ON $\mathbf{PC}_\Delta$-CLASSES IN THE THEORY OF MODELS

by

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Introduction

In this paper we present some results concerning certain special $\mathbf{PC}_\Delta$-classes.[1]

In § 1 we enumerate notations, definitions and some wellknown results to be used in the paper.

In § 2 we expose a generalization of a theorem of KLEENE [5]. KLEENE's theorem asserts the following. Let Σ be a set of sentences in the first order predicate calculus over a language $\mathbf{L}$ containing only finitely many predicate and function symbols and suppose that Σ satisfies the following conditions: (a) Σ contains its all consequences, (b) Σ is recursively enumerable with respect to a natural Gödel numbering. In this case the theory Σ is „finitely axiomatizable using additional predicate symbols" i.e. we can give a formula F in an enlarged language $\mathbf{L}' \supset \mathbf{L}$, such that for any formula G of the original language $\mathbf{L}$ G is derivable from F if and only if $G \in \Sigma$. The derivability notion used here is based upon a usual formal system of the first order predicate calculus; the identity symbol is treated as the other predicate symbols.

A first step in strengthening KLEENE's theorem would be to require from the class of the $\mathbf{L}$-reducts of all models of F to be identical with the class of all models of Σ in the language $\mathbf{L}$. This strong form is not true, only the weaker statement that an F exists such that the *infinite* relational systems of the two mentioned classes are the same.

We make also a second step in the generalization essential for the applications, namely allow Σ to contain denumerable infinitely many additional symbols besides the finitely many symbols of $\mathbf{L}$. Our requirement that Σ is recursively enumerable has to have the meaning that Σ is recursively enumerable under a natural Gödel numbering based upon an enumeration of the additional symbols, in which the number of arguments of the i-th predicate is a recursive function of i. In this way we shall introduce the class $\mathbf{PC}_{\Delta\mathrm{rec}}$ of classes of relational systems as follows. $\mathbf{K} \in \mathbf{PC}_{\Delta\mathrm{rec}}$. if $\mathbf{K}$ is the class of the $\mathbf{L}$-reducts of the models of a recursively enumerable set Σ of sentences of an enlarged language $\mathbf{L}'$. The recursive enumeration mentioned in this definition is based upon an enumeration of the symbols of $\mathbf{L}'$ as above.

So we can formulate our generalization of KLEENE's theorem as follows (Theorem 1 in § 2). *If* $\mathbf{L}_0$ *is a finite language*, $\mathbf{K}$ *is a class of relational systems*

[1] See [6] and § 1 of the present paper for a definition of $\mathbf{PC}$ and $\mathbf{PC}_\Delta$-class.

of $\mathbf{L}_0$, *and* $\mathbf{K} \in \mathsf{PC}_{\Delta\mathrm{rec}}$ *then* $\mathbf{K}^\infty \in \mathsf{PC}$ *where* $\mathbf{K}^\infty$ *is the class of the infinite systems of* $\mathbf{K}$.

In later sections there will be applications of this theorem.

Our proof is based upon the same idea as KLEENE's proof, namely we treat formulae as elements which are able to form values of the individual variables by the help of a Gödel numbering. The main point in the construction of the above F is a reproduction of the inductive semantical definition of the notion of a sequence of elements satisfying a formula in a relational system.

A trivial example shows that the conclusion $\mathbf{K}^\infty \in \mathsf{PC}$ of the theorem cannot be improved in general to $\mathbf{K} \in \mathsf{PC}$, even if we require Σ to be a recursive set of sentences of $\mathbf{L}$. However we do not know whether the similar improvements in Corollary 3, 5, 5′, 9 hold. Our conjecture is that they do not hold.

The main work in KLEENE [5] is devoted to a strictly constructive treatment. KLEENE proves also a variant of the mentioned theorem for the intuitionistic predicate calculus. Naturally our proof is of no constructive character, consequently our theorem does not imply KLEENE's results in a strict sense.

Our proof technically differs from KLEENE's one. We use ROBINSON's system as described in [4] to deal with recursive functions and predicates and thus we need to adjoin only eight new symbols to $\mathbf{L}_0$ to get $\mathbf{L}$.

In § 3 we introduce the following construction of relational systems. Let $\mathbf{L}$ be a language containing only predicate symbols and no function symbols. Let $\mathfrak{A}$ be a relational system, $F(x)$ a formula of the corresponding language containing no free variable except x. Let us denote by $\mathfrak{A} \,||\, F(x)$ the subsystem $\mathfrak{B}$ of $\mathfrak{A}$ whose domain is the set of those elements of the domain of $\mathfrak{A}$ which satisfy the formula $F(x)$ in $\mathfrak{A}$. We consider $\mathfrak{A} \,||\, F(x)$ as defined only if the latter set is not empty, i.e. if $(\exists x)\, F(x)$ holds in $\mathfrak{A}$. We put for a class $\mathbf{K}$ of systems $\mathbf{K} \,||\, F(x) = \{\mathfrak{A} \,||\, F(x) : \mathfrak{A} \in \mathbf{K}\}$. We prove (Theorem 2 (a) in § 3) that if $\mathbf{K} \in \mathsf{PC}_\Delta$ then $\mathbf{K} \,||\, F(x) \in \mathsf{PC}_\Delta$ for any formula $F(x)$ of the corresponding language provided that $\mathbf{K} \,||\, F(x)$ is defined. Further we prove (Theorem 2 (b)) that if $\mathbf{K} \in \mathsf{PC}$, then $\mathbf{K} \,||\, F(x) \in \mathsf{PC}_{\Delta\mathrm{rec}}$. So we obtain (Corollary 3) that if $\mathbf{K} \in \mathsf{PC}$, then $(\mathbf{K} \,||\, F(x))^\infty \in \mathsf{PC}$ (using Theorem 1 of § 2).

In § 4 we prove by using Theorem 2 (a), that if $\mathbf{K} \in \mathsf{PC}_\Delta$ then $\mathsf{H}(\mathbf{K}) \in$ $\in \mathsf{PC}_\Delta$ (where $\mathsf{H}(\mathbf{K})$ denotes the class of the homomorphic images of the systems of $\mathbf{K}$) (Corollary 4'). The question whether this is true is left open in TARSKI [7] We have also the result that $\mathbf{K} \in \mathsf{PC}$ implies $(\mathsf{H}(\mathbf{K}))^\infty \in \mathsf{PC}$ (Corollary 5′).

Let $\mathbf{K} \in \mathsf{EC}_\Delta$ (or $\mathbf{K} \in \mathsf{PC}_\Delta$). The main content of § 4 is to give an axiomatization Σ for the class $\mathsf{H}(\mathbf{K})$ using additional function symbols so that each formula of Σ is in some normal form (Theorem 7). This normal form is established in such a way, that any set of sentences having this normal form is ,,preserved" under homomorphism in a natural sense (see more precisely Theorem 6).

In the whole section we consider the more general notion of $\mathbf{F}$-homomorphism instead of (simple) homomorphism. This notion is defined in KEISLER [2].

In § 5 we deal with endomorphisms. The relational system $\mathfrak{A}$ is said to be an endomorphic image of $\mathfrak{B}$ if $\mathfrak{A}$ is a subsytem of $\mathfrak{B}$ and at the same time also a homomorphic image of $\mathfrak{B}$, $\mathsf{End}(\mathbf{K})$ will denote the class of all endomorphic images of the systems of $\mathbf{K}$. We state that if $\mathbf{K} \in \mathsf{PC}_\Delta$ then

$\mathbf{End}(\mathbf{K}) \in \mathbf{PC}_\Delta$ and if $\mathbf{K} \in \mathbf{PC}$ then $(\mathbf{End}(\mathbf{K}))^\infty \in \mathbf{PC}$ (Corollaries 8 and 9). The proofs are very similar to those of Corollaries 4 and 5 and are omitted. Next an analogon of LYNDON's theorem on homomorphisms is proved concerning endomorphisms and thus we obtain the corollary, that a first order sentence is preserved under endomorphism if and only if it is equivalent to a sentence of the form $\bigwedge_{i=1}^{n} (F_i^1 \wedge F_i^2)$ where F_i^1 is a positive sentence, F_i^2 is a universal sentence for each i (Corollary 11).

§ 1. Preliminaries

We shall distinguish between sets and classes but we shall consider also classes of classes as a third type. We shall use the usual set theoretical notations. We mention only that if A and B are sets then A^B denotes the set of all (unary) functions on A into B, 2^A denotes the set of all (unary) functions on A with possible values 0 and 1, A^n (where n is a natural number) denotes the set of ordered n-tuples of elements of A. We identify a (unary) function φ with the set all ordered pairs $(a, \varphi(a))$ where a is an element for which $\varphi(a)$ is defined, $\varphi(a)$ being the value of φ at the argument a. We make similar conventions for functions of more variables. If $\varphi \in A^B$ then we write sometimes $\varphi\colon A \to B$. If $\varphi\colon A \to B$, $\psi\colon B \to C$ then $\psi \circ \varphi$ denotes the composition of φ and ψ, i.e. $\psi\circ\varphi(a) = \psi(\varphi(a))$ for $a \in A$. If φ is a one-to-one function, φ^{-1} denotes its inverse.

The following well known set theoretical lemma is applied in § 4.

Lemma 1. *Let A be a set, let α be a function defined on A so that $\alpha(x)$ for $x \in A$ is a finite set. If X is an arbitrary finite subset of A* (i.e. $X \in A^{[\omega]}$) *let $\beta(X)$ be a set of unary functions defined on X* (the elements of $\beta(X)$ are the „good" functions defined on X) *and if $\varepsilon \in \beta(X)$ then $\varepsilon(x) \in \alpha(x)$ for $x \in X$,* (i.e. the good functions take values only from a fixed finite set $\alpha(x)$ for each argument $x \in A$). *Now suppose (a) for arbitrary $X \in A^{[\omega]}$, $\beta(X)$ is not empty, (b) if $X \subset Y \in A^{[\omega]}$, $\varepsilon \in \beta(Y)$ then $\varepsilon \restriction X \in \beta(X)$* (i.e. the restriction of a good function is a good one too). *Under these hypotheses then there is a function δ on A such that for each $X \in A^{[\omega]}$ $\delta \restriction X \in \beta(X)$.* (i.e. there exists a function defined on the whole set A whose restriction to each finite subset is a good function).

We shall mean by a *language* $\mathbf{L}$ a set of certain symbols certain of which are *predicate symbols*, the others are *function symbols*. Notations $P \in \mathbf{L}$ and $f \in \mathbf{L}$ will always imply that P is a predicate symbol and f is a function symbol of $\mathbf{L}$. To each $P \in \mathbf{L}$ and $f \in \mathbf{L}$ there is associated a natural number $\nu(P) \geq 0$ and $\nu(f) \geq 0$ and P and f are said to be a $\nu(P)$-ary predicate symbol and a $\nu(f)$-ary function symbol respectively. If $\nu(f) = 0$ then f is an *(individual) constant*. A *relational system* or more briefly a *system* $\mathfrak{A}$ *of* the language $\mathbf{L}$ is a pair (A, Δ) of a non empty set A and a function Δ defined on L such that $\Delta(P)$ is a $\nu(P)$-ary relation on the set A (i.e. an element of $2^{A^{\nu(P)}}$ [2]) and $\Delta(f)$ is a $\nu(f)$-ary function on A with values from A (i.e. an element of $A^{A^{\nu(f)}}$ for any $P, f \in \mathbf{L}$. A is said to be

[2] The relations are usually considered as truth functions. That will be consistent with our convention if we identify the truth value true and false with 1 and 0 resp. We write $R(a_1, \ldots, a_n)$ instead of $R(a_1, \ldots, a_n) = 0$ for a relation $R(x_1, \ldots, x_n)$.

the *domain* of $\mathfrak{A}$ and denoted by $|\mathfrak{A}|$ and we write $P_{\mathfrak{A}}, f_{\mathfrak{A}}$ for $\Delta(P)$ and $\Delta(f)$ respectively. If $\nu(f) = 0$ then $f_{\mathfrak{A}}$ is identified with an element of A. The class of all systems of $\mathbf{L}$ is denoted by $\mathfrak{S}(\mathbf{L})$. We say that the system $\mathfrak{A}$ is infinite if the set $|\mathfrak{A}|$ is infinite. For a class $\mathbf{K}$ of systems let $\mathbf{K}^{\infty}$ denote the subclass of $\mathbf{K}$ consisting of all infinite systems of $\mathbf{K}$. Sometimes we shall use notations of the form $(A; R, \ldots, \varphi, \ldots)$ to denote a system $\mathfrak{A}$ such that $|\mathfrak{A}| = A$ and $R = P_{\mathfrak{A}}, \ldots \varphi = f_{\mathfrak{A}}, \ldots$ where $P, \ldots, f, \ldots$ are given uniquely by the context.

We define the first order logic with equality associated with $\mathbf{L}$ in the well known way by fixing denumerable many *(individual) variables* $v_0, v_1, \ldots,$ the propositional connectives $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (equivalence), the quantifiers $(\exists x)$ (existential quantifier), (x) (universal quantifier) where x is a variable; and the identity symbol $=$. In § 2 we shall consider only $\neg, \wedge, (x)$ as primitive symbols, the other logical operations will be used as abbreviations in the well known way. The *terms* and *formulae of* $\mathbf{L}$ are defined in the usual way; a *prime formula of* $\mathbf{L}$ is a formula of the form $P(t_1, \ldots, t_{\nu(p)})$ or $t_1 = t_2$ ($P \in L$; $t_1, \ldots, t_{\nu(p)}$ are terms). The use of $P(t_1, \ldots, t_n)$ and $f(t_1, \ldots, t_n)$ always implies $n = \nu(P)$ and $n = \nu(f)$ respectively. The set of all formulae of L and the set of the formulae of $\mathbf{L}$ not containing any free variables (the *sentences* of $\mathbf{L}$) are denoted by $\mathfrak{F}(\mathbf{L})$ and $\mathfrak{F}_0(\mathbf{L})$ respectively. A formula is *open* if it contains no quantifier. If F is a formula $Cl(F)$ denotes the universal closure of F, i.e. the formula obtained by prefixing to F universal quantifiers (x_i) for each free variable x_i of F in some order. If Σ is a set of formulae, so $Cl(\Sigma) = \{Cl(F) : F \in \Sigma\}$.

