RANDOM SPACE-FILLING IN ONE DIMENSION

by

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1. Introduction

We construct a model of a random car-parking procedure as follows. Take the segments $[0, x]$ $(x \geqq 0)$ of the x-axis. If $x < 1$, this segment is considered to be a "gap". If $x \geqq 1$, choose a random number $t_1 \in [0, x-1]$ (i.e. t_1 is a random variable, uniformly distributed over $[0, x-1]$). We now consider the unit interval $[t_1, t_1+1]$ to be "covered" and so turn our attention to the remaining segments $[0, t_1]$, $[t_1+1, x]$. If $t_1 \geqq 1$, choose a random number $t_2 \in [0, t_1-1]$ and then regard the unit interval $[t_2, t_2+1]$ as "covered". If $t_1 < 1$, then the segment $[0, t_1]$ is left "uncovered" and we regard this segment as a "gap". Similarly, if $x - t_1 - 1 \geqq 1$, choose a random number $t_3 \in [t_1+1, x-1]$ and so "cover" the unit interval $[t_3, t_3+1]$. If $x - t_1 - 1 < 1$, we regard the segment $[t_1+1, x]$ as a "gap". We continue to "cover", in this random way, all the remaining segments with unit intervals until each such remaining segment is of length strictly less than one. Let $n(x)$ denote the number of unit intervals so placed on the segment $[0, x]$. Note that the possibility of overlapping of the unit intervals, either between themselves or over the ends of the original segment $[0, x]$, has been excluded. We have also made the convention that when one of the remaining segments, as yet "uncovered", is of length equal to one, then in a deterministic way we regard this segment as "covered" at the next stage of the process.

Ambartsumian [1], Bánkövi [2], Griffiths [3], Ney [4], Rényi [5], Robbins and Dvoretzky [6] and Smalley [7] have all studied this problem, Ambartsumian, Griffiths and Smalley reproducing some of the results first proved by Rényi. Rényi has derived equations for both the expectation and the variance of the random variable $n(x)$, and he obtained an asymptotic expression for the expectation, valid as $x \to \infty$. It should be noted that D. G. Kendall has pointed out a mistake in Rényi's equation, 5. 4, for the variance, which however does not affect any of his conclusions. It is Rényi's work which is the most relevant to this present paper.

This paper is only part of work done to prove that, asymptotically, the distribution of $n(x)$ is normal. The proof was based on a study of the moments of $n(x)$. However Robbins and Dvoretzky also claim to have proved the asymptotic normality, and are about to publish their proof.[2] (We have not, as yet, seen their work.) We shall therefore content ourselves

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[1] University of Cambridge.

[2] Cf. the paper of Robbins and Dvoretzky in this issue, pp. 209.

here with a study of the second moment of $n(x)$, which is, of course, of special interest. The study of the higher moments is rather similar.

Results of a Monte Carlo experiment, which simulated the parking procedure, will also be recorded.

2. Asymptotic behaviour of the variance of n(x)

The main result obtained by RÉNYI was that for any positive integer m

$$\mathbf{E}\{n(x)\} = cx + c - 1 + O(1/x^m) \qquad (x \to \infty), \tag{1}$$

where $\mathbf{E}$ denotes expectation and

$$c = \int_0^\infty \exp\left(-2\int_0^s \frac{1-e^{-t}}{t}\,dt\right) ds \approx 0.74759 .$$

The computing of this estimate of the number c was carried out by the Mathematical Laboratory, University of Cambridge.

