimes b),$$

so that by non-degeneracy of the right A-module $\operatorname{Hom}_{(A)}^{(A)}(A, V) \otimes A$, the expression $(\varpi_V \otimes A)\lambda(v \otimes a)$ is equal to zero as needed.

The third equality in (7.5) follows by the following reasoning. By (1.6), the antimultiplicativity of S (cf. [2, Theorem 6.12]) and (6.14) in [2] (cf. 1.15), for any $a, b, c \in A$

$$c(\mu(A \otimes S)T_3(a \otimes b)) = \mu(A \otimes S)((c \otimes 1)T_3(a \otimes b)) = \mu(A \otimes S)((1 \otimes b)T_2(c \otimes a))$$
$$= (\mu(A \otimes S)T_2(c \otimes a))S(b) = c \sqcap^L (a)S(b)$$

so that $\mu(A \otimes S)T_3 = \mu(\Box^L \otimes S)$. Using this identity in the third equality, [2, Lemma 6.14] in the fourth one and Lemma 4.9 (6) in the penultimate one, we obtain

$$\varpi_V(v \cdot a)(S(b)c) \stackrel{(6.4)}{=} (v \cdot a)^{\varrho} \cdot S(b^{\varrho})c \stackrel{(7.1)}{=} v^{\varrho} \cdot (\mu(A \otimes S)T_3(a \otimes b^{\varrho}))c = v^{\varrho} \cdot \sqcap^L(a)S(b^{\varrho})c$$
$$= v^{\varrho} \cdot S(b^{\varrho}\overline{\sqcap^R}(a))c = (v\overline{\sqcap^R}(a))^{\varrho} \cdot S(b^{\varrho})c = \varpi_V(v\overline{\sqcap^R}(a))(S(b)c),$$

for any $v \in V$ and $a, b, c \in A$, where V is regarded as a right $R = \sqcap^R(A) = \overline{\sqcap}^R(A)$ -module as in Theorem 4.5 and the implicit summation index notation $\varrho(v \otimes b) =: v^{\varrho} \otimes b^{\varrho}$ is used. Applying [2, Proposition 6.13], this proves $\varpi_V(v \cdot a) = \varpi_V(v \overline{\sqcap}^R(a))$ hence the third equality in (7.5). The fourth equality in (7.5) follows by Lemma 4.9 (1), the penultimate equality follows by [2, Lemma 3.7 (3)] and the last one follows since λ is a right A-module map, see Proposition 2.1 (1).

Theorem 7.12. Consider a regular weak multiplier bialgebra A with left and right full comultiplication possessing an antipode S. The functors in Proposition 7.4 and Proposition 7.10 are mutually inverse equivalences.

Proof. For any firm right L-module P,

$$(P \otimes_L A)^c \cong P \otimes_L A^c \cong P \otimes_L L \cong P$$

as L-modules. In the second step we applied Example 7.6 and in the last step we used that P is firm. This isomorphism is clearly natural in P.

For a right-right Hopf module $(V, \cdot, \lambda, \rho)$, we claim that

$$\xi: V^c \otimes_L A \to V, \qquad \varpi_V(v) \otimes_L a \mapsto \varpi_V(v)(a) = \cdot \lambda^{S21}(v \otimes a)$$

is a natural isomorphism of Hopf modules. It is a morphism of Hopf modules since $\varpi_V(v) : A \to V$ is so by Proposition 7.5. It is natural in V since for a morphism $f : V \to V'$ of Hopf modules and $a \in A$,

$$\xi'(f^c \otimes_L A)(\varpi_V(v) \otimes_L a) = f \varpi_V(v)(a) = f \xi(\varpi_V(v) \otimes_L a).$$

The candidate to be the inverse of ξ is

$$\zeta: V \to V^c \otimes_L A, \qquad v \cdot a \mapsto \pi(\varpi_V \otimes A) \lambda(v \otimes a),$$

where π is the canonical epimorphism $V^c \otimes A \to V^c \otimes_L A$. It is a well-defined linear map by Lemma 7.11. It is the inverse of ξ since by Lemma 6.4 (3) and (7.3), for any $v \in V$ and $a \in A$

$$\xi\zeta(v\cdot a) = \cdot \lambda^{S21}\lambda(v\otimes a) = \cdot G_1(v\otimes a) = v\cdot a$$

and by Lemma 6.4 (2), identity (2.3) in [2], (7.4) and by [2, Lemma 3.7 (3)], for any $v \in V$ and $a, b \in A$

$$\begin{aligned} \zeta\xi(\varpi_V(v)\otimes_L ab) &= \pi(\varpi_V\otimes A)\lambda\lambda^{S^{21}}(v\otimes ab) = \pi(\varpi_V\otimes A)E_1(v\otimes ab) \\ &= \pi(\varpi_V\otimes A)(((\overline{\sqcap}^R\otimes A)T_1(a\otimes b))(v\otimes 1)) \\ &= \pi((\varpi_V(v)\otimes 1)((\sqcap^L\otimes A)T_1(a\otimes b))) \\ &= \varpi_V(v)\otimes_L \mu(\sqcap^L\otimes A)T_1(a\otimes b) = \varpi_V(v)\otimes_L ab. \end{aligned}$$