If $F(x_1, \ldots, x_n)$ (briefly F) is a formula, $t(x_1, \ldots, x_n)$ (briefly t) is a term, $x_1, \ldots, x_n$ are distinct (free) variables or constants, then $F(t_1, \ldots, t_n)$ and $t(t_1, \ldots, t_n)$ denote the formula and the term respectively arising from F and t by substituting the term t_i for x_i for each $i = 1, \ldots, n$. We write

$$\left|\frac{x_1, \ldots, x_n}{t_1, \ldots, t_n}\right. F \quad \text{or} \quad \left|\frac{x_i}{t_i}\right. F \tag{1}$$

and

$$\left|\frac{x_1, \ldots, x_n}{t_1, \ldots, t_n}\right. t \quad \text{or} \quad \left|\frac{x_i}{t_i}\right. t \tag{2}$$

for $F(t_1, \ldots, t_n)$ and $t(t_1, \ldots, t_n)$ respectively.

Let $x_1, \ldots, x_n$ be distinct variables, let every free variable of F and t occur among $x_1, \ldots, x_n$. We write

$$\mathfrak{A}\left|\frac{x_1, \ldots, x_n}{a_1, \ldots, a_n}\right. F \quad \text{or} \quad \mathfrak{A}\left|\frac{x_i}{t_i}\right. F \tag{3}$$

for the statement that the elements $a_1, \ldots, a_n$ of $|\mathfrak{A}|$ *satisfy* the formula F *in* the system $\mathfrak{A}$ *under the correspondence* $x_1 \rightarrow a_1, \ldots, x_n \rightarrow a_n$ and

$$\mathfrak{A}\left|\frac{x_1, \ldots, x_n}{a_1, \ldots, a_n}\right. t \quad \text{or} \quad \mathfrak{A}\left|\frac{x_i}{a_i}\right. t \tag{4}$$

to denote the *value* of t in $\mathfrak{A}$ *under the correspondence* $x_1 \rightarrow a_1, \ldots, x_n \rightarrow a_n$.

The use of the notations (4) and (3) will always imply that our conditions hold for F, t, $x_1, \ldots, x_n$.

If F is a sentence, we write

$$\mathfrak{A} \models F \tag{5}$$

for the statement that $\mathfrak{A}$ *satisfies* F or F *holds in* $\mathfrak{A}$ or $\mathfrak{A}$ is a *model of* F. We assume that the notion of satisfaction as used in (3) and (5) is known. We mention only that the interpretation of the identity symbol $\mathbf{=}$ is always the real identity $=$ in the case of relational systems. However we shall need occassionally so called pseudosystems. Roughly speaking a pseudosystem $\mathfrak{A}$ differs from a system only in that the realization $\mathbf{=}_\mathfrak{A}$ of the identity symbol is not necessarily the real identity. More precisely a *pseudosystem* $\mathfrak{A}$ *of* $\mathbf{L}$ is a pair (A, Δ) where A is a set $(A = |\mathfrak{A}|)$, Δ is a mapping of $L \cup \{\mathbf{=}\}$ such that $\Delta(P)$ $(= P_\mathfrak{A})$ and $\Delta(f)$ $(= f_\mathfrak{A})$ for $P, f \in \mathbf{L}$ are as before and $\Delta(\mathbf{=})$ denoted by $\mathbf{=}_\mathfrak{A}$ is a binary relation on $\mathfrak{A}$. In the case of pseudosystems we must modify the notation of satisfaction in the natural way. We shall use the notations (3), (4), (5) in connection with a pseudosystem $\mathfrak{A}$ in the appropriate sense.

Let Σ be a set of sentences of $\mathbf{L}$ (or *an axiom system of* $\mathbf{L}$). Let $\mathbf{M}_\mathbf{L}(\Sigma)$ denote the class of all models of Σ in the language $\mathbf{L}$, i.e. $\mathbf{M}_\mathbf{L}(\Sigma) = \{\mathfrak{A} : \mathfrak{A} \in \mathfrak{S}(\mathbf{L}),$ $\mathfrak{A} \models F$ for every $F \in \Sigma\}$. We write $\mathbf{M}_\mathbf{L}(F)$ instead of $\mathbf{M}_\mathbf{L}(\{F\})$. If $\mathbf{K} =$ $= \mathbf{M}_\mathbf{L}(\Sigma)$ then we write $\mathbf{K} \in \mathbf{EC}_\Delta$; if in addition $\Sigma = \{F\}$ then $\mathbf{K} \in \mathbf{EC}$.

Let $\mathbf{L}$, $\mathbf{L}'$ be two languages, $\mathbf{L} \subset \mathbf{L}'$, let $\mathfrak{A}$ be a system or pseudosystem of $\mathbf{L}'$. Let $\mathfrak{A} \mid \mathbf{L}$ denote the uniquely determined system or pseudosystem $\mathfrak{B}$ of $\mathbf{L}$ such that $|\mathfrak{B}| = |\mathfrak{A}|$, $P_\mathfrak{B} = P_\mathfrak{A}$, $f_\mathfrak{B} = f_\mathfrak{A}$ for any $P, f \in \mathbf{L}$ and $\mathbf{=}_\mathfrak{A} = \mathbf{=}_\mathfrak{B}$ in the case of pseudosystems. Let $\mathbf{K}' \mid \mathbf{L} = \{\mathfrak{A} \mid \mathbf{L} : \mathfrak{A} \in \mathbf{K}\}$ for a class $\mathbf{K}' \subset$ $\subset \mathfrak{S}(\mathbf{L}')$. If in addition $\mathbf{K}' \in \mathbf{EC}_\Delta$ then $\mathbf{K} = \mathbf{K}' \mid \mathbf{L} \in \mathbf{PC}_\Delta$ (or $\mathbf{K}$ is a $\mathbf{PC}_\Delta$-class), and if $\mathbf{K}' \in \mathbf{EC}$ then $\mathbf{K} \in \mathbf{PC}$.

If $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ then $\mathbf{Th}(\mathbf{K})$ denotes the set of sentences of $\mathbf{L}$ holding in every system of $\mathbf{K}$; $\mathbf{Th}(\mathfrak{A}) = \mathbf{Th}(\{\mathfrak{A}\})$.

A sentence F is a consequence of the set Σ of sentences or of the sentence G (notation: $\Sigma \vdash F$, $G \vdash F$ resp.) if every model of Σ or of G is a model of F. The set of all consequences of Σ is denoted by $\mathbf{Cn}(\Sigma)$. If F, G are formulae and $Cl(F \leftrightarrow G)$ is identically true, i.e. it is true in every system of the corresponding language then F and G are said to be equivalent and we write $F \sim G$. Besides we shall use the sign $\sim$ to denote the real equivalence, i.e. if A and B are statements (having truth values) $A \sim B$ will mean that A and B have the same truth value.

We shall denote languages by $\mathbf{L}$; relational systems or pseudosystems by $\mathfrak{A}$, $\mathfrak{B}$,; formulae by $E, F, G, H, \Gamma, \Phi, \Psi$; sets of formulae by Σ, Θ; predicate symbols by M, N, P, Q, R; function symbols by f, g, h, l; functions by φ, ψ, ε, δ, r; sets by A, B, U, V, W, X, Z; variables by v, w, x, y, z; terms by t, u; classes of relational systems by $\mathbf{K}$; natural numbers by i, j, k, l, m, n, s. All these notations can occur with indices or superscripts having similar meaning. In § 2 and § 3 we shall use several bold type letters to denote variables to emphasize their correspondence with certain elements.

Let $\mathfrak{A}$ be a pseudosystem of $\mathbf{L}$. $\mathbf{=}_\mathfrak{A}$ is said to be a *congruence relation on* $\mathfrak{A}$ if $\mathbf{=}_\mathfrak{A}$ is an equivalence relation and for any $P, f \in \mathbf{L}$ the closures of the formulas

$$x_1 \mathbf{=} y_1 \wedge \ldots \wedge x_n \mathbf{=} y_n \rightarrow (P(x_1, \ldots, x_n) \leftrightarrow P(y_1, \ldots, y_n))$$
$$x_1 \mathbf{=} y_1 \wedge \ldots \wedge x_n \mathbf{=} y_n \rightarrow f(x_1, \ldots, x_n) \mathbf{=} f(y_1, \ldots, y_n)$$

hold in $\mathfrak{A}$, ($x_1, \ldots, x_n, y_1, \ldots, y_n$ being distinct variables). In this case we define the factor system $\mathfrak{B} = \mathfrak{A}/{=_{\mathfrak{A}}}$ in the well known way as follows. $\mathfrak{B}$ is a relational system of $\mathbf{L}$, $|\mathfrak{B}|$ is the set of all equivalence classes $a/{=_{\mathfrak{A}}}$ formed by elements a of $|\mathfrak{A}|$ with respect to the equivalence relation $=_{\mathfrak{A}}$ and if $P, f \in \mathbf{L}$ then $P_{\mathfrak{B}}, f_{\mathfrak{B}}$ are defined by

$$P_{\mathfrak{B}}(a_1/{=_{\mathfrak{A}}}, \ldots, a_n/{=_{\mathfrak{A}}}) \sim P_{\mathfrak{A}}(a_1, \ldots, a_n)$$
$$f_{\mathfrak{B}}(a_1/{=_{\mathfrak{A}}}, \ldots, a_n/{=_{\mathfrak{A}}}) = f(a_1, \ldots, a_n)/{=_{\mathfrak{A}}}.$$

Lemma 2. *If $\mathfrak{A}$ is a pseudosystem of $\mathbf{L}$, $\mathfrak{A}$ satisfies all sentences of the set $\Sigma \subset \mathfrak{F}_0(L)$ and $=_{\mathfrak{A}}$ is a congruence relation on $\mathfrak{A}$ then $\mathfrak{A}/{=_{\mathfrak{A}}} \in \mathbf{M}_{\mathbf{L}}(\Sigma)$.*

A formula is said to be in prenex normal form (pnf) if it is of the following form:

$$(6) \qquad (x_1) \ldots (x_{k_1}) (\exists y_1) \ldots (x_{k_{n-1}+1}) \ldots (x_{k_n}) (\exists y_n) (x_{k_n+1}) \ldots (x_{k_{n+1}}) \Phi$$

where Φ contains no quantifier. It can happen that for some $i = 1, \ldots n$ $k_i - k_{i-1} = 0$ i.e. no universal quantifier occurs between $(\exists y_{i-1})$ and $(\exists y_i)$, or that $n = 0$ i.e. no existential quantifier occurs in the prefix. To every formula F there exists a formula G of the same language so that G contains the same free variables as F, $G \sim F$ and G is in pnf. Let H be a sentence in pnf i.e. of the form (6). We define an open formula H^* in an enlarged language $\mathbf{L}_1^*$ as follows. Let f_i^H be new function symbols for $i = 1, \ldots, n$ with $\nu(f_i^H) = k_i$ Let

$$(7) \qquad F^* = \left| \frac{y_i}{f_i^H(x_1, \ldots, x_{k_i})} \right. \Phi .$$

Lemma 3. *If $\mathfrak{A}$ is a system or pseudosystem of $\mathbf{L}$ then $\mathfrak{A} \vdash H$ if and only if there exists a system or pseudosystem $\mathfrak{A}^*$ of $\mathbf{L}^*$ for which $\mathfrak{A}^* \vdash Cl(H^*)$ and* $\mathfrak{A}^* | \mathbf{L} = \mathfrak{A}$. That is the well known procedure of introducing the Skolem functions.

If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(\mathbf{L})$ then $\mathfrak{A}$ is said to be a subsystem of $\mathfrak{B}$ if and only if $|\mathfrak{A}| \subset |\mathfrak{B}|$, $P_{\mathfrak{A}} \subset P_{\mathfrak{B}}$ and $f_{\mathfrak{A}} \subset f_{\mathfrak{B}}$ for any $P, f \in \mathbf{L}$. In this case we write $\mathfrak{A} \subset \mathfrak{B}$. The class of all subsystems of a system $\mathfrak{B}$ is denoted by $\mathbf{S}(\mathfrak{B})$ and we put $\mathbf{S}(K) = \bigcup_{\mathfrak{B} \in K} \mathbf{S}(\mathfrak{B})$. If A is a non empty subset of $|\mathfrak{B}|$ and for any $a_1, \ldots, \ldots, a_n \in A$, $f \in \mathbf{L}$ we have $f_{\mathfrak{A}}(a_1, \ldots, a_n) \in A$ then $\mathfrak{B}[A]$ denotes the unique subsystem $\mathfrak{A}$ of $\mathfrak{B}$ for which $|\mathfrak{A}| = A$. We shall consider $\mathfrak{B}[A]$ holding defined only if our conditions.

If $\mathfrak{A}_n \in \mathfrak{S}(\mathbf{L})$ and $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$ for $n < \omega$ then $\bigcup_{n<\omega} \mathfrak{A}_n$ is the system $\mathfrak{B}$ for which

$$|\mathfrak{B}| = \bigcup_{n<\omega} |\mathfrak{A}_n|, \; P_{\mathfrak{B}} = \bigcup_{n<\omega} P_{\mathfrak{A}_n}, \; f_{\mathfrak{B}} = \bigcup_{n<\omega} f_{\mathfrak{A}_n}$$

for any $P, f \in \mathbf{L}$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(\mathbf{L})$. $\mathfrak{A}$ is an *elementary subsystem* of $\mathfrak{B}$ ($\mathfrak{A} \prec \mathfrak{B}$), or $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$ if $\mathfrak{A} \subset \mathfrak{B}$ and for any $F \in \mathfrak{F}(\mathbf{L})$ and $a_1, \ldots, \ldots, a_n \in |\mathfrak{A}|$

$$\mathfrak{A} \left| \frac{x_i}{a_i} \right. F \sim \mathfrak{B} \left| \frac{x_i}{a_i} \right. F$$

Let N be a predicate symbol. We shall abbreviate $(x)(N(x) \to F)$ as $(x)_N F$ and $(\exists x)(N(x) \wedge F)$ as $(\exists x)_N F$. F^N is said to be the *relativized of F* *to* N and F^N arises from F by replacing each quantifier (x) by $(x)_N$ and $(\exists x)$ by $(\exists x)_N$.

Lemma 4. *If* $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$, $F \in \mathfrak{F}_0(L)$, $\{x: N_{\mathfrak{A}}(x)\} = B$ *and* $\mathfrak{A}[B]$ *is defined then* $\mathfrak{A}[B] \vdash F$ *if and only if* $\mathfrak{A} \vdash F^N$.