Put

$$n(x) = cx + c - 1 + r(x) . \tag{2}$$

Since $n(x) \leq x$ for $x > 1$, $n(1) = 1$, and $n(x) = 0$ for $0 \leq x < 1$, we observe that $|\, r(x) \,| \leqq x$ for $x \geqq 1$, and $|\, r(x) \,| < 1$ for $0 \leqq x < 1$, whence $\mathbf{E}\,|\, r(x) \,|^k < 1 + x^k$ for all $x \geqq 0$. We then deduce from RÉNYI's result (1) that

$$\mathbf{E}[n(x) - \mathbf{E}\{n(x)\}]^2 = \mathbf{E}[r(x) - \mathbf{E}\{r(x)\}]^2 = \mathbf{E}\{r(x)\}^2 + o(1) \qquad (x \to \infty) .$$

It turns out that $\mathbf{E}\{r(x)\}^2 = R_2(x)$, say, is much easier to deal with than $\mathbf{E}\{n(x)\}^2$. We also write $\mathbf{E}\{r(x)\} = R_1(x)$. Denote by $r(x \mid t)$ the number $r(x)$ $(x \geqq 1)$ conditional on t being the first random number chosen in the model described in the introduction. Then

$$r(x+1 \mid t) = r(t) + r(x-t) \qquad (0 \leq t \leq x),$$

where $r(t)$ and $r(x-t)$ are independent. It follows that for $x > 0$,

$$R_2(x+1) = \mathbf{E}\{r(x+1)\}^2 =$$

$$= \frac{1}{x}\int_0^x \mathbf{E}\{r(x+1 \mid t)\}^2\,dt =$$

$$= \frac{1}{x}\int_0^x \mathbf{E}\{r(t) + r(x-t)\}^2\,dt =$$

$$= \frac{2}{x}\int_0^x [R_2(t) + R_1(t)\,R_1(x-t)]\,dt .$$

Thus

$$R_2(x) = (1 - c - cx)^2 \qquad (0 \leq x < 1)$$

$$R_2(1) = 4(1 - c)^2$$

$$(3) \qquad R_2(x+1) = \frac{2}{x}\int_0^x R_2(t)\,dt + \frac{2}{x}\int_0^x R_1(t)\,R_1(x-t)\,dt \qquad (x > 0)\,.$$

(3) is a much simplified and corrected version of RÉNYI's equation [1] for the variance. Similarly it is easy to see that $R_1(.)$ satisfies

$$R_1(x) = 1 - c - cx \qquad (0 \leq x < 1)$$

$$R_1(1) = 2(1 - c)$$

$$(4) \qquad R_1(x+1) = \frac{2}{x}\int_0^x R_1(t)\,dt \qquad (x > 0)\,.$$

We see from these formulae that $R_k(x)$ $(k = 1, 2)$ is continuously differentiable in the intervals $(0 \leq x < 1)$, $(1 < x < 2)$, $(2 < x < \infty)$. There is a simple jump discontinuity at $x = 1$, with $R_k(1+) = R_k(1)$ and

$$R_k(1+) - R_k(1-) = (2 - 2c)^k - (1 - 2c)^k = j_k > 0\,.$$

$R_k(x)$ is continuous but not differentiable at $x = 2$.

Put

$$\Phi_k(s) = \int_0^\infty e^{-sx} R_k(x)\,dx \qquad (s > 0;\ k = 1,2)\,.$$

This integral exists since $|R_k(x)| < x^k + 1$ $(0 \leq x < \infty)$. Now

$$e^s\left[\Phi_2(s) - \int_0^1 e^{-sx} R_2(x)\,dx\right] = e^s \int_1^\infty e^{-sx} R_2(x)\,dx =$$

$$= \int_0^\infty e^{-sx} R_2(x+1)\,dx =$$

$$= 2\int_0^\infty e^{-sx}\frac{dx}{x}\left[\int_0^x R_2(t)\,dt + \int_0^x R_1(t)\,R_1(x-t)\,dt\right].$$

