ON THE LOTTERY PROBLEM

by

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The general form of the lottery-game — as it is well known — is the following:

On each lottery-ticket there are the integers $1, 2, \ldots, n$ from which one has to select k numbers. After this l numbers are drawn out from $1, 2, \ldots, n$. If the set of numbers selected on a lottery ticket has exactly d common elements with the set of l numbers which have been drawn ($d \leq l \leq k$), we say, that we obtained a d-hit on the lottery-ticket.

The so-called lottery-problem in question is the following: what is the minimal of lottery-tickets, so, that suitably selecting the k numbers on them, we can be sure to have at least one d-hit? (In the case of the lottery in Hungary $n = 90$, $k = l = 5$, $1 \leq d \leq 5$).

The general combinatorial problem according to this is the following:

Let k, l, d, n be positive integers, $1 \leq k \leq n$, $1 \leq l \leq n$, $1 \leq d \leq \min(k, l)$ and E a set with n elements. We call a subset of k elements of the set E a k-tuple of E. Let S be a system of k-tuples of E. We say, that S *has property* P, if to each l-tuple L of E there exists at least one k-tuple of E belonging to S, which has at least d common elements with L. (We can say, that the d-tuples of the k-tuples belonging to S represent all l-tuples of E.) Denote by N the number of k-tuples belonging to S. The problem is as follows:

What is the minimum of N, depending on n, k, l, d?

We call an S-system with property P a minimal-system $S_0(n, k, l, d)$, if for this the value $N_0(n, k, l, d)$ is the possible smallest.

We give in this paper a lower bound for N_0 in case $d = 2$, and an asymptotic formula for it in case for fixed k, l, $d = 2$ and $n \to \infty$. We can determine the exact value of N_0 and the minimal system S_0 only in the case $k \leq 5$ $d = 2$ and for special values of n satisfying some congruences. (For example for the case $n = 84$ or $n = 100$ and $k = 5$). So we can consider the lottery-problem essentially solved only in the case, when we want to be sure of a 2-hit.

Theorem. *Given a set E of n elements, integers $k, l \geq 2$ and a minimal-system S_0 $(n, k, l, 2)$ (with property P) then for the number N_0 of k-tuples in S_0 we have the inequality*

$$N_0 \geq \frac{n(n-l+1)}{k(l-1)^2} \tag{1}$$

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and

(2) $$\lim_{n\to\infty} N_0 \Big/ \frac{n(n-l+1)}{k(l-1)^2} = 1$$

If $k \leqq 5$ *and* $\frac{n}{l-1}$ *is integer, further*

$$\frac{n}{l-1} \equiv 1 \bmod k(k-1)$$

(3) *or*

$$\frac{n}{l-1} \equiv k \bmod k(k-1)$$

hen there exists a minimal-system S_0 *for which the equality*

(4) $$N_0 = \frac{n(n-l+1)}{k(l-1)^2}$$

holds.[4]

Proof. For the proof we use the following three results:

1. *Theorem of* TURÁN P. ([1]): (in a specialized form). Let E be a set with n elements, and let an integer l be prescribed ($3 \leqq l \leqq n$, $l-1|n$). If B is a system of N pairs of elements from E with the property, that each subset of E with l elements contains at least one pair belonging to B, then the inequality

$$N \geqq (l-1)\binom{\frac{n}{l-1}}{2}$$

holds.

Equality holds for and only for the following system B: we divide the elements of E into mutually disjoint subsets each having $\frac{n}{l-1}$ elements. The minimal system B_0 contains all pairs (and only those) whose elements are from the same class.