We shall need the possibility of replacing functions by predicates in the following form. We associate a new predicate symbol Q^f with each function symbol $f \in \mathbf{L}$ with $\nu(Q^f) = \nu(f) + 1$. Let $\overline{\overline{\mathbf{L}}}$ denote the language consisting of the predicate symbols of $\mathbf{L}$ and of the predicate symbols Q^f for all $f \in \mathbf{L}$. Now let $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$. The system $\overline{\overline{\mathfrak{A}}}$ of $\overline{\overline{\mathbf{L}}}$ is defined by the following conditions: $|\overline{\overline{\mathfrak{A}}}| = |\mathfrak{A}|$, $P_{\overline{\overline{\mathfrak{A}}}} = P_{\mathfrak{A}}$ for $P \in \mathbf{L}$, $(Q^f)_{\overline{\overline{\mathfrak{A}}}}(a_1, \ldots, a_n, a_{n+1}) \sim f_{\mathfrak{A}}(a_1, \ldots, a_n) = a_{n+1}$ for any $a_1, \ldots, a_n$, $a_{n+1} \in |\mathfrak{A}|$, $f \in \mathbf{L}$. Let $\overline{\overline{\mathbf{K}}} = \{\overline{\overline{\mathfrak{A}}}: \mathfrak{A} \in \mathbf{K}\}$. Let now $F \in \mathfrak{F}_0(\mathbf{L})$. Let $\overline{\overline{F}}$ denote the formula obtained from F by „replacing" each $f \in \mathbf{L}$ by Q^f in a well known way such that $\mathfrak{A} \vdash F$ is equivalent to $\overline{\overline{\mathfrak{A}}} \vdash \overline{\overline{F}}$. If Σ is a set of sentences of $\mathbf{L}$, we define $\overline{\overline{\Sigma}}$ as the set of the formulae $\overline{\overline{F}}$ for $F \in \Sigma$ and of the formulae

$$(x_1) \ldots (x_n)\, (\exists y)\, (z)\, (Q^f(x_1, \ldots, x_n, y) \wedge (Q^f(x_1, \ldots, x_n, z) \rightarrow z = y))$$

for every $f \in \mathbf{L}$.

Lemma 5. $\overline{\overline{\mathbf{M}_{\mathbf{L}}(\Sigma)}} = \mathbf{M}_{\overline{\overline{\mathbf{L}}}}(\overline{\overline{\Sigma}})$.

Lemma 6. *If for a* $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ *we have either* $\overline{\overline{\mathbf{K}}} \in \mathbf{EC}_\Delta$ *or* $\overline{\overline{\mathbf{K}}} \in \mathbf{EC}$ *or* $\overline{\overline{\mathbf{K}}} \in \mathbf{PC}_\Delta$ *or* $\overline{\overline{\mathbf{K}}} \in \mathbf{PC}$ *then the same holds for* $\mathbf{K}$. In order to obtain the desired axiom system for proving Lemma 6 we need only replace each part $Q^f(x_1, \ldots, x_n, y)$ of the corresponding formulae by $f(x_1, \ldots, x_n) = y$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(\mathbf{L})$. The mapping φ of $|\mathfrak{B}|$ onto $|\mathfrak{A}|$ is said to be a *homomorphism of* $\mathfrak{B}$ *onto* $\mathfrak{A}$, if for any $P, f \in \mathbf{L}$ and $a_1, \ldots, a_n \in |\mathfrak{B}|$ $P_{\mathfrak{B}}(a_1, \ldots, a_n)$ implies $P_{\mathfrak{A}}(\varphi(a_1), \ldots, \varphi(a_n))$ and $\varphi(f_{\mathfrak{A}}(a_1, \ldots, a_n)) = f_{\mathfrak{B}}(\varphi(a_1), \ldots, \varphi(a_n))$. In this case $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$, in notation $\mathfrak{A} \in \mathbf{H}(\mathfrak{B})$. We write also $\mathbf{H}(\mathbf{K})$ for $\bigcup\limits_{\mathfrak{B} \in \mathbf{K}} \mathbf{H}(\mathfrak{B})$.

A sentence F is universal if $F = Cl(\Phi)$ where Φ contains no quantifier. F is positive if F does not contain $\neg, \rightarrow, \longleftrightarrow$. It is trivial and well known that universal sentences are preserved under taking subsystems and positive sentences are preserved under homomorphism. $\mathbf{Pos}(\Sigma)$ denotes the set of all positive consequences of Σ.

In § 5 we shall need a theorem of LYNDON [6].

Lemma 7. *(Theorem of* LYNDON*). If* $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$, $\mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma)$ *and* $\mathfrak{A} \in \mathbf{M}_{\mathbf{L}}(\mathbf{Pos}(\Sigma))$ *then there exists an* $\mathfrak{A}'$ *for which* $\mathfrak{A} \prec \mathfrak{A}'$ *and* $\mathfrak{A}' \in \mathbf{H}(\mathbf{K})$.

In the following we briefly describe the notion of *ultrapower* and *strong limit ultrapower* to be used in § 5. These notions are special cases of important recent constructions in the theory of models. For more details we refer to [1] and [3].

Let $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$, $A = |\mathfrak{A}|$, I be a non empty set, D be an ultrafilter on I (i.e. a maximal dual ideal of the Boolean algebra of all subsets of I). For any function $\varphi, \psi \in A^I$ we write $\varphi \approx_D \psi$ if and only if $\{i \in I: \varphi(i) = \psi(i)\} \in D$. $\approx_D$ is an equivalence relation on the set A^I. For each $\varphi \in A^I$ let $\varphi/D = \{\psi : \varphi \approx_D \psi\}$ the equivalence class of φ with respect to $\approx_D$. We define

$A_D^I = \{\varphi/D : \varphi \in A^I\}$. We define the system $\mathfrak{A}_D^I$ of $\mathbf{L}$ as follows. Let $|\mathfrak{A}_D^I| = A_D^I$,

$$P_{\mathfrak{A}_D^I}(\varphi_1/D, \ldots, \varphi_n/D) \sim \{i \in I : P_{\mathfrak{A}}(\varphi_1(i), \ldots, \varphi_n(i))\} \in D$$

and

$$f_{\mathfrak{A}_D^I}(\varphi_1/D, \ldots, \varphi_n/D) = f_{\mathfrak{A}}(\varphi_1, \ldots, \varphi_n)/D\,.$$

Now we define an auxiliary notion to make easier the definition of strong limit ultrapower. Let c_a denote the constant function of A^I which takes the value $a \in A$ for each $i \in I$, let $A_{I,D}$ denote the set $\{c_a/D : a \in A\}$. It is well known that $\mathfrak{A}_D^I[A_{I,D}]$ is an elementary subsystem of $\mathfrak{A}_D^I$ and it is isomorphic to $\mathfrak{A}$ by the mapping $c_a/D \to a$. Let $A^{[I/D]} = (A_D^I - A_{I,D}) \cup A$ and let $d_A^{I,D}$ (briefly d) be the onto mapping $d : A^{[I/D]} \to A_D^I$ for which $d(\varphi/D) = \varphi/D$ if $\varphi \in A^I$ and $\varphi/D \notin A_{I,D}$ and $d(a) = c_a/D$ for $a \in A$. We define the system $\mathfrak{A}^{[I/D]}$ of $\mathbf{L}$ such that $d_A^{I,D}$ is an isomorphism from $\mathfrak{A}^{[I/D]}$ onto $\mathfrak{A}_D^I$. Consequently we have $\mathfrak{A} \prec \mathfrak{A}^{[I/D]}$.

Lemma 8. *If $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(\mathbf{L})$ and $\mathfrak{A}$ satisfies every universal sentence holding in $\mathfrak{B}$ then $\mathfrak{A}$ is isomorphic to a subsystem $\mathfrak{A}'$ of $\mathfrak{B}^{[I/D]}$ for some I and D as before.*

That is well known and is an immediate consequence of Theorem 1.15 of [1].

Now let $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$, let I_n be a non empty set, D_n an ultrafilter on I_n for $n < \omega$. We define by induction

$$\mathfrak{A}_0 = \mathfrak{A} \tag{8}$$

$$\mathfrak{A}_{n+1} = \mathfrak{A}_n^{[I_n/D_n]} \ (n = 0, 1, \ldots) \tag{9}$$

Lemma 9. $\mathfrak{A} \prec \bigcup_{n<\omega} \mathfrak{A}_n$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(L)$, α be a mapping from $|\mathfrak{B}|$ into $|\mathfrak{A}|$. Let α_D^I denote the mapping of B_D^I into A_D^I ($A = |\mathfrak{A}|$, $B = |\mathfrak{B}|$) such that

$$\alpha_D^I(\varphi/D) = (\alpha \circ \varphi)/D \quad \text{for any } \varphi \in B^I$$

(we note that in this case $\alpha \circ \varphi \in A^I$) Further we define $\alpha^{[I/D]} : B^{[I/D]} \to A^{[I/D]}$ such that

$$d_A^{I,D} \circ \alpha^{[I,D]} = \alpha_D^I \circ d_B^{I,D}$$

We see that $\alpha^{[I/D]}(a) = \alpha(a)$ for $a \in B$, in other words $\alpha \subset \alpha^{[I/D]}$.

Now we put

$$\mathfrak{B}_0 = \mathfrak{B}$$

$$\mathfrak{B}_{n+1} = \mathfrak{B}_n^{[I_n/D_n]}$$

$$\alpha_0 = \alpha$$

$$\alpha_{n+1} = \alpha_n^{[I_n/D_n]}$$

in addition to (8) and (9).

Lemma 10. *If α is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ then $\bigcup_{n<\omega} \alpha_n$ is a homomorphism of $\bigcup_{n<\omega} \mathfrak{B}_n$ onto $\bigcup_{n<\omega} \mathfrak{A}_n$.*

We shall use the following

Lemma 11. Compactness Theorem. *If $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$ and $F \in \mathbf{Cn}(\Sigma)$ then there is a finite subset Σ_0 of Σ for which $F \in \mathbf{Cn}(\Sigma_0)$.*

We suppose the notion of (general) recursive function, (general) recursive predicate of the natural numbers, recursive or recursively enumerable set of natural numbers to be known. The following axiom system called Robinson's system (see KLEENE [4], p. 197) will be used in § 2. It contains the constants $\mathbf{0}, \mathbf{1}$ and the binary function symbols $+, \cdot$. Let the set of the letter be $\mathbf{L}^0$.

$$(R)\begin{cases} (x)\,(y)\,(x + \mathbf{1} = y + \mathbf{1} \to x = y) \\ (x)\,(x + \mathbf{0} = x) \\ (x)\,(y)\,(x + (y + \mathbf{1}) = (x + y) + \mathbf{1}) \\ (x)\,(\neg\, x + \mathbf{1} = \mathbf{0}) \\ (x)\,(x \cdot \mathbf{0} = \mathbf{0}) \\ (x)\,(y)\,(x \cdot (y + \mathbf{1}) = x \cdot y + x) \\ (x)\,(\exists y)\,(y + \mathbf{1} = x \vee x = \mathbf{0}) \end{cases}$$

If k is a natural number, $\underline{k}$ will denote the corresponding *numeral*, i.e. if $k = 0$ then $\underline{k} = \mathbf{0}$, and $\underline{k+1} = (\underline{k}) + \mathbf{1}$. In the following lemma $\vdash_{(R)} F$ will mean that the sentence F is derivable from (R) in a usual formal system of the first order predicate calculus with equality axioms.

Lemma 12. (a) *For each recursive predicate $R(x_1, \ldots, x_n)$ there is a formula $F(x_1, \ldots, x_n)$ of $\mathbf{L}^0$ such that for any natural numbers $k_1, \ldots, k_n$*

$$R(k_1, \ldots, k_n) \sim \vdash_{(R)} F(\underline{k_1}, \ldots, \underline{k_n})$$

$$\neg\, R(k_1, \ldots, k_n) \sim \vdash_{(R)} \neg F(\underline{k_1}, \ldots, \underline{k_n})$$

(b) *If specially $R(x_1, \ldots, x_n)$ is $\varphi(x_1, \ldots, x_{n-1}) = x_n$ for a number theoretic function φ then in addition to* (a) *we have for any natural numbers $k_1, \ldots, k_{n-1}$ and $k_n = \varphi\ (k_1, \ldots, k_{n-1})$*

$$\vdash_{(R)} (x)\,\big(F(\underline{k_1}, \ldots, \underline{k_{n-1}}, x) \to x = \underline{k_n}\big)$$

(c) *For each natural number n*

$$\vdash_{(R)} (x)\,\big(x < \underline{n} \to (x = \mathbf{0} \vee x = \mathbf{1} \vee \ldots \vee x = \underline{n - 1})\big)$$

where $x < y$ is an abbreviation of $(\exists z)\,\big((z + \mathbf{1}) + x = y\big)$ (see [4] Corollary of Theorem 32 (p. 296) for (a), Theorem 32 (p. 295) for (b) and *166 (p. 197) or its proof for (c)).

§ 2. A generalization of a theorem of Kleene

For the sake of simplicity we assume in the following definition that $\mathbf{L}$ contains only predicate symbols and no function symbol.

Definition. The language $\mathbf{L}$ is said to be *recursive by* the enumeration $\mu = (P_i)_{i<\omega}$ of all predicate symbols of $\mathbf{L}$ if $r(i) = \nu(P_i)$ is a recursive function of i.

We define the Gödel number $Nu_\mu(F) = Nu(F)$ of formulae F of L by induction on the number of logical operations contained in F.

(i) If $F = v_i = v_j$ $(i, j < \omega)$ then let

$$Nu(F) = 2^1 \cdot 3^i \cdot 5^j$$

(ii) If $F = P_i(v_{i_1}, \dots, v_{i_{r(i)}})$ then let

$$Nu(F) = 2^1 \cdot \prod_{k=1}^{r(i)} p_{i+1}^{i_k}\,{}^3$$

(iii) If $F = \neg G$ then let

$$Nu(F) = 2^3 \cdot 3^{Nu(G)}$$

(iv) If $F = F_1 \wedge F_2$ then let

$$Nu(F) = 2^4 \cdot 3^{Nu(F_1)} \cdot 5^{Nu(F_2)}$$

(v) If $F = (v_i)\, G$ then let

$$Nu(F) = 2^5 \cdot 3^i \cdot 5^{Nu(G)}$$

We remark that only $\neg$, $\wedge$, (v_i) are considered as primitive operations, the others will be used as abbreviations. A set $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$ is said to be *recursively enumerable by* μ if $\{ Nu_\mu(F) \colon F \in \Sigma\}$ is recursively enumerable.

We define the class $\mathbf{PC}_{\Delta\mathrm{rec}}$ of classes of relational systems as follows. Let $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$, $\mathbf{L}$ contain no function symbol.

Definition. $\mathbf{K} \in \mathbf{PC}_{\Delta\mathrm{rec}}$ *if and only if there exist a language* $\mathbf{L}'$ *recursive by an enumeration* μ *and a set* $\Sigma \subset \mathfrak{F}_0(\mathbf{L}')$ *such that* $\mathbf{L} \subset \mathbf{L}'$ *and* Σ *is recursively enumerable by* μ *and* $\mathbf{K} = \mathbf{M}_{\mathbf{L}'}(\Sigma) \mid \mathbf{L}$.