Since

$$\int_0^\infty \frac{de^{-sx}}{ds}\,\frac{dx}{x}\left[\int_0^x R_2(t)\,dt + \int_0^x R_1(t)\,R_1(x-t)\,dt\right]$$

is uniformly convergent for $0 < s < \infty$, we may differentiate the above equation to obtain

$$\frac{d}{ds}\left\{e^s\left[\Phi_2(s) - \int_0^1 e^{-sx} R_2(x)\,dx\right]\right\} = -2\int_0^\infty e^{-sx}\,dx\left[\int_0^x R_2(t)\,dt + \right.$$

$$\left. + \int_0^x R_1(t)\,R_1(x-t)\,dt\right].$$

The repeated integral on the right hand side is absolutely convergent, hence, by Fubini's theorem, we may invert the order of integration. Thus

$$\frac{d}{ds}\left\{e^s\left[\Phi_2(s) - \int_0^1 e^{-sx} R_2(x)\,dx\right]\right\} = \frac{-2\,\Phi_2(s)}{s} - 2\,\Phi_1^2(s)\,.$$

Observing that $\exp\left(-2\int_s^\infty \frac{e^{-t}}{t}\,dt\right)$ is an integrating fatcor for this equation, we see that

$$\frac{d}{ds}\left\{e^s\left[\Phi_2(s) - \int_0^1 e^{-sx} R_2(x)\,dx\right]\exp\left(-2\int_s^\infty \frac{e^{-t}}{t}\,dt\right)\right\} =$$

$$= \left[-\frac{2}{s}\int_0^1 e^{-sx} R_2(x)\,dx - 2\,\Phi_1^2(s)\right]\exp\left(-2\int_s^\infty \frac{e^{-t}}{t}\,dt\right).$$

Denoting the right hand side of this equation by $Y(s)$ and integrating both sides from u to u' $(0 < u < u')$

$$\left[e^s\left[\Phi_2(s) - \int_0^1 e^{-sx} R_2(x)\,dx\right]\exp\left(-2\int_s^\infty \frac{e^{-t}}{t}\,dt\right)\right]_u^{u'} = \int_u^{u'} Y(s)\,ds\,.$$

We then note that

$$\lim_{u'\to\infty} e^{u'}\left[\Phi_2(u') - \int_0^1 e^{-u'x} R_2(x)\,dx\right]\exp\left(-2\int_{u'}^\infty \frac{e^{-t}}{t}\,dt\right) = 0\,,$$

so that

$$-\,e^u\left[\Phi_2(u) - \int_0^1 e^{-ux} R_2(x)\,dx\right]\exp\left(-2\int_u^\infty \frac{e^{-t}}{t}\,dt\right) = \int_u^\infty Y(s)\,ds\,.$$

Thus

$$(5) \qquad u^2 e^u[\Phi_2(u) - \int_0^1 e^{-ux} R_2(x)\,dx] =$$

$$= \int_u^\infty \left(2s \int_0^1 e^{-sx} R_2(x)\,dx + 2s^2 \Phi_1^2(s)\right) A(u,s)\,ds\,,$$

where $A(u, s) = \exp\left(-2\int_u^s \frac{1-e^{-t}}{t}\,dt\right)$.

Similarly

$$(6) \qquad u^2 e^u\left[\Phi_1(u) - \int_0^1 e^{-ux} R_1(x)\,dx\right] = \int_u^\infty 2s\left(\int_0^1 e^{-sx} R_1(x)\,dx\right) A(u,s)\,ds\,.$$

Put

$$c(u) = \int_u^\infty \exp\left(-2\int_u^s \frac{1-e^{-t}}{t}\,dt\right) ds\,.$$

Then

$$c(u) = c + (2c - 1)\,u + \left(\frac{3}{2}c - 1\right)u^2 + O(u^3) \qquad (u \downarrow 0)\,.$$

Now

$$u^2 e^u\left[\Phi_1(u) - \int_0^1 e^{-ux} R_1(x)\,dx\right] = c(u) - c + (1 - 2c)\,u\,.$$