2. *Theorem of* HANANI ([2]): Let the set E have m elements and let H be a system of k-tuples of E with the property, that each pair of elements from E is contained at least in one k-tuple of H. Then from the number M of k-tuples in H we have obviously

(5) $$M \geqq \frac{m(m-1)}{k(k-1)}$$

If $k \leqq 5$ and

$$m \equiv 1 \bmod k(k-1)$$

[4] Equality holds also in the case $k = p$, and $\frac{n}{l-1} = p^\nu$ or $k = p+1$ and $\frac{n}{l-1} + p + p^\nu$ where p is a power of a prime and ν an arbitrary positive integer. The proof goes on the same way only instead of HANANI's theorem in [2] we have to use a result from [5].

or

$$m \equiv k \bmod k\,(k-1)$$

then there exist minimal systems H_0 for which equality holds in (5)[5].

3. *Theorem of* ERDÖS—HANANI [3] (see also [6]): With the above notations when M_0 is the number of k-tuples in a minimal-system H_0:

$$\lim_{m\to\infty} \frac{M_0}{\dfrac{m(m-1)}{k(k-1)}} = 1 .$$

To prove our theorem, let us suppose that we have a system S of k-tuples with property P. Then, if we consider all the pairs of elements in these k-tuples; they represent all the l-tuples of E; i.e. each l-tuple of E contains at least one pair from these. But then, according to Turán's theorem, the number of different pairs in the k-tuples of S is at least $(l-1).\binom{\frac{n}{l-1}}{2}$. Equality can hold only in the case, if these pairs are the following: We divide the n numbers into $l-1$ equal classes, and the k-tuples in S contain all the pairs — and only these — the two elements of which belong to the same class. Since each k-tuple contains $\binom{k}{2}$ pairs, and the "best" case is, when all the pairs in the k-tuples are different — (no two k-tuple in S has two common elements) — S has obviously at least

$$\frac{(l-1)\binom{\frac{n}{l-1}}{2}}{\binom{k}{2}}$$

k-tuples. This proves (1).

In case when (3) holds, using Hanani's theorem and constructing a minimal-system H_0 with $m = \dfrac{n}{l-1}$, for each of the $l-1$ classes we get a minimal-system S_0 which has

$$\binom{\frac{n}{l-1}}{2} \Big/ \binom{k}{2}$$

k-tuples, and this proves (4).

As to the asymptotic case, using the theorem of ERDŐS and HANANI again for each of the $l-1$-classes and $m = \dfrac{n}{l-n}$ we get (2).

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[5] For the construction of such a system see [2]. Evidently for such a minimal system each pair of elements of E is contained in exactly one k-tuple of H.

Remark. If in the lottery-problem we want to construct in a similar way a minimal-system S which assure a 3,4 or 5-hit, we would need a generalization of TURÁN's theorem[6] and a generalisation of HANANI's theorem and construction.

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О ЛОТЕРНОЙ ЗАДАЧЕ

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Резюме

Пусть будут k, l, d, n положительные целые, $1 \leqq k, l \leqq n, 1 \leqq d \leqq \leqq \min(k, l)$ и E — множество, состоящее из n элементов. Пусть будет s некоторая система подмножеств E, состоящих из k элементов. Мы говорим, что s имеет «представительное свойство», если для каждого подмножества L, состоящего из l элементов от E, существует элемент от s, который имеет с L общие элементы не меньше d. Пусть обозначает $N_0 = N_0(n, k, l, d)$ наименьшее число элементов s с «представительным свойством». В случае $d = 2$ доказана.

Теорема.

$$N_0 \geqq \frac{n(n-l+1)}{k(l-1)^2}$$

и

$$\lim_{n\to\infty} \frac{N_0}{\dfrac{n(n-l+1)}{k(l-1)^2}} = 1\,.$$

Если $k \leqq 5$, $\dfrac{n}{l-1}$ целое, и

$$\frac{n}{l-1} \equiv 1 \pmod{k(k-1)}$$

тогда

$$N_0 = \frac{n(n-l+1)}{k(l-1)^2}\,.$$

[6] The necessity of generalizing TURÁN's theorem, which is raised in [4], turned up already in several questions.