Theorem 1. *If* $\mathbf{L}_0$ *is a finite language*, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L}_0)$ *and* $\mathbf{K} \in \mathbf{PC}_{\Delta\mathrm{rec}}$ *then* $\mathbf{K}^\infty \in \mathbf{PC}$ ($\mathbf{K}^\infty$ *being the class of infinite systems of* $\mathbf{K}$).

Proof. According to the hypothesis we have the language $\mathbf{L} \supset \mathbf{L}_0$, the enumeration $(P_i)_{i<\omega}$ of all predicate symbols of $\mathbf{L}$, the set Σ of sentences of $\mathbf{L}$ and the recursive predicate $R(n, m)$ such that $r(i) = \nu(P_i)$ is a recursive function of i, $\mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma) \mid \mathbf{L}_0$ and $n = Nu(F)$ for some $F \in \Sigma$ if and only if there exists a natural number m for which $R(n, m)$ holds. We may assume that the predicate symbols of $\mathbf{L}_0$ are $P_0, \dots, P_{n_0-1}$. The language $\mathbf{L}_1$ is defined as the set of the predicate symbols $P_0, \dots, P_{n_0-1}$ and of the following additional symbols:

$\mathbf{0}$, $\mathbf{1}$ constants
$+$, $\cdot$ binary function symbols
N unary predicate symbol
h binary function symbol
l unary function symbol
M binary predicate symbol

Let us consider the relativization of Robinson's theory to the predicate N (see § 1), i.e. the following sentences

$N(\mathbf{0})$
$N(\mathbf{1})$
$(v_0)\,(v_1)\,\big(N(v_0) \wedge N(v_1) \to N(v_0 + v_1)\big)$
$(v_0)\,(v_1)\,\big(N(v_0) \wedge N(v_1) \to N(v_0 \cdot v_1)\big)$
$(v_0)_N\,(v_1)_N\,(v_0 + \mathbf{1} = v_1 + \mathbf{1} \to v_0 = v_1)$
$(v_0)_N\,(v_0 + \mathbf{0} = v_0)$

[3] p_i denotes the i-th prime number ($p_0 = 2$, $p_1 = 3$, ...)

$$(v_0)_N\,(v_1)_N\,(v_0 + (v_1 + \mathbf{1}) = (v_0 + v_1) + \mathbf{1})$$
$$(v_0)_N(\neg\, v_0 + \mathbf{1} = \mathbf{0})$$
$$(v_0)_N\,(v_0 \cdot \mathbf{0} = \mathbf{0})$$
$$(v_0)_N\,(v_1)_N\,(v_0 \cdot (v_1 + \mathbf{1}) = v_0 \cdot v_1 + v_0)$$
$$(v_0)_N\,(\exists\, v_1)_N\,(v_1 + \mathbf{1} = v_0 \vee v_0 = \mathbf{0})$$

Let R^N denote the conjunction of these formulae.

Let us consider the following number theoretical predicates:

$Neg(n, m)$	$n = Nu(\neg\, \Phi), m = Nu(\Phi)$ for some $\Phi \in \mathfrak{F}(L)$	$NEG(\boldsymbol{n}, \boldsymbol{m})$
$Conj(n, m_1, m_2)$	$n = Nu(\Phi_1 \wedge \Phi_2), m_1 = Nu(\Phi_1)$ $m_2 = Nu(\Phi_2)$ for some $\Phi_1, \Phi_2 \in \mathfrak{F}(L)$	$CONJ(\boldsymbol{n}, \boldsymbol{m}_1, \boldsymbol{m}_2)$
$Quant(n, m, i)$	$n = Nu\big((v_i)\Phi\big), m = Nu(\Phi)$ for some $\Phi \in \mathfrak{F}(L)$	$QUANT(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{i})$
$Eq(n)$	$n = Nu(v_{j_0} \equiv v_{j_1})$ for some j_0, j_1	$EQ(\boldsymbol{n})$
$Pim(n, i)$	$n = Nu(P_i(v_{j_0}, \ldots, v_{j_{r(i)-1}})$ for some $j_0, \ldots, j_{r(i)-1}$	$PRIM(\boldsymbol{n}, \boldsymbol{i})$
$Eq(n, j, k)$	$n = Nu(v_{j_0} \equiv v_{j_1})$, $k = 0$ or $k = 1$, and $j = j_k$	$EQ(\boldsymbol{n}, \boldsymbol{j}, \boldsymbol{k})$
$Prim(n, i, j, k)$	$n = Nu(P_i(v_{j_0}, \ldots, v_{j_{r(i)-1}})$ $0 \leq k \leq r(i) - 1, j = j_k$	$PRIM(\boldsymbol{n}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$
$Pr(n, i)$	$n = Nu(P_i(v_0, \ldots, v_{r(i)-1})$	$PR(\boldsymbol{n}, \boldsymbol{i})$
$k < i$		$S_0(\boldsymbol{k}, \boldsymbol{i})$
$k < r(i)$		$S_1(\boldsymbol{k}, \boldsymbol{i})$
$k \geq r(i)$		$S_2(\boldsymbol{k}, \boldsymbol{i})$
$k \geq lh(n)$	$n = Nu(\Phi)$ and the maximal natural number i for which v_i is a a free variable of Φ is $lh(n) - 1$	$S_3(\boldsymbol{k}, \boldsymbol{n})$
$i = \max(j, k)$		$MAX(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$
$R(n, m)$		$RE(\boldsymbol{n}, \boldsymbol{m})$

Lemma 13. *The number theoretical predicates listed above are all recursive.*

That is trivial by the definition of $Nu(\Phi)$ and by our hypothesis that $r(i)$ is recursive. If n is a natural number then $\underline{n}$ denotes the corresponding numeral (formal term of $\mathbf{L}_1$) (see § 1).

Lemma 14. *Let $R(x_1, \ldots, x_n)$ be a recursive number theoretic predicate. Then there exists a formula $F^N(x_1, \ldots, x_n)$ of the language $\mathbf{L}^0 \subset \mathbf{L}_1$ such that for every model $\mathfrak{A}$ of R^N and for any natural numbers $k_1, \ldots, k_n$ $R(k_1, \ldots, k_n)$ is true if and only if $\mathfrak{A} \vdash F^N(\underline{k}_1, \ldots, \underline{k}_n)$.*

Proof. Our assertion is a direct consequence of Lemma 12(a). The desired formula F^N can be obtained by relativizing the formula F of Lemma 12 to N.

The formula F^N to each of the predicates listed before Lemma 1 is denoted by the corresponding notation standing on the right side of the above list.

Lemma 15. *For every model* $\mathfrak{A}$ *of* R^N *and for arbitrary natural numbers* n, i, s

$$\mathfrak{A} \vdash (\boldsymbol{j})_N\big(S_0(\boldsymbol{j}, \underline{n}) \to (\boldsymbol{j} = \boldsymbol{0} \vee \boldsymbol{j} = \boldsymbol{1} \vee \ldots \vee \boldsymbol{j} = \underline{n-1})\big) \tag{1}$$

$$\mathfrak{A} \vdash (\boldsymbol{k})_N\big(S_1(\boldsymbol{k}, \underline{i}) \to (\boldsymbol{k} = \boldsymbol{0} \vee \boldsymbol{k} = \boldsymbol{1} \vee \ldots \vee \boldsymbol{k} = \underline{r(i)-1})\big) \tag{2}$$

$$\mathfrak{A} \vdash (\boldsymbol{k})_N\big(MAX(\boldsymbol{k}, \underline{s}, \underline{i}) \to \boldsymbol{k} = \underline{\max(s, i)}\big) \tag{3}$$

Proof. (1) follows from Lemma 12(*c*), (3) from Lemma 12(*b*), (2) follows from Lemma 12(b) and (c) if we specialize

$$S_1(\boldsymbol{k}, \boldsymbol{i}) = (\exists \boldsymbol{j})_N\big(S_0(\boldsymbol{k}, \boldsymbol{j}) \wedge R_0^N(\boldsymbol{i}, \boldsymbol{j})\big) \tag{4}$$

where $R_0(i,j)$ is a formula representing $j = r(i)$ as intended to get in Lemma 1 (b) i.e. we have

$$\mathfrak{A} \vdash R_0^N(\underline{i}, \underline{j}) \sim j = r(i) \tag{5}$$

and

$$\mathfrak{A} \vdash (\boldsymbol{k})_N\big(R_0^N(\underline{i}, \boldsymbol{k}) \to \boldsymbol{k} = \underline{r(i)}\big) \tag{6}$$

for natural numbers i, j.

For $S_1(\boldsymbol{k}, \boldsymbol{i})$ defined by (4) we must show (2), furthermore also

$$\mathfrak{A} \vdash S_1(\underline{k}, \underline{i}) \sim k < r(i) \tag{7}$$

i.e. that $S_1(\boldsymbol{k}, \boldsymbol{i})$ satisfies Lemma 14 too.

From (5) and Lemma 14 for $S_0(\boldsymbol{k}, \boldsymbol{j})$ it follows easily that $k < r(i)$ implies $\mathfrak{A} \vdash S_1(\underline{k}, \underline{i})$. Conversely if $\mathfrak{A} \vdash S_1(\underline{k}, \underline{i})$ then from (4) and (6) it follows that $\mathfrak{A} \vdash S_0(\underline{k}, \underline{r(i)})$ i.e. $k < r(i)$, consequently (7) is proved. (2) follows from (4), (6) and (1) similarly.

Now we give a finite axiom system Σ_1 for which we shall prove

$$\mathbf{K}^\infty = \mathbf{M}_{\mathbf{L}_1}(\Sigma_1) \,|\, \mathbf{L}_0 \tag{8}$$

We use the abreviations $m_i = Nu\big(P_i(v_0, \ldots, v_{r(i)-1})\big)$ $(i = 0, 1, \ldots)$; $e_0 = Nu(v_0 = v_1)$

A 1 R^N

A 2 $l(\boldsymbol{0}) = \boldsymbol{0}$

A 3 $(\boldsymbol{a})\, N\big(l(\boldsymbol{a})\big)$

A 4 $(\boldsymbol{a})\,(\boldsymbol{x})\,(\exists\, \boldsymbol{b})\big(l(\boldsymbol{b}) = l(\boldsymbol{a}) + \boldsymbol{1} \wedge (\boldsymbol{j})_N\big(S_0(\boldsymbol{j}, l(\boldsymbol{a})) \to$

$\to h(\boldsymbol{a}, \boldsymbol{j}) = h(\boldsymbol{b}, \boldsymbol{j})\big) \wedge h\big(\boldsymbol{b}, l(\boldsymbol{a})\big) = \boldsymbol{x}\big)$

A 5 $(\boldsymbol{a})\,(\boldsymbol{b})\,\{[l(\boldsymbol{a}) = l(\boldsymbol{b}) \wedge (\boldsymbol{j})_N\big(S_0(\boldsymbol{j}, l(\boldsymbol{a})) \to h(\boldsymbol{a}, \boldsymbol{j}) = h(\boldsymbol{b}, \boldsymbol{j})\big)] \to$

$\to \boldsymbol{a} = \boldsymbol{b}\}$

A 6 $(\boldsymbol{a})\big(l(\boldsymbol{a}) = \underline{2} \to \big(M(\underline{e_0}, \boldsymbol{a}) \longleftrightarrow h(\boldsymbol{a}, \boldsymbol{0}) = h(\boldsymbol{a}, \boldsymbol{1})\big)$

A 7 $\bigwedge_{i=0}^{n_0-1} (\boldsymbol{a})\,[l(\boldsymbol{a}) = \underline{r(i)} \rightarrow (M(\underline{m_i}, \boldsymbol{a}) \longleftrightarrow P_i(h(\boldsymbol{a}, \boldsymbol{0}), \ldots, h(\boldsymbol{a}, \underline{r(i)-1})))]$

A 8 $(\boldsymbol{n})_N (\boldsymbol{a})\,(\boldsymbol{b})\,\{[EQ(\boldsymbol{n}) \wedge S_2(l(\boldsymbol{a}), \underline{2}) \wedge S_2(l(\boldsymbol{b}), \underline{2}) \wedge$
$\wedge\ (\boldsymbol{j})_N (\boldsymbol{k})_N ((EQ(\boldsymbol{n}, \boldsymbol{j}, \boldsymbol{k}) \wedge S_0(\boldsymbol{j}, l(\boldsymbol{a})) \wedge (\boldsymbol{k} = \boldsymbol{1} \vee \boldsymbol{k} = \underline{2})) \rightarrow$
$\rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{j}))] \rightarrow (M(\boldsymbol{n}, \boldsymbol{a}) \longleftrightarrow M(\boldsymbol{m}, \boldsymbol{b}))\}$

A 9 $(\boldsymbol{n})_N (\boldsymbol{m})_N (\boldsymbol{i})_N (\boldsymbol{a})\,(\boldsymbol{b})\,\{[PRIM(\boldsymbol{n}, \boldsymbol{i}) \wedge PR(\boldsymbol{m}, \boldsymbol{i}) \wedge S_2(l(\boldsymbol{a}), \boldsymbol{i}) \wedge$
$\wedge\ S_2(l(\boldsymbol{b}), \boldsymbol{i}) \wedge (\boldsymbol{j})_N (\boldsymbol{k})_N ((PRIM\,(\boldsymbol{n}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \wedge S_0(\boldsymbol{j}, l(\boldsymbol{a})) \wedge$
$\wedge\ S_1(\boldsymbol{k}, \boldsymbol{i})) \rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{j}))] \rightarrow (M(\boldsymbol{n}, \boldsymbol{a}) \longleftrightarrow M(\boldsymbol{m}, \boldsymbol{b}))\}$

A 10 $(\boldsymbol{n})_N (\boldsymbol{m})_N (\boldsymbol{a})\,[NEG(\boldsymbol{n}, \boldsymbol{m}) \wedge S_3(l(\boldsymbol{a}), \boldsymbol{n}) \rightarrow$
$\rightarrow (M(\boldsymbol{n}, \boldsymbol{a}) \longleftrightarrow \neg\, M(\boldsymbol{m}, \boldsymbol{a}))]$

A 11 $(\boldsymbol{n})_N (\boldsymbol{m}_1)_N (\boldsymbol{m}_2)_N (\boldsymbol{a})\,[CONJ\,(\boldsymbol{n}, \boldsymbol{m}_1, \boldsymbol{m}_2) \wedge S_3(l(\boldsymbol{a}), \boldsymbol{n}) \rightarrow$
$\rightarrow (M(\boldsymbol{n}, \boldsymbol{a}) \longleftrightarrow (M(\boldsymbol{m}_1, \boldsymbol{a}) \wedge M(\boldsymbol{m}_2, \boldsymbol{a}))]$