Thus, it is easy to see that

$$\lim_{u \downarrow 0} \Phi_1(u) = 0\,.$$

For small s the integrand on the right hand side of (5) is $O(s)$, (when $s > 0$). Thus

$$\lim_{u \downarrow 0} u^2 \Phi_2(u) = 2\int_0^\infty \left\{s\int_0^1 e^{-sx} R_2(x)\,dx + s^2\Phi_1^2(s)\right\}$$

$$\exp\left(-2\int_0^s \frac{1-e^{-t}}{t}\,dt\right) ds\,.$$

We note that the integrand here is strictly positive for $s > 0$. Hence $\lim_{u \downarrow 0} u^2\Phi_2(u) = c_2$ (say), where $0 < c_2 < \infty$; in fact we have

$$c_2 \approx 0.035672,$$

this estimate having been computed from the above formula by Mrs. M. O. MUTCH of the Mathematical Laboratory, University of Cambridge. I am particularly indebted to Mrs. MUTCH since the computing of c_2 proved to be a very delicate matter, involving a great deal of patience and ingenuity.

3. Asymptotic behaviour of the variance of n(x) (continued)

We may not immediately deduce, from the fact that

$$u^2 \Phi_2(u) \sim c_2 \qquad (u \downarrow 0),$$

that

$$R_2(x) \sim c_2 x \qquad (x \to \infty).$$

We need some monotonicity property of $R_2(x)$. This we shall proceed to establish.

RÉNYI showed that for any positive integer m,

$$R_1(x) = O\left(\frac{1}{x^m}\right) \qquad (x \to \infty), \tag{7}$$

$$R_1'(x) = O\left(\frac{1}{x^m}\right) \qquad (x \to \infty). \tag{8}$$

For $x > 1$ we may differentiate (3) to obtain

$$x\, R_2'(x+1) + R_2(x+1) = 2\, R_2(x) + 2(1-c)\, R_1(x) + 2\, R_1(x-1)$$

$$+ 2 \oint_0^x R_1(t)\, R_1'(x-t)\, dt\,, \tag{9}$$

where $\oint_0^x$ denotes integration over $([0, x] - \{1, 2\})$. Thus from (7) and (8) we see that, for any positive integer m,

$$\frac{x^2}{2} R_2'(x+1) = x\, R_2(x) - \int_0^x R_2(t)\, dt + O(1/x^m) \qquad (x \to \infty).$$

Put

$$f(x) = x\, R_2(x) - \int_0^x R_2(t)\, dt\,,$$

Then

$$f'(x) = x\, R_2'(x) \qquad (x > 2),$$

and f' will be continuous for $x > 2$. Thus

$$\frac{x^2 f'(x+1)}{2(x+1)} = f(x) + O(1/x^m) \qquad (x \to \infty). \tag{10}$$

Now

$$u \int_0^\infty e^{-ux}\, R_2(x)\, dx = \oint_0^\infty e^{-ux}\, R_2'(x)\, dx + (3 - 4\,c)\, e^{-u} + (1-c)^2.$$

Also, for $2 < A < \infty$,

$$\lim_{u \downarrow 0} u \oint_0^\infty e^{-ux}\, R_2'(x)\, dx = \lim_{u \downarrow 0} u \int_A^\infty e^{-ux}\, R_2'(x)\, dx\,.$$

Thus

$$\lim_{u \downarrow 0} u \int_A^\infty e^{-ux} R_2'(x)\, dx = c_2 , \tag{11}$$

Let $0 < \lambda < c_2$. Suppose that, for all $x > A$,

$$R_2'(x) \leqq \lambda .$$

Then

$$\lim_{u \downarrow 0} u \int_A^\infty e^{-ux} R_2'(x)\, dx \leqq \lambda .$$

This is impossible from (11); thus we may assert that for any $A > 2$, there is an $x > A$ such that

$$R_2'(x) > \lambda .$$

Denote the remainder term in (10) by $h(x)$. Then

$$\frac{x^2 f'(x+1)}{2(x+1)} = f(x) + h(x) . \tag{12}$$

Let $\varepsilon > 0$. We may certainly choose $A_\varepsilon > 1$ such that, for all $x > A_\varepsilon$,

$$| h(x) | < \varepsilon .$$

We now choose x_0 such that

(i) $x_0 > A_\varepsilon + 4$,

(ii) $\frac{(x_0 - 4)^2}{2(x_0 - 1)} \left[\frac{(x_0 - 2)^2}{2\, x_0} \left\{ \frac{\lambda\, x_0^2}{2} - 2\,\varepsilon \right\} - 2\,\varepsilon \right] - \varepsilon > x_0^3$,

(iii) $R_2'(x_0 + 1) > \lambda$.