References

- G. Böhm, S. Caenepeel and K. Janssen, Weak bialgebras and monoidal categories, Comm. Algebra 39 (2011), no. 12, 4584–4607.
- G. Böhm, J. Gómez-Torrecillas and E. López-Centella, Weak multiplier bialgebras, Trans. Amer. Math. Soc., in press. Preprint available at http://arxiv.org/abs/1306.1466.
- [3] G. Böhm and J. Gómez-Torrecillas, Firm Frobenius monads and firm Frobenius algebras, Bull. Math. Soc. Sci. Math. Roumanie 56(104) (2013), no. 3, 281–298.
- [4] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras. I. Integral theory and C*-structure, J. Algebra 221 (1999), no. 2, 385–438.
- [5] G. Böhm and K. Szlachányi, Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace, J. Algebra 233 (2000), 156-212.
- [6] G. Böhm and J. Vercruysse, Morita theory of comodules, Comm. Algebra 37 (2009), no. 9, 3207–3247.
- [7] J. Dauns, Multiplier rings and primitive ideals, Trans. Amer. Math. Soc. 145 (1969), 125–158.
- [8] K. Janssen and J. Vercruysse, Multiplier bi- and Hopf algebras, J. Algebra Appl. 9 (2010), no. 2, 275–303.
- [9] B. Kahng and A. Van Daele, *The Larson-Sweedler theorem for weak multiplier Hopf algebras*, in preparation.
- [10] H. Kurose, A. Van Daele and Y. Zhang, Corepresentation theory of multiplier Hopf algebras II., Int. J. Math. 11, (2000), 233–278.
- [11] F. Nill, Axioms for weak bialgebras, preprint available at http://arxiv.org/abs/math/9805104.
- [12] Phung Hô Hai, Tannaka-Krein duality for Hopf algebroids, Israel J. Math. 167 (2008), 193–225.
- [13] P. Schauenburg, Weak Hopf algebras and quantum groupoids, [in:] "Noncommutative geometry and quantum groups" P.M. Hajac and W. Pusz (eds.), pp. 171–188, Banach Center Publ. 61 Polish Acad. Sci., Warsaw, 2003.
- [14] P. Schauenburg, Duals and doubles of quantum groupoids (×_R-Hopf algebras), [in:] "New trends in Hopf algebra theory", N. Andruskiewitsch, W.R. Ferrer Santos and H-J. Schneider (eds.), pp. 273–299 AMS Contemporary Mathematics 267, Amer. Math. Soc. Providence 2000.
- [15] K. Szlachányi, Finite quantum groupoids and inclusions of finite type, [in:] "Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects", R. Longo (ed.), pp. 393–407, Fields Inst. Comm. 30, AMS and Fields Institute 2001.
- [16] M. Takeuchi, Groups of algebras over $A \otimes \overline{A}$, J. Math. Soc. Japan 29 (1977), no. 3, 459–492.
- [17] A. Van Daele, Multiplier Hopf algebras, Trans. Amer. Math. Soc. 342 (1994), no. 2, 917–932.
- [18] A. Van Daele and S. Wang, Weak Multiplier Hopf Algebras. Preliminaries, motivation and basic examples, in: Operator Algebras and Quantum Groups. W. Pusz and P.M. Soltan (eds.), Banach Center Publications (Warsaw), vol. 98 (2012), 367–415.
- [19] A. Van Daele and S. Wang, Weak Multiplier Hopf Algebras. The main theory, Journal f
 ür die reine und angewandte Mathematik (Crelle's Journal), in press, DOI: 10.1515/crelle-2013-0053. Preprint available at http://arxiv.org/abs/1210.4395.
- [20] A. Van Daele and S. Wang, Weak Multiplier Hopf Algebras III. Integrals and duality, in preparation.

[21] A. Van Daele and Y. Zhang, Corepresentation theory of multiplier Hopf algebras I., Int. J. Math. 10 (1999), 503–539.

WIGNER RESEARCH CENTRE FOR PHYSICS, H-1525 BUDAPEST 114, P.O.B. 49, HUNGARY *E-mail address:* bohm.gabriella@wigner.mta.hu