A 12 $(\boldsymbol{n})_N (\boldsymbol{m})_N (\boldsymbol{i})_N (\boldsymbol{a})\,[QUANT(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{i}) \wedge S_3(l(\boldsymbol{a}), \boldsymbol{n})) \rightarrow$
$\rightarrow [M(\boldsymbol{n}, \boldsymbol{a}) \longleftrightarrow (\boldsymbol{b})\,(\{MAX(l(\boldsymbol{b}), l(\boldsymbol{a}), \boldsymbol{i}) \wedge$
$\wedge\ (\boldsymbol{k})_N ((\neg\, \boldsymbol{k} = \boldsymbol{i} \wedge S_0(\boldsymbol{k}, l(\boldsymbol{a}))) \rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{k}))\} \rightarrow$
$\rightarrow M(\boldsymbol{m}, \boldsymbol{b}))]]$

A 13 $(\boldsymbol{n})_N (\boldsymbol{m})_N (RE(\boldsymbol{n}, \boldsymbol{m}) \rightarrow M(\boldsymbol{n}. \boldsymbol{0}))$

I. Proof of

(9) $$\mathbf{M}_{\mathbf{L}_1}(\Sigma_1) \,|\, \mathbf{L}_0 \supset \mathbf{K}^\infty$$

Let $\mathfrak{A}_0 \in \mathbf{K}^\infty$, i.e. let $|\,\mathfrak{A}_0\,|$ be infinite and $\mathfrak{A}_0 = \mathfrak{A} \,|\, \mathbf{L}_0$ for $\mathfrak{A} \in \mathbf{M}_{\mathbf{L}}(\Sigma)$

We have to define $\mathfrak{A}_1$ such that

(10) $$\mathfrak{A}_1 \in \mathbf{M}_{\mathbf{L}_1}(\Sigma_1)$$

and

(11) $$\mathfrak{A}_0 = \mathfrak{A}_1 \,|\, \mathbf{L}_0$$

Let B be a subset of A of power ω and let $N_{\mathfrak{A}_1}(a)$ be true if and only if $a \in B$.

Let $\mathbf{0}_{\mathfrak{A}_1}$, $\mathbf{1}_{\mathfrak{A}_1}$ be two elements of B, $+_{\mathfrak{A}_1}$, $\cdot_{\mathfrak{A}_1}$ binary functions on $A = |\,\mathfrak{A}_0\,|$ so that $\mathfrak{B} = (A;\ \mathbf{0}_{\mathfrak{A}_1}, \mathbf{1}_{\mathfrak{A}_1}, +_{\mathfrak{A}_1}, \cdot_{\mathfrak{A}_1})\ [B]$ is defined and isomorphic to $(\omega;\ 0, 1, +, \cdot\,)$ where the latter is the system of the natural numbers with the usual constants and operations. We may and shall suppose that $\mathfrak{B}$ is identical with $(\omega;\ 0, 1, +, \cdot)$. If $a, b \in A$ and $a \notin \omega$ or $b \notin \omega$ then $a +_{\mathfrak{A}_1} b$, $a \cdot_{\mathfrak{A}_1} b$ can be defined arbitrarily, e.g. $a +_{\mathfrak{A}_1} b = a \cdot_{\mathfrak{A}_1} b = 0$.

We establish a one-to-one mapping φ of $A - \{0\}$ onto the set of all finite sequences (with at least one element) of A. Let $l_{\mathfrak{A}_1}(0) = 0$ and $l_{\mathfrak{A}_1}(a) =$ the length of $\varphi(a)$ ($=$ the number of elements of the sequence $\varphi(a)$) for $a \in A - \{0\}$. We say that a *represents* the sequence $\varphi(a)$.

If $\varphi(a) = (a_0, \ldots, a_{s-1})$ and $k \in \omega$, $k < s$ then let $h_{\mathfrak{A}_1}(a, k) = a_k$. Otherwise let $h_{\mathfrak{A}_1}(a, b) = 0$.

Let Φ be an arbitrary formula of $\mathbf{L}$, $n = Nu(\Phi)$, $a \in A$, $l(a) \geq lh(n)$, $\varphi(a) = (a_0, \ldots, a_{s-1})$. We define

$$M_{\mathfrak{A}_1}(n, a) \sim \mathfrak{A} \left| \frac{v_0, \ldots, v_{lh(n)-1}}{a_0, \ldots, a_{lh(n)-1}} \right. \Phi$$

Specially if $a = 0$ and Φ is a closed formula

$$M_{\mathfrak{A}_1}(n, 0) \sim \mathfrak{A} \vdash \Phi$$

In all other cases $x, y \in A$ $M_{\mathfrak{A}_1}(x, y)$ may be arbitrary, e.g. $M_{\mathfrak{A}_1}(x, y) = 0$.

Considering (11) $\mathfrak{A}_1$ has been completely defined.

We have to verify that $\mathfrak{A}_1$ satisfies the axioms $A1 - A13$. This verification is straightforward. We give only a sketch of it. We make advantage of the fact that the special formulae listed before Lemma 13 take the same truth value as the corresponding number theoretic predicates for natural number arguments (i.e. for elements a of A satisfying $N_{\mathfrak{A}_1}(a)$).

A1 is true because the system of the natural numbers with the usual operations satisfies the axioms of Robinson's theory (and see Lemma 4 in § 1). A4 expresses that for every sequence $(a_0, \ldots, a_{s-1})$ represented by a and every element x there is a $b \in A$ representing $(a_0, \ldots, a_{s-1}, x)$. A5 expresses that there is only one element of A representing a given sequence.

Now let us observe the definition of $M_{\mathfrak{A}_1}(n, a)$. A6, A7 are obvious. A9 expresses that $P_i(v_{i_0}, \ldots, v_{i_{r(i)-1}})$ takes the same truth value for a sequence $(a_0, \ldots, a_{s-1})$ represented by a which is taken by $P_i(v_0, \ldots, v_{r(i)-1})$ for the sequence $(a_{i_0}, \ldots, a_{i_{r(i)-1}})$ represented by b. A8 is similar for the identity. A10, A11 express the meaning of the negation and conjunction. Let $n = Nu(\Phi)$, $a \in A$, $l_{\mathfrak{A}_1}(a) \geq lh(n)$. A12 expresses that a sequence $(a_0, \ldots, a_{s-1})$ represented by a satisfies the formula $(v_i)\Phi$ if and only if for every b $\varphi(b)$ satisfies Φ provided that the following hold: $\varphi(b) = (b_0, \ldots, b_{s'-1})$, $s' = \max(s, i)$ and $b_j = a_j$ if $j < s$ and $j \neq i$. A13 expresses that the sentences F for which there exists an m with $R(Nu(F), m)$ i.e. the sentences of Σ are true in $\mathfrak{A}$.

We have proved (10) and (11), consequently also (9), qu. e.d.

II. Proof of

$$\mathbf{M}_{\mathbf{L}_1}(\Sigma_1) \,|\, \mathbf{L}_0 \subset \mathbf{K}^\infty \tag{12}$$

Let $\mathfrak{A}_1 \in \mathbf{M}_{\mathbf{L}_1}(\Sigma_1)$, $\mathfrak{A}_0 = \mathfrak{A}_1 \,|\, \mathbf{L}_0$, $|\mathfrak{A}_1| = A$. A is trivially infinite because the values of the numerals in $\mathfrak{A}_1$ must be different.

In order to prove (12) we must construct a system $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$ for which

$$\mathfrak{A} \,|\, \mathbf{L}_0 = \mathfrak{A}_0 \tag{13}$$

and

$$\mathfrak{A} \in \mathbf{M}_{\mathbf{L}}(\Sigma) \tag{14}$$

Let us denote the set of the values of the numerals in $\mathfrak{A}_1$ by B. B is a (possibly proper) subset of $\{a \colon N_{\mathfrak{A}_1}(a)\}$ From the fact that in $\mathfrak{A}_1$ A1 holds it follows that $\mathfrak{B} = (A; \mathbf{0}_{\mathfrak{A}_1}, \mathbf{1}_{\mathfrak{A}_1}, \boldsymbol{+}_{\mathfrak{A}_1}, \cdot_{\mathfrak{A}_1})\ [B]$ is defined and isomorphic to $(\omega; 0, 1, +, \cdot)$. We may and shall suppose that $\mathfrak{B}$ is identical with the latter system.

Lemma 16. *For every natural number $n \geqq 0$ and every sequence $(a_0, a_1, \ldots, a_{n-1})$ of n elements of A (possibly the empty sequence) there is exactly one element a of A for which $l_{\mathfrak{A}_1}(a) = n$ and $h_{\mathfrak{A}_1}(a, k) = a_k$ for $k < n$.* Let $[a_0, a_1, \ldots, a_{n-1}]$ denote this a.

Proof. First we show the existence of a by induction on n. If $n = 0$ then $a = 0$ is suitable by A2. Let $n \geq 1$ and let us assume that $a \in A$, $l_{\mathfrak{A}_1}(a) = n - 1$, $h_{\mathfrak{A}_1}(a, k) = a_k$ for $k < n - 1$. Using A4 („substituting" a_{n-1} for x) we obtain $b \in A$ for which $l_{\mathfrak{A}_1}(b) = n$, $h_{\mathfrak{A}_1}(b, n-1) = a_{n-1}$ and

$$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} \right. (\boldsymbol{j})_N \big(S_0(\boldsymbol{j}, l(\boldsymbol{a})) \rightarrow h(\boldsymbol{a}, \boldsymbol{j}) = h(\boldsymbol{b}, \boldsymbol{j})\big) \tag{15}$$

By Lemma 14 $\mathfrak{A}_1 \vdash S_0(\mathbf{0}, \underline{n-1}), \ldots, S_0(\underline{n-2}, \underline{n-1})$ consequently (15) implies $h_{\mathfrak{A}_1}(b, k) = h_{\mathfrak{A}_1}(a, k) = a_k$ for $k < n - 1$ qu. e.d. Secondly we prove the unicity. Suppose $a, b \in A$; $l_{\mathfrak{A}_1}(a) = l_{\mathfrak{A}_1}(b) = n \in \omega$, $h_{\mathfrak{A}_1}(a, k) = h_{\mathfrak{A}_1}(b, k)$ for $k < n - 1$. $a = b$ will follow by $\mathfrak{A}_1 \vdash$ A6 if we show that

$$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} \right. (\boldsymbol{j})_N \big(S_0(\boldsymbol{j}, \underline{n}) \rightarrow h(\boldsymbol{a}, \boldsymbol{j}) = h(\boldsymbol{b}, \boldsymbol{j})\big)$$

But that is a consequence of Lemma 15 (1) and of our hypothesis.

To define $\mathfrak{A}$ we give $(P_i)_{\mathfrak{A}}$ for $i \geqq n_0$ as follows. For any $a_0, \ldots, a_{r(i)-1} \in A$ let

$$(P_i)_{\mathfrak{A}}(a_0, \ldots, a_{r(i)-1}) \sim M_{\mathfrak{A}_1}(m_i, [a_0, \ldots, a_{r(i)-1}]) \tag{16}$$

We remark that by $\mathfrak{A}_1 \vdash$ A7 (16) holds also for $i < n_0$. Similarly we have by $\mathfrak{A}_1 \vdash$ A6 that

$$a_0 = a_1 \sim M_{\mathfrak{A}_1}(e_0, [a_0, a_1]) \tag{17}$$

Lemma 17. *If $\Phi \in \mathfrak{F}(\mathbf{L})$, $n = Nu(\Phi)$, $a_0, \ldots, a_{s-1} \in A$, $lh(n) \leq S$ then*

$$\mathfrak{A} \left| \frac{v_0, v_1, \ldots, v_{s-1}}{a_0, a_1, \ldots a_{s-1}} \right. \Phi \sim M_{\mathfrak{A}_1}(n, [a_0, \ldots, a_{s-1}]) \tag{18}$$

Proof. The proof proceeds by induction on the number of the logical operators (i.e. $\neg$, $\wedge$, (x)) occurring in Φ.

1. Let first Φ be $v_{j_0} \equiv v_{j_1}$ or $P_i(v_{j_0}, \ldots, v_{j_{r(i)-1}})$. We consider only the second case. The first one can be treated similarly using A8 and (17) instead of A9 and (16).

Let $a = [a_0, \ldots, a_{s-1}]$, $b = [a_{j_0}, \ldots, a_{j_{r(i)-1}}]$, $m = m_i$. By Lemma 14 we have

$$\mathfrak{A}_1 \vdash PRIM(\underline{n}, \underline{i}), PR(\underline{m}, \underline{i}), S_2(\underline{s}, \underline{i}), S_2(\underline{r(i)}, \underline{i}) \tag{19}$$

further

$$\mathfrak{A}_1 \vdash PRIM(\underline{n}, \underline{i}, \underline{j_k}, \underline{k}) \quad \text{for} \quad k < r(i) \tag{20}$$

and

$$\mathfrak{A}_1 \vdash \neg PRIM(\underline{n}, \underline{i}, \underline{j}, \underline{k}) \quad \text{for} \quad j < s, j \neq j_k \tag{21}$$

By using Lemma 15 (1) and (2), (20) and (21) we obtain

$$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} (\boldsymbol{j})_N (\boldsymbol{k})_N \big((S_1(\boldsymbol{k}, \underline{i}) \wedge S_0(\boldsymbol{j}, \underline{s})\big) \rightarrow \right.$$
$$\rightarrow \big(PRIM(\underline{n}, \underline{i}, \boldsymbol{j}, \boldsymbol{k}) \rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{j}))\big)$$

From this and (19) and $\mathfrak{A}_1 \vdash$ A8 it follows

(22) $$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} \big(M(\underline{n}, \boldsymbol{a}) \longleftrightarrow M(\underline{m}, \boldsymbol{b})\big) \right.$$

Applying (16) gives

$$(P_i)_{\mathfrak{A}}(a_{j_0}, \ldots, a_{j_{r(i)-1}}) \sim M_{\mathfrak{A}_1}(m, b)$$

This and (22) imply

$$(P_i)_{\mathfrak{A}}(a_{j_0}, \ldots, a_{j_{r(i)-1}}) \sim M_{\mathfrak{A}_1}(n, a)$$

which is exactly (18) as was to be shown.