Thus $f'(x_0 + 1) > \lambda\, (x_0 + 1) > 0$. From (iii) and (12) we see that

$$f(x_0) + h(x_0) > \frac{\lambda\, x_0^2}{2} .$$

Thus, from (ii),

$$f(x_0) - \varepsilon > \frac{\lambda\, x_0^2}{2} - 2\,\varepsilon > 0 .$$

Let us suppose that, for all $x \in [x_0, x_0 + 1], f'(x) > 0$. Then, for all $x \in [x_0, x_0 + 1]$,

$$f(x) + h(x) > f(x_0) - \varepsilon > 0 ,$$

and so, for all $x \in [x_0 + 1,\ x_0 + 2]$,

$$f'(x) > 0 .$$

If $f'(x) \leqq 0$ for some $x > x_0 + 2$, then $f'(x)$ must (by continuity) vanish for some such x, in which case we write

$$z = \inf \{x : x > x_0 + 2,\ f'(x) = 0\}.$$

Then, from (12), $f(z-1)+h(z-1)=0$. Yet since $f'(x)>0$ $(x_0 \leqq x < z)$, we also have

$$f(x)+h(x)>f(x_0)-\varepsilon>0 \qquad (x_0 \leqq x < z),$$

which, in particular, implies that $f(z-1)+h(z-1)>0$. This is a contradiction: we thus deduce that if $f'(x)>0$, for all $x \in [x_0, x_0+1]$, then $f'(x)>0$ for all $x \geqq x_0$. We proceed to show that this is in fact the case. We recall that $f'(x)$ is positive at x_0+1, and if it is not positive throughout $[x_0, x_0+1]$ then, by continuity, it must vanish at at least one point in this interval.

Suppose then that there is an x_1 $(x_0 \leqq x_1 < x_0+1)$ such that $f'(x_1)=0$. Then, from (12), $f(x_1-1)+h(x_1-1)=0$. Also by the first mean-value theorem, there is an x_2 $(x_1-1<x_2<x_0)$ such that

$$f'(x_2)=\frac{f(x_0)-f(x_1-1)}{x_0-x_1+1}>f(x_0)-\varepsilon>\frac{\lambda x_0^2}{2}-2\varepsilon>0,$$

Hence, from (12),

$$f(x_2-1)+h(x_2-1)>\frac{(x_2-1)^2}{2x_2}\left(\frac{\lambda x_0^2}{2}-2\varepsilon\right).$$

Thus, from (ii),

$$f(x_2-1)-\varepsilon>\frac{(x_0-2)^2}{2x_0}\left(\frac{\lambda x_0^2}{2}-2\varepsilon\right)-2\varepsilon>0.$$

It is now impossible for $f'(x)$ to be positive throughout $[x_2-1, x_2]$, since otherwise, by a repetition of the preceding argument, $f'(x)$ would be positive for all $x \geqq x_2-1$, and we have assumed that $f'(x_1)=0$ (and $x_1>x_2-1$). Thus there is an x_3 $(x_2-1 \leqq x_3 < x_2)$ such that $f'(x_3)=0$, by continuity and the fact that $f'(x_2)>0$. Then, from (12), $f(x_3-1)+h(x_3-1)=0$. Again, from the first mean-value theorem, there is an x_4 $(x_3-1<x_4<x_2-1)$ such that

$$f'(x_4)=\frac{f(x_2-1)-f(x_3-1)}{x_2-x_3}>f(x_2-1)-\varepsilon>0.$$

Thus, from (12) and (ii),

$$\begin{aligned} f(x_4-1) &> \frac{(x_4-1)^2}{2x_4}[f(x_2-1)-\varepsilon]-\varepsilon> \\ &> \frac{(x_0-4)^2}{2(x_0-1)}\left[\frac{(x_0-2)^2}{2x_0}\left\{\frac{\lambda x_0^2}{2}-2\varepsilon\right\}-2\varepsilon\right]-\varepsilon> \\ &> x_0^3 \\ &> (x_4-1)^3, \end{aligned}$$

where $x_4-1>A_\varepsilon>1$. But we see from the definition of $f(x)$, and the fact that $0 \leqq R_2(x) \leqq x^2$ $(x \geqq 1)$, that $f(x) \leqq x^3$ $(x \geqq 1)$. We thus have a contradiction, and we conclude that $f'(x)$, and hence $R_2'(x)$, is positive for all $x \geqq x_0$.

Put

$$\begin{aligned} R(x) &\equiv R_2(x) \qquad (x \geqq x_0) \\ &\equiv R_2(x_0) \qquad (0 \leqq x < x_0) \end{aligned}$$

so that R is a monotonic increasing function on $[0, \infty)$. Also, when $s \downarrow 0$,

$$c_2 + o(1) = s^2 \Phi_2(s) =$$

$$= s \int_0^\infty R_2(x)\, d(-e^{-sx}) =$$

$$= sR_2(0) + s \int_0^\infty e^{-sx}\, dR_2(x) =$$

$$= sR_2(0) + s \int_0^{x_0} e^{-sx}\, d\big(R_2(x) - R(x)\big) + s \int_0^\infty e^{-sx}\, dR(x) =$$

$$= o(1) + s \int_0^\infty e^{-sx}\, dR(x)\,.$$

Thus, from KARAMATA's Tauberian theorem ([8], Chapter V, Th. 4.3) we conclude that

$$R(x) \sim c_2\, x \qquad (x \to \infty)\,,$$

and hence that

$$R_2(x) \sim c_2 x \qquad (x \to \infty)\,.$$

We also see, from (9), that

$$R_2'(x) \to c_2 \qquad (x \to \infty)\,.$$

It follows then that

$$\mathbf{D}^2[n(x)] \sim c_2 x \qquad (x \to \infty)\,,$$

where it will be recalled that

$$c_2 \approx 0.035672.$$

Note also that $\mathbf{D}^2[n(x)]$ is ultimately an increasing function of x, since

$$\mathbf{D}^2[n(x)] = R_2(x) - [R_1(x)]^2,$$

and, from (7) and (8)

$$\frac{d}{dx}\,\mathbf{D}^2[n(x)] = R_2'(x) - 2\,R_1(x)\,R_1'(x) \to c_2 > 0 \qquad (x \to \infty)\,.$$

In a rather similar piece of work (which we do not intend to publish) we have proved that

$$M_{2k-1}(x) = o(x^{k-\frac{1}{2}}) \qquad (x \to \infty)$$

and that

$$M_{2k}(x) \sim \frac{(2\,k)!\; c_2^k\, x^k}{k!\; 2^k} \qquad (x \to \infty)\,,$$

$(k = 1, 2, \ldots)$ where

$$M_r(x) = \mathbf{E}[n(x) - \mathbf{E}\{n(x)\}]^r.$$

It is then an immediate consequence of the moments convergence theorem of FRÉCHET and SHOHAT that

$$\frac{n(x) - cx}{\sqrt{c_2 x}}$$

is asymptotically $(x \to \infty)$ normally distributed with mean zero and standard deviation one.