2. Let $\Phi = \neg \Psi$ or $\Phi = \Psi_1 \wedge \Psi_2$. These induction cases can be treated by using $\mathfrak{A}_1 \vdash$ A10, A11. Put $m = Nu(\Psi)$ or $m_1 = Nu(\Psi_1)$ and $m_2 = Nu(\Psi_2)$. Let us observe that by Lemma 14 we have $\mathfrak{A}_1 \vdash NEG(\underline{n}, \underline{m})$ or $\mathfrak{A}_1 \vdash CONJ(\underline{n}, \underline{m_1}, \underline{m_2})$ and in both cases $\mathfrak{A}_1 \vdash S_3(\underline{l_{\mathfrak{A}_1}(a)}, \underline{n})$.

3. Let $\Phi = (v_i)\,\Psi$, $m = \overline{Nu\,\Psi}$. To fix the notations we suppose $i \geq s$. The other case $i < s$ can be treated without essential change. Let $a = [a_0, \ldots, a_{s-1}]$.

(a) First we suppose

(23) $$\mathfrak{A} \left| \frac{v_0, v_1, \ldots, v_{s-1}}{a_0, a_1, \ldots, a_{s-1}} \Phi \right.$$

We have to show

(24) $$M_{\mathfrak{A}_1}(n, a)$$

By $\mathfrak{A}_1 \vdash$ A12 and $\mathfrak{A}_1 \vdash QUANT(\underline{n}, \underline{m}, \underline{i}), S_3(\underline{s}, \underline{n})$ it is sufficient to prove that

(25) $$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}}{a} (\boldsymbol{b}) \big[(MAX(l(\boldsymbol{b}), l(\boldsymbol{a}), \underline{i}) \wedge (\boldsymbol{k})_N \big((\neg\, \boldsymbol{k} = \underline{i} \wedge S_0(\boldsymbol{k}, l(\boldsymbol{a}))\big) \rightarrow \right.$$
$$\rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{k})\big) \rightarrow M(\underline{m}, \boldsymbol{b})\big]$$

Let $b \in A$ and suppose

(26) $$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} MAX\big(l(\boldsymbol{b}), l(\boldsymbol{a}), \underline{i}\big) \right.$$

and

(27) $$\mathfrak{A}_1 \left| \frac{\boldsymbol{a}, \boldsymbol{b}}{a, b} (\boldsymbol{k})_N \big((\neg\, \boldsymbol{k} = \underline{i} \wedge S_0(\boldsymbol{k}, l(\boldsymbol{a}))\big) \rightarrow h(\boldsymbol{b}, \boldsymbol{k}) = h(\boldsymbol{a}, \boldsymbol{k})\big) \right.$$

(26) implies by Lemma 15 (3)

$$l_{\mathfrak{A}_1}(b) = \max(s, i) = s\,.$$

From Lemma 16 and (27) and $\mathfrak{A}_1 \vdash S_0(\underline{k}, s)$ for $k < s$ it follows that $b =$

$= [a_0, \ldots, a_{s-1}, a'_s, \ldots, a'_i]$ for some $a'_s, \ldots, a'_i \in A$. From (23) we infer

$$\mathfrak{A} \left| \frac{v_0, \ldots, v_{s-1}, \ldots, v_i}{a_0, \ldots, a_{s-1}, \ldots, a'_i} \right. \Psi .$$

This and the induction hypothesis for Ψ imply $M_{\mathfrak{A}_1}(m, b)$ consequently we have proved (25).

(b) Secondly we suppose (24) and prove (23).

By $\mathfrak{A}_1 \vdash QUANT(\underline{n}, \underline{m}, \underline{i}), S_3(\underline{s}, \underline{n})$ and A12 we have now (25). Let a'_i be an arbitrary element of A, $a'_s, \ldots, a'_{i-1}$ be elements of A (these latter are unessential) and $b = [a_0, \ldots, a_{s-1}, a'_s, \ldots, a'_i]$. By Lemma 14 we have (26) and by Lemma 15 (1) we have (27). From (26), (27) and (25) it follows $M_{\mathfrak{A}_1}(m, b)$ which implies

$$\mathfrak{A} \left| \frac{v_0, \ldots, v_{s-1}, \ldots, v_i}{a_0, \ldots, a_{s-1}, \ldots, a'_i} \right. \Psi \tag{28}$$

by the induction hypothesis. In other words, for arbitrary $a'_i \in A$ (28) holds. But this means exactly that (23) holds.

So we have finished the proof of Lemma 17.

Now we prove (14). Let $F \in \Sigma$, $n = Nu(F)$. According to our hypothesis we have a natural number m for which $R(n, m)$ holds. By Lemma 3 this implies $\mathfrak{A}_1 \vdash RE(\underline{n}, \underline{m})$ hence by using $\mathfrak{A}_1 \vdash$A13 we obtain $M_{\mathfrak{A}_1}(n, 0)$. Consequently by Lemma 17 we have $\mathfrak{A} \vdash F$. We have thus shown (14), and sonsequently also (12).

(9) and (12) give (8), hence $\mathbf{K}^\infty \in$ **PC**.

So we have finished the proof of Theorem 1.

We give a counterexample showing that the conclusion $\mathbf{K}^\infty \in$ **PC** cannot be improved in general to $\mathbf{K} \in$ **PC**. Let $\mathbf{L}_0$ be the empty set. The systems of $\mathbf{L}_0$ can be identified with sets. The formulae of $\mathbf{L}_0$ contain only the identity symbol besides variables and the logical operations. We can construct a sentence $F_n \in \mathfrak{F}_0(\mathbf{L}_0)$ for any $n \in \omega$ such that $\mathfrak{A} \in \mathbf{M}_{\mathbf{L}_0}(F_n)$ is equivalent to $\mathfrak{A} \cong n$ $(n = \{0, 1, \ldots, n-1\})$. Let H be a recursive but not primitive recursive set of natural numbers $\neq 0$. Consider $\Sigma = \{\neg F_n: n \in H\}$. Then $\Sigma \subset \mathfrak{F}_0(\mathbf{L}_0)$, Σ is recursive and furthermore the set of the natural numbers n such that $n \in \mathbf{K} = \mathbf{M}_{\mathbf{L}_0}(\Sigma)$ is identical with $\omega - H$ (the complement of H).

Let F be a formula of an arbitrary language $\mathbf{L}(\supset \mathbf{L}_0)$. It is trivial that the set of all n such that $n \in \mathbf{M}_\mathbf{L}(F) \mid \mathbf{L}_0$ is primitive recursive. Hence if $\mathbf{K} \in$ **PC** held then $\omega - H$ would be primitive recursive, that is not true.

§ 3. An operation on relational systems

We suppose that the language $\mathbf{L}$ contains only predicate symbols and no function symbol.

Let $\mathfrak{A}$ be a relational system of $\mathbf{L}$, $F(x)$ a formula of $\mathbf{L}$ with the single free variable x. We define $\mathfrak{A} \,||\, F(x)$ as the subsystem $\mathfrak{B}$ of $\mathfrak{A}$ such that $|\mathfrak{B}|$ is the set of those elements a of $|\mathfrak{A}|$ which satisfy $F(x)$ in $\mathfrak{A}$, i.e. $\mathfrak{A} \left| \frac{x}{a} \right. F(x)$. Since we do not allow relational systems with empty domain, we consider $\mathfrak{A} \,||\, F(x)$ as defined only if $\mathfrak{A} \vdash (\exists x)\, F(x)$.

This construction has a rather general character, as it will turn out in §§ 4 and 5 in which we apply it together with Theorem 2.

We define for a class $\mathbf{K}$ of relational systems

$$\mathbf{K} \,||\, F(x) = \{\mathfrak{A} \,||\, F(x) \colon \mathfrak{A} \in \mathbf{K}\}$$

Theorem 2(a). *If* $\mathbf{L}$ *is an arbitrary language,* $F(x) \in \mathfrak{F}(\mathbf{L})$ *with the single free variable* x, $\mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma)$ *for some* $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$ *and* $(\exists x)\, F(x)$ *is a consequence of* Σ *then*

$$\mathbf{K} \,||\, F(x) \in \mathbf{PC}_{\Delta} \tag{1}$$

(b) *If in addition* $\mathbf{L}$ *is finite, and* Σ *is a one element set then*

$$\mathbf{K} \,||\, F(x) \in \mathbf{PC}_{\Delta\mathrm{rec}} \tag{2}$$

Proof. We prove the theorem by constructing an axiom system Σ' in a language $\mathbf{L}' \supset \mathbf{L}$ such that

$$\mathbf{K} \,||\, F(x) = \mathbf{M}_{\mathbf{L}'}(\Sigma') \,|\, \mathbf{L} \tag{3}$$

We shall see that if the hypothesis of (b) holds, then $\mathbf{L}'$ and Σ' will satisfy the further requirements of (b).

We suppose that $F(x)$ is a prime formula, say $Q(x)$. To reduce the general case to this we have to adjoin a new unary predicate symbol Q to $\mathbf{L}$ and to add the axiom $(x)\big(Q(x) \leftrightarrow F(x)\big)$ to Σ. Let the resulting language and axiom system be $\mathbf{L}_1$ and Σ_1 resp. Obviously we have $\mathbf{K} \,||\, F(x) = (\mathbf{M}_{\mathbf{L}_1}(\Sigma_1) \,||\, Q(x)) \,|\, \mathbf{L}$ and so we obtain (1) or (2) if we apply the same to $\mathbf{L}_1$, Σ_1, $Q(x)$.

We may assume that each formula $H \in \Sigma$ has the following prenex normal form

$$(x_1) \ldots (x_{k_1})\, (\exists y_1) \ldots (x_{k_{n-1}+1}) \ldots (x_{k_n})\, (\exists y_n)\, (x_{k_n+1}) \ldots (x_{k_{n+1}})\, \Phi \tag{4}$$

where Φ is an open formula of $\mathbf{L}$. For each $H \in \Sigma$ we introduce distinct new function symbols $f_1^H, \ldots, f_n^H$ corresponding to the variables $y_1, \ldots, y_n$ bound by existential quantifiers in H, with $\nu(f_i^H) = k_i$ $(i = 1, \ldots, n)$. By adjoining every f_i^H to $\mathbf{L}$ for each $H \in \Sigma$ we obtain the enlarged language $\mathbf{L}_1$. We associate an open formula H^* of $\mathbf{L}_1$ with each $H \in \Sigma$ by putting

$$H^* = \left| \frac{y_i}{f_i(x_1, \ldots, x_{k_i})} \right. \Phi\,. \tag{4'}$$

Let T be a prime formula of $\mathbf{L}_1$. T is said to be a *free prime formula* (briefly fpr) if (i) each variable of T occurs in only one argument place of T and (ii) the argument places of T are occupied by $v_0, \ldots, v_{m-1}$ in that natural order in which these argument places follow each other from left to right in T. It is obvious how to give a rigorous inductive definition for the notion of the free prime formula. At any rate the following lemma is evident.

Lemma 18. *To each prime formula* T *of* $\mathbf{L}_1$ *there is a unique fpr* T_0 *such that we get* T *from* T_0 *by substituting a variable of* T *for each variable* v_i *of* T_0 *Moreover, this latter sustitution is uniquely determined.*

We associate a predicate symbol P^T with each fpr T with the following stipulations. The symbols P^T are different from each other for different fpr-s, and they are different from the predicate symbols of $\mathbf{L}$ with the following exception: if T is a prime formula of L, i.e. of the form $v_0 = v_1$ or $P(v_0, \ldots, v_{n-1})$ for $P \in \mathbf{L}$ then let P^T be identical with $=$ or P respectively. We put $\nu(P^T)$ to be the number of the variables in T.

Let $\mathbf{L}' = \{P^T : T \text{ is an fpr}\} - \{=\}$. Obviously we have $\mathbf{L} \subset \mathbf{L}'$.

Now we define $\overline{\Psi}$ for each open formula Ψ of $\mathbf{L}_1$ as follows. Let first Ψ be a prime formula of $\mathbf{L}_1$. By Lemma 18 Ψ arises from a unique fpr Ψ_0 by well determined substitutions for variables of Ψ_0, i.e. $\Psi = \left| \dfrac{v_0, v_1, \ldots, v_{m-1}}{x_0, x_1, \ldots, x_{m-1}} \right. \Psi_0$ where $x_0, \ldots, x_{m-1}$ are variables. Let $\overline{\Psi}$ be the formula $P^{\Psi_0}(x_0, x_1, \ldots, x_{m-1})$ If Ψ is an arbitrary open formula of $\mathbf{L}_1$, then $\overline{\Psi}$ is obtained from Ψ by replacing each prime formula part Φ in Ψ by $\overline{\Phi}$.

We remark that $\overline{\Psi}$ has the same variables as Ψ and if $\Psi \in \mathfrak{F}(\mathbf{L})$ then $\overline{\Psi} = \Psi$.

Let I be the set of the following open formulae

$$I \quad \begin{cases} v_0 = v_0 \\ v_0 = v_1 \rightarrow v_1 = v_0 \\ (v_0 = v_1 \wedge v_1 = v_2) \rightarrow v_0 = v_2 \\ (v_0 = v_n \wedge v_1 = v_{n+1} \wedge \ldots \wedge v_{n-1} = v_{2n-1}) \rightarrow (P(v_0, \ldots, v_{n-1}) \rightarrow P(v_n, \ldots, v_{2n-1}) \\ \qquad \text{for arbitrary } P \in L \end{cases}$$

Let $\Sigma^* = \{H^* : H \in \Sigma\} \cup I$.

Let Σ_1 be the set of all formulae $\left| \dfrac{x_1, \ldots, x_k}{t_1, \ldots, t_k} \right. F$ where $F \in \Sigma^*$, $x_1, \ldots, x_k$ are distinct variables, $t_1, \ldots, t_k$ are terms of $\mathbf{L}_1$. Let $\overline{\Sigma}_1 = \{\overline{\Psi} : \Psi \in \Sigma_1\}$ and $\Sigma_2 = Cl(\overline{\Sigma}_1)$.

Let t be a term of $\mathbf{L}_1$, x be a variable not occurring in t. We define the formula E_t by

$$E_t = Cl\left(\overline{Q(t)} \rightarrow (\exists x)\,(\overline{x = t})\right) \tag{5}$$

Finally we put

$$\Sigma' = \Sigma_2 \cup \{E_t : t \text{ is a term of } \mathbf{L}_1\} \cup \{(x)\, Q(x)\}$$

For $\mathbf{L}'$ and Σ' so defined we shall prove (3).

We remark that if the hypothesis of (b) holds then in virtue of the „effectiveness" of our construction we trivially have an enumeration $\mu = (P_i)_{i<\omega}$ of all predicate symbols of $\mathbf{L}'$ for which $\nu(P_i)$ is a primitive recursive function of i and $\{Nu_\mu(F) : F \in \Sigma'\}$ is a primitive recursive set of natural numbers. Thus (3) will imply (2).