4. The „Monte Carlo" experiment

A "Monte Carlo" experiment, simulating the parking procedure described in the introduction, was performed by Edsac II, until recently the electronic computer of the Mathematical Laboratory, University of Cambridge. The machine's "random" numbers were supplied by a pseudo-random-number mechanism. This mechanism takes an assigned number (chosen from a "table of random numbers" and fed into the machine), raises it to a high power and selects the middle portion of the number obtained as the first generated "random" number. In this way also, the second generated "random" number is obtained from the first, the third from the second, and so on. We took $x = 1000$ and 2000 runs were performed. The results are shown, in histogram form, in Figure 1. The sample mean, m, and sample variance, s^2, were respectively 747,447 and 38,5000. The shape of the histogram for $n(1000)$ is very nearly normal. Assuming that $n(1000)$ is, in fact, normal with mean 747,5 and variance s^2, we calculated a value of χ^2 from our observations, which were grouped into 31 parts. We obtained

$$\chi^2_{28} = 32{,}408.$$

Since the upper 5 per cent level for χ^2 with 28 degrees of freedom is 41,337, there is no significant evidence here for non-normality.

If we assume that $n(1000)$ is normally distributed, we may inquire into the question of whether our asymptotic formulae are, in practice, usable when $x = 1000$. Let us assume then that $n(1000)$ is normally distributed. Write $\mu = \mathbf{E}[n(1000)]$. Then we have the following 95 per cent confidence interval for μ:

$$747{\cdot}17 < \mu < 747{\cdot}72.$$

Since $\mu^* \equiv 1000c + c - 1 = 747{,}34$ lies in the above interval, it is not unreasonable to suppose that the approximate formula

$$\mathbf{E}[n(x)] \doteq cx + c - 1 \qquad (x \geqq 1000)$$

is quite accurate, as the order of Rényi's remainder term would suggest.

Again, still supposing that $n(1000)$ is normally distributed, we know then that $(n-1)\, s^2/\sigma^2$ has a x^2 distribution with $n - 1$ degrees of freedom, where $n = 2000$ and $\sigma^2 = \mathbf{D}^2[n(1000)]$. Since n is large, we may suppose that

$$\sqrt{2(n-1)\, s^2/\sigma^2} - \sqrt{2(n-1) - 1}$$

is normally distributed with zero mean and unit variance. This gives the following 95 per cent confidence interval for σ:

$$6{,}02 < \sigma < 6{,}41.$$

However $\sigma^* \equiv \sqrt{1000 c_2} = 5{,}97$, and so there is an indication that the uninvestigated remainder term in the approximation $\sigma \approx \sqrt{c_2 x}$ is not quite negligible when $x = 1000$.

It is not surprising that so haphazard a method of space-filling should lead to about a 25 per cent wastage of the space available, but it may be thought curious that when $x = 1000$ the standard deviation of the number of cars accomodated should be less than one per cent of the expected number of cars. The space filling process seems to have more rigidity than one would intuitively ascribe to it.

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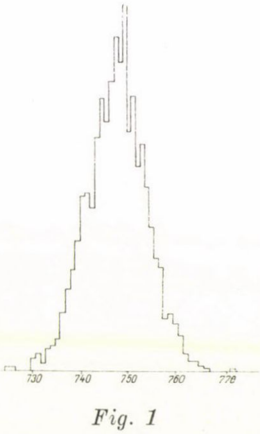

Fig. 1

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СЛУЧАЙНОЕ ЗАПОЛНЕНИЕ ИНТЕРВАЛА

D. MANNION

Резюме

В работе проводятся дальнейшие исследования проблемы случайного заполнения интервала изучаемой A. RÉNYI и др. Пусть $n(x)$ означает числа интервалов длины 1 при случайном заполнении с такими интервалами интервала длины x. Автор показывает, что $\mathbf{D}^2(n(x))$ монотонно растет, а $\mathbf{D}^2(n(x)) \sim c_2 x$, где $c_2 = 0{,}035672\ldots$

В связи с этой асимптотической формулой в работе разобраны результаты одних вычислений, проведенных по методу Монте-Карло ($x = 1000$, 2000 опытов).