I. Proof of $\mathbf{K} \,||\, Q(x) \subset \mathbf{M}_{\mathbf{L}'}(\Sigma') \,|\, \mathbf{L}$

Let $\mathfrak{A} \in \mathbf{K} \,||\, Q(x)$. We have to show

$$\mathfrak{A} \in \mathbf{M}_{\mathbf{L}'}(\Sigma') \,|\, \mathbf{L} \tag{6}$$

We have a system $\mathfrak{B} \in \mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma)$ for which $\mathfrak{A} \subset \mathfrak{B}$ and

$$|\,\mathfrak{A}\,| = A = \{a : a \in B = |\,\mathfrak{B}\,|, Q_{\mathfrak{B}}(a)\} \tag{7}$$

We define the system $\mathfrak{B}_1 \in \mathfrak{S}(\mathbf{L}_1)$. Let $|\mathfrak{B}_1| = B$, $P_{\mathfrak{B}_1} = P_{\mathfrak{B}}$ for any $P \in \mathbf{L}$. Let $H \in \Sigma$ of the form (4). Since $\mathfrak{B} \vdash H$, by Lemma 3 in § 1 we have a $\mathfrak{B}^H \in \mathfrak{S}(\mathbf{L}_1)$ for which $\mathfrak{B} = \mathfrak{B}^H \,|\, \mathbf{L}$ and $\mathfrak{B}^H \vdash Cl(H^*)$.

Let $(f_i^H)_{\mathfrak{B}_1} = (f_i^H)_{\mathfrak{B}^H}$ for $i = 1, \ldots, n$ and for each $H \in \Sigma$. So we have

$$\mathfrak{B}_1 \,|\, \mathbf{L} = \mathfrak{B} \text{ and } \mathfrak{B}_1 \vdash Cl(H^*) \text{ for every } H \in \Sigma \tag{8}$$

We define the system $\mathfrak{A}'$ of $\mathbf{L}'$ by putting $|\mathfrak{A}'| = A$ and

$$(P^T)_{\mathfrak{A}'}(a_0, \ldots, a_{m-1}) \sim \mathfrak{B}_1 \left| \frac{v_0, \ldots, v_{m-1}}{a_0, \ldots, a_{m-1}} \right. T \tag{9}$$

for any fpr T and $a_0, \ldots, a_{m-1} \in A$. If we replace T by $P(v_0, \ldots, v_{n-1})$ for $P \in \mathbf{L}$ in (8) we obtain

$$P_{\mathfrak{A}'}(a_0, \ldots, a_{n-1}) \sim P_{\mathfrak{B}_1}(a_0, \ldots, a_{n-1}) \sim P_{\mathfrak{A}}(a_0, \ldots, a_{n-1})$$

i.e.

$$\mathfrak{A}' \,|\, \mathbf{L} = \mathfrak{A} \tag{10}$$

Now we show

$$\mathfrak{A}' \in \mathbf{M}_{\mathbf{L}'}(\Sigma) \tag{11}$$

If Φ is an arbitrary open formula of $\mathbf{L}_1$, $a_1, \ldots, a_{k-1}$ are elements of A then

$$\mathfrak{A}' \left| \frac{x_0, \ldots, x_{k-1}}{a_0, \ldots, a_{k-1}} \right. \overline{\Phi} \sim \mathfrak{B}_1 \left| \frac{x_0, \ldots, x_{k-1}}{a_0, \ldots, a_{k-1}} \right. \Phi \,. \tag{12}$$

That is a direct consequence of (9) and the definition of $\overline{\Phi}$.

Let $\Psi \in \Sigma_1$. We show

$$\mathfrak{B}_1 \vdash Cl(\Psi) \tag{13}$$

Now we have $\Psi = \left| \frac{x_0, \ldots, x_{m-1}}{t_0, \ldots, t_{m-1}} \right. F$ where $F = H^*$ for some $H \in \Sigma$ or $F \in I$. In the first case (13) follows from (8), in the second one (13) follows from the trivial fact that $\mathfrak{B}_1 \vdash Cl(F)$.

Now let $E \in \Sigma_2$, i.e. $E = Cl(\overline{\Psi})$ for some $\Psi \in \Sigma_1$. It follows from (12) and (13) that $\mathfrak{A}' \vdash E$.

Secondly let $E = E_t$ of the form (5). We prove $\mathfrak{A}' \vdash E_t$. Let $x_1, \ldots, x_m$ be all the distinct variables of t and $a_1, \ldots, a_m \in A$. We have to show $\mathfrak{A}' \left| \frac{x_i}{a_i} \right. (\overline{Q(t)} \to (\exists x)\, \overline{x = t})$. Suppose $\mathfrak{A}' \left| \frac{x_i}{a_i} \right. \overline{Q(t)}$. This implies by (12) $\mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. Q(t)$ i.e. $Q_{\mathfrak{B}}(\tau)$ where $\tau = \mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. t$, consequently by (7) $\tau \in A$. But trivially $\mathfrak{B}_1 \left| \frac{x, x_1, \ldots, x_n}{\tau, a_1, \ldots, a_n} \right. x = t$ and so again by (12) we have $\mathfrak{A}' \left| \frac{x, x_1, \ldots, x_n}{\tau, a_1, \ldots, a_n} \right. \overline{x = t}$ i.e. $\mathfrak{A}' \left| \frac{x_i}{a_i} \right. (\exists x)\, \overline{x = t}$ and that had to be shown.

Finally we have trivially $\mathfrak{A}' \vdash (x)\, Q(x)$.

Thus we have shown (11), (10) and (11) gives (6) qu. e.d.

II. Proof of $\mathbf{K} \,||\, Q(x) \subset \mathbf{M}_{\mathbf{L}'}(\Sigma') \,|\, \mathbf{L}$

Let

$$\mathfrak{A}' \in \mathbf{M}_{\mathbf{L}'}(\Sigma') \tag{14}$$

$$\mathfrak{A} = \mathfrak{A}' \,|\, \mathbf{L} \tag{15}$$

We have to construct a $\mathfrak{B}$ for which

$$\mathfrak{B} \in \mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma) \tag{16}$$

$$\mathfrak{A} \subset \mathfrak{B} \tag{17}$$

and if $|\,\mathfrak{A}\,| = A, |\,\mathfrak{B}\,| = B$

$$A = \{a : a \in B, Q_{\mathfrak{B}}(a)\} \tag{18}$$

We associate a new constant c_a with each $a \in A$. We suppose that $c_a \notin \mathbf{L}_1$ and $c_{a_1} \neq c_{a_2}$ if $a_1 \neq a_2$. Let $A_1 = \{c_a : a \in A\}$. We define the auxiliary language $\mathbf{L}_1^A = \mathbf{L}_1 \cup A_1$. Let B_1 be the set of all closed terms of $\mathbf{L}_1^A$, i.e. which contain no variable.

We define the pseudosystem $\mathfrak{B}_1$ of $\mathbf{L}_1$ as follows. Let $|\,\mathfrak{B}_1\,| = B_1$. Let $b_1, \ldots, b_n \in B_1$, let $c_{a_0}, \ldots, c_{a_{m-1}}$ be all the different elements of A_1 occuring in some of the b_i-s, $t_i = \left|\frac{c_{a_0}, \ldots, c_{a_{m-1}}}{v_0, \ldots, v_{m-1}}\right. b_i$ for $i = 1, \ldots, n$. t_i is a term of $\mathbf{L}_1$.

Now we define

$$b_1 =_{\mathfrak{B}_1} b_2 \sim \mathfrak{A}' \left|\frac{v_0, \ldots, v_{m-1}}{a_0, \ldots, a_{m-1}}\right. \overline{t_1 = t_2} \tag{19}$$

$$P_{\mathfrak{B}_1}(b_1, \ldots, b_n) \sim \mathfrak{A}' \left|\frac{v_0, \ldots, v_{m-1}}{a_0, \ldots, a_{m-1}}\right. \overline{P(t_1, \ldots, t_n)} \tag{20}$$

for $P \in \mathbf{L}$ and

$$f_{\mathfrak{B}_1}(b_1, \ldots, b_n) = f(b_1, \ldots, b_n) \tag{21}$$

for $f \in \mathbf{L}_1$.

Using similar notations (19), (20), (21) imply that

$$\mathfrak{B}_1 \left|\frac{x_1, \ldots, x_n}{b_1, \ldots, b_n}\right. \Phi \sim \mathfrak{A}' \left|\frac{v_0, \ldots, v_{m-1}}{a_0, \ldots, a_{m-1}}\right. \overline{\left(\left|\frac{x_1, \ldots, x_n}{t_1, \ldots, t_n}\right. \Phi\right)} \tag{22}$$

for any open formula Φ of $\mathbf{L}_1$.

Now we put $\mathfrak{B}_2 = \mathfrak{B}_1 \,|\, \mathbf{L}$, consequently $\mathfrak{B}_2$ is a pseudosystem of $\mathbf{L}$. We prove that $\mathfrak{B}_2$ is a pseudomodel of Σ.

Let $H \in \Sigma$. By Lemma 3 it is sufficient to show

$$\mathfrak{B}_1 \vdash Cl(H^*) \tag{23}$$

for proving $\mathfrak{B}_2 \vdash H$. We have to show that if $b_1, \ldots, b_{k_{n+1}}$ are arbitrary elements of B_1 then

$$\mathfrak{B}_1 \left|\frac{x_1, \ldots, x_{k_{n+1}}}{b_1, \ldots, b_{k_{n+1}}}\right. H^* \tag{24}$$

(we suppose H to be of the form (4)).

Let $c_{a_0}, \ldots, c_{a_{m-1}}$ denote all the different elements of A_1 occurring in some of $b_1, \ldots, b_{k_{n+1}}$, let us replace each c_{a_j} by v_j in each b_i and let $t_1, \ldots, \ldots, t_{k_{n+1}}$ be the resulting terms of L_1. Let Ψ be $\left|\frac{x_1, \ldots, x_{k_{n+1}}}{t_1, \ldots, t_{k_{n+1}}}\right. H^*$. For proving (24) it is sufficient to show by (22) that $\mathfrak{A}' \left|\frac{v_0, \ldots, v_{m-1}}{a_0, \ldots, v_{m-1}}\right. \overline{\Psi}$. But that is true since $\Psi \in \Sigma_1$ and consequently $Cl(\overline{\Psi}) \in \Sigma'$ and so we have (14). So we have proved (24) and (23).

Moreover we assert that $=_{\mathfrak{B}_2}$ is a congruence relation on $\mathfrak{B}_2$. We have to show that $\mathfrak{B}_1 \vdash Cl(\Phi)$ for $\Phi \in I$ where I was defined above (equality axioms). This follows similarly as before from the fact that if $\Phi \in I, t_1, \ldots, t_n$ are terms of $\mathbf{L}_1$ then $Cl\left(\left|\frac{x_1, \ldots, x_n}{t_1, \ldots t_n}\right. \Phi\right) \in \Sigma'$.

Let $\mathfrak{B}$ be the factor system $\mathfrak{B}_2/=_{\mathfrak{B}_2}$. By Lemma 2 and from that what we have just proved it follows (16).

Let A_2 denote the set of the equivalence classes $c_a/=_{\mathfrak{B}_2} = [a]$ $(a \in A)$. We assert that the subsystem $\mathfrak{B}[A_2]$ is isomorphic to $\mathfrak{A}$ by the natural mapping $[a] \to a$. Indeed we have $[a_0] = [a_1] \sim c_{a_0} =_{\mathfrak{B}_1} c_{a_1} \sim \mathfrak{A}' \left|\frac{v_0, v_1}{a_0, a_1}\right. \overline{v_0 = v_1} \sim$ $\sim \mathfrak{A}' \left|\frac{v_0, v_1}{a_0, a_1}\right. v_0 = v_1 \sim a_0 = a_1$, hence the given mapping is one-to-one and moreover

$$P_{\mathfrak{B}[A_2]}([a_0], \ldots, [a_{n-1}]) \sim P_{\mathfrak{B}}([a_0], \ldots, [a_{n-1}]) \sim P_{\mathfrak{B}_1}(c_{a_0}, \ldots, c_{a_{n-1}}) \sim$$

$$\sim \mathfrak{A}' \left|\frac{v_0, \ldots, v_{n-1}}{a_0, \ldots, a_{n-1}}\right. \overline{P(v_0, \ldots, v_{n-1})} \sim \mathfrak{A}' \left|\frac{v_0, \ldots, v_{n-1}}{a_0, \ldots, a_{n-1}}\right. P(v_0, \ldots, v_{n-1}) \sim$$

$$\sim P_{\mathfrak{A}'}(a_0, \ldots, a_{n-1}) \sim P_{\mathfrak{A}}(a_0, \ldots, a_{n-1}) \text{ as desired.}$$

We can identify $\mathfrak{B}[A']$ with $\mathfrak{A}$, hence we can consider $\mathfrak{A}$ as a subsystem of $\mathfrak{B}$.

To complete the proof we have only to show that $A = X$ if X denotes $\{b: Q_{\mathfrak{B}}(b)\}$. Being $\mathfrak{A}$ a model of $(x)\, Q(x)$ and at the same time a subsystem of $\mathfrak{B}$ we have $A \subset X$. To prove $A \supset X$ let $b \in X$. Then $b = t(c_{a_0}, \ldots, c_{a_{n-1}})/ =_{\mathfrak{B}_1}$ where $t(v_0, \ldots, v_{n-1})$ is a term of $\mathbf{L}_1$. It follows from (12) and $Q_{\mathfrak{B}_1}(x)$ that

$$\mathfrak{A}' \left|\frac{v_0, \ldots, v_{n-1}}{a_0, \ldots, a_{n-1}}\right. \overline{Q(t(v_0, \ldots, v_{n-1}))} \tag{25}$$

But $E_t \in \Sigma'$ (see (5)), consequently $\mathfrak{A}' \vdash E_t$ hence (25) implies

$$\mathfrak{A}' \left|\frac{v_0, \ldots, v_{n-1}}{a_0, \ldots, a_{n-1}}\right. (\exists x)\ \overline{x = t(v_0, \ldots, v_{n-1})}\,.$$

Let $a \in A$ for which

$$\mathfrak{A}' \left|\frac{v_0, \ldots, v_{n-1}, x}{a_0, \ldots, a_{n-1}, a}\right. \overline{x = t(v_0, \ldots, v_{n-1})}\,.$$

By definition this is equivalent to $c_a =_{\mathfrak{B}_1} t(c_{a_0}, \ldots, c_{a_{n-1}})$ i.e. $b = [c_a] = a \in A$. Thus we have proved (16), (17), (18) qu. e.d.

Corollary 3. *If* $\mathbf{K} \in \mathbf{PC}$ *and* $(\exists x)\, F(x)$ *holds in all systems of* $\mathbf{K}$ *then we have*

$$(\mathbf{K} \,\|\, F(x))^{\infty} \in \mathbf{PC}$$

Proof. By Theorem 1 and Theorem 2 (b).

Now we want to give an example of a class $\mathbf{K} \in \mathbf{EC}$ and a formula $F(x)$ such that $\mathbf{K} \,\|\, F(x) \notin \mathbf{EC}_\Delta$. Our example is a slight modification of the one given by Lyndon [3] for showing that $\mathbf{K} \in \mathbf{EC}$ does not imply $\mathbf{H}(\mathbf{K}) \in \mathbf{EC}_\Delta$. Let the language $\mathbf{L}$ consist of the predicate symbols P, Q, R with $v(P) = v(Q) = 1, v(R) = 2$. Let H be the conjunction of the following formulae

$$(\boldsymbol{x})\,(Q(\boldsymbol{x}) \to \neg\, P(\boldsymbol{x}))$$

(26) $$(\exists \boldsymbol{x})\,(\boldsymbol{y})\,(P(\boldsymbol{x}) \wedge (P(\boldsymbol{y}) \to \boldsymbol{x} = \boldsymbol{y}))$$

(27) $$(\boldsymbol{x})\,(P(\boldsymbol{x}) \to R(\boldsymbol{x}, \boldsymbol{x}))$$

(28) $$(\boldsymbol{x})\,(P(\boldsymbol{x}) \to (\boldsymbol{y})\,(R(\boldsymbol{x}, \boldsymbol{y}) \to (\exists \boldsymbol{z})\,(R(\boldsymbol{x}, \boldsymbol{z}) \wedge R(\boldsymbol{y}, \boldsymbol{z}) \wedge Q(\boldsymbol{z})))) .$$

Let $\mathbf{K} = \mathbf{M}_\mathbf{L}(H)$.

We assert that $\mathbf{K} \,\|\, Q(x)$ consists of the relational systems $\mathfrak{A}$ for which

$$(*) \quad \begin{cases} \mathfrak{A} \vdash (x)\, Q(x) \\ \mathfrak{A} \vdash (x) \neg\, P(x) \\ \text{and there exists an infinite sequence } z_1, z_2, \ldots, z_n, \ldots \text{ such that} \\ R_\mathfrak{B}\,(z_{n-1}, z_n) \text{ for each } n = 2, 3, \ldots \end{cases}$$

Let first $\mathfrak{B} \in \mathbf{K}$, $\mathfrak{A} = \mathfrak{B} \,\|\, Q(x)$. Then there exists exactly one element x_0 of $|\mathfrak{B}| = B$ for which $P_\mathfrak{B}(x_0)$ is true. Let us replace x_0 for $\boldsymbol{x}$ and $\boldsymbol{y}$ in (28). Then (27) and (28) say that there exists a $z_1 \in A = |\mathfrak{A}|$ such that

(29) $$R_\mathfrak{B}(x_0, z_1) \text{ and } Q_\mathfrak{B}(z_1)$$

hence $z_1 \in A$.

Taking x_0 for $\boldsymbol{x}$, z_1 for $\boldsymbol{y}$ in (28) we see by (29) that there exist a $z_2 \in A$ with $R_\mathfrak{B}(x_0, z_2)$, $R_\mathfrak{B}(z_1, z_2)$.

Continuing in this manner we get the sequence $z_1, z_2, \ldots$ such that $R_\mathfrak{A}(z_n, z_{n+1})$ and $R_\mathfrak{B}(x_0, z_n)$ hold for each $n \geq 1$.

Secondly let $\mathfrak{A}$ satisfy (*). We choose a new element $x_0 \notin A = |\mathfrak{A}|$ and define $|\mathfrak{B}| = A \cup \{x_0\}$. We define $P_\mathfrak{B}(x)$ to be true in $\mathfrak{B}$ if and only if $x = x_0$; $R_\mathfrak{B}(x, y)$ if and only if $x = y = x_0$, or $x = x_0$ and $y = z_n$, or $x = z_{n-1}$ and $y = z_n$ for some n; $Q_\mathfrak{B}(x)$ if and only if $x \in A$. It can easily be checked that $\mathfrak{B} \vdash H$ and $\mathfrak{A} = \mathfrak{B} \,\|\, Q(x)$.

Now we show that $\mathbf{K} \,\|\, Q(x) \notin \mathbf{EC}_\Delta$. It is sufficient to exhibit a system $\mathfrak{A}$ such that $\mathfrak{A}$ does not satisfy (*) but an ultrapower $\mathfrak{A}^I_D$ of $\mathfrak{A}$ does satisfy (*).

Let $|\mathfrak{A}| = A$ be the set of the distinct elements z_{ik} for natural numbers i, k with $i, k \geq 1$, $i \leq k$ and let P, Q, R be defined in $\mathfrak{A}$ as follows.

$Q_\mathfrak{A}(x)$ is identically true

$P_\mathfrak{A}(x)$ is identically false

$R_\mathfrak{A}(z_{ik}, z_{jl})$ is true if and only if $k = l$ and $j = i + 1$.

It can be seen that $\mathfrak{A}$ does not satisfy (*).

Let I be the set of the positive integers, D be a non principal ultrafilter on I. We can write the elements of A_D^I as $(x_1, x_2, \ldots)/D$. Let us put $z_i = (x_1, \ldots, x_{i-1}, z_{ii}, z_{ii+1}, \ldots)/D$. Then $R_{\mathfrak{A}}(z_{i-1}, z_i)$ is always true, i.e. $\mathfrak{A}_I^D$ satisfies (*) que.d.

§ 4. Homomorphisms

Let $\mathbf{F}$ be an arbitrary set of formulae of the language $\mathbf{L}$, let $\mathfrak{A}$, $\mathfrak{B}$ be systems of $\mathbf{L}$, φ be a mapping of $|\mathfrak{B}| = B$ onto $|\mathfrak{A}| = A$.

Definition. *φ is said to be an F-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ if for every $F \in \mathbf{F}$ and arbitrary elements $b_1, \ldots, b_n$ of B*

$$\mathfrak{B}\left|\frac{x_1, \ldots, x_n}{b_1, \ldots, b_n} F \quad \textit{implies} \quad \mathfrak{A}\right|\frac{x_1, \ldots, x_n}{\varphi(b_1), \ldots, \varphi(b_n)} F .$$

This notion is due to KEISLER [2]. KEISLER requires $\mathbf{F}$ to have some specia properties (to be a Generalized Atomic set of formulae) but we do not need such restrictions. Moreover we can and shall suppose without any loss of generality that $v_0 \equiv v_1$ is an element of $\mathbf{F}$, because for any mapping φ of B $b_0 = b_1$ implies $\varphi(b_0) = \varphi(b_1)$.

In the described case $\mathfrak{A}$ is said to be an $\mathbf{F}$-homomorphic image of $\mathfrak{B}$ ($\mathfrak{A} \in \mathbf{H}_{\mathbf{F}}(\mathfrak{B})$). We put $\mathbf{H}_{\mathbf{F}}(\mathbf{K}) = \bigcup_{\mathfrak{B} \in \mathbf{K}} \mathbf{H}_{\mathbf{F}}(\mathfrak{B})$.

The notion of (simple) homomorphism is a special case of that of $\mathbf{F}$-homomorphism. In order to see this we have to take $\mathbf{F}$ to be the set of all formulae of the form $v_0 \equiv v_1$, $P(v_0, \ldots, v_{n-1})$ and $f(v_0, \ldots, v_{n-1}) = v_n$ for $P, f \in \mathbf{L}$.

Corollary 4. *If $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$, $\mathbf{F} \subset \mathfrak{F}(\mathbf{L})$ and $\mathbf{K} \in \mathbf{EC}_\Delta$ then $\mathbf{H}_{\mathbf{F}}(\mathbf{K}) \in \mathbf{PC}_\Delta$.*

Corollary 5. *If $\mathbf{L}$ is a finite language. $\mathbf{F}$ is an arbitrary recursively enumerable set of formulae of $\mathbf{L}$, $\mathbf{K} \subset \mathfrak{S}(\mathbf{L})$ and $\mathbf{K} \in \mathbf{EC}$ then $(\mathbf{H}_{\mathbf{F}}(\mathbf{K}))^\infty \in \mathbf{PC}$.*

Corollary 4′. *If $\mathbf{K} \in \mathbf{PC}_\Delta$ or $\mathbf{K} \in \mathbf{EC}_\Delta$ then $\mathbf{H}(\mathbf{K}) \in \mathbf{PC}_\Delta$.*

Corollary 5′. *If $\mathbf{K} \in \mathbf{PC}$ or $\mathbf{K} \in \mathbf{EC}$ then $(\mathbf{H}(\mathbf{K}))^\infty \in \mathbf{PC}$.*

Proofs. Corollary 4′ can be derived from Corollary 4 as follows. We consider the class $\mathbf{K}' \subset \mathfrak{S}(\mathbf{L}')$ for some $\mathbf{L}' \supset \mathbf{L}$ such that $\mathbf{K}' \in \mathbf{EC}_\Delta$ and $\mathbf{K} = \mathbf{K}' \mid \mathbf{L}$. Let $\mathbf{F}$ be the set of the formulae $v_0 \equiv v_1$, $P(v_0, \ldots, v_{n-1})$, $f(v_0, \ldots, v_{n-1}) \equiv v_n$ for any $P, f \in L$. Then trivially $\mathbf{H}(\mathbf{K}) = \mathbf{H}_{\mathbf{F}}(\mathbf{K}') \mid \mathbf{L} \in \mathbf{PC}_\Delta$.
Corollary 5′ can be proved similarly by the help of Corollary 5.

Now we are going to prove Corollaries 4 and 5 at the same time.

By assumption we have $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$ such that $\mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma)$. In case of Corollary 5 Σ consists of a single formula. Let us associate a new predicate symbol P^+ with each $P \in \mathbf{L}$ and a new function symbol f^+ with each $f \in \mathbf{L}$. We take $\nu(P^+) = \nu(P)$, $\nu(f^+) = \nu(f)$. Besides we take the new unary function symbol h and unary predicate symbol A. Let $\mathbf{L}'$ be the language which we get by adjoining every P^+, f^+ and A and h to $\mathbf{L}$.

Let us define F^+ for an arbitrary formula F of $\mathbf{L}$ as the formula resulting from F by substituting P^+ and f^+ for any $P, f \in \mathbf{L}$ in F. F^A denotes the relativized of F to A (see Lemma 4 in § 1.)

Now we define the set Σ' as the collection of the following sentences:

$$(1) \qquad (x)\, A\big(h(x)\big)$$

$$(2) \qquad (x)\, \big(A(x) \to (\exists y)\, \big(h(y) = x\big)\big)$$

$$(3) \qquad (x_1) \ldots (x_n)\, [F^+(x_1, \ldots, x_n) \to F^A\big(h(x_1), \ldots, h(x_n)\big)]$$

for each $F(x_1, \ldots, x_n) \in F$

$$G^+$$

for each $G \in \Sigma$

$$(4) \qquad (x_1) \ldots (x_n)\, \big((A(x_1) \wedge \ldots \wedge A(x_n)) \to A\big(f(x_1, \ldots, x_n)\big)\big)$$

for each $f \in L$

Now we shall use the notations introduced before Lemma 5. We assert that

$$(5) \qquad \overline{\overline{\mathbf{H}_{\mathbf{F}}(\mathbf{K})}} = \big(\overline{\overline{\mathbf{M}_{\mathbf{L}'}(\Sigma')}} \,\|\, A(x)\big) \,|\, \bar{\bar{\mathbf{L}}}$$

I. First let $\mathfrak{A} \in \mathbf{H}_{\mathbf{F}}(\mathbf{K})$ i.e. $\mathfrak{B} \in \mathbf{K}$ and φ be an $\mathbf{F}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. We define the system $\mathfrak{B}'$ of $\mathbf{L}'$ by as follows. Let $|\,\mathfrak{B}'\,| = |\,\mathfrak{B}\,| = B$, $(P^+)_{\mathfrak{B}'} = P_{\mathfrak{B}}$, $(f^+)_{\mathfrak{B}'} = f_{\mathfrak{B}}$ for any $P, f \in \mathbf{L}$ and $h_{\mathfrak{B}'} = \varphi$. Let further A' be a subset of B of the same power as $|\,\mathfrak{A}\,|$ and we define the relations and functions $P_{\mathfrak{B}'}, f_{\mathfrak{B}'}$ such that if $a_1, \ldots, a_n \in A'$ then $f_{\mathfrak{B}'}(a_1, \ldots, a_n) \in A'$ and the subsystem $(\mathfrak{B}' \,|\, \mathbf{L})\,[A']$ is isomorphic to $\mathfrak{A}$. Finally we require $A_{\mathfrak{B}'}(a) \sim a \in A'$ for any $a \in B$. It can be attained by an "exchange" procedure that $(\mathfrak{B}' | \mathbf{L})\,[A']$ is identical with $\mathfrak{A}$. Now it is trivial to verify that $\mathfrak{B}' \in \mathbf{M}_{\mathbf{L}'}(\Sigma')$ and $\bar{\bar{\mathfrak{A}}} = (\bar{\bar{\mathfrak{B}}}' \,\|\, A(x))\, \bar{\bar{\mathbf{L}}}$.

II. Secondly let $\mathfrak{B}' \in \mathbf{M}_{\mathbf{L}'}(\Sigma')$, $\mathfrak{A}_1 = (\bar{\bar{\mathfrak{B}}}' \,\|\, A(x))\, \bar{\bar{\mathbf{L}}}$. Since the formulae of the form of (4) are in Σ', $\mathfrak{A}_1 = \bar{\bar{\mathfrak{A}}} \in \mathfrak{S}(\bar{\bar{\mathbf{L}}})$ for a unique $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$. We define $\mathfrak{B} \in \mathfrak{S}(\mathbf{L})$ such that $|\,\mathfrak{B}\,| = |\,\mathfrak{B}'\,|$, $P_{\mathfrak{B}} = (P^+)_{\mathfrak{B}'}$, $f_{\mathfrak{B}} = (f^+)_{\mathfrak{B}'}$ for $P, f \in \mathbf{L}$. Then $h_{\mathfrak{B}'}$ will be an $\mathbf{F}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ and $\mathfrak{B} \in \mathbf{K}$. Thus we have shown (5).

By Lemma 5 (§ 1) (5) implies $\overline{\overline{\mathbf{H}_{\mathbf{F}}(\mathbf{K})}} = (\mathbf{M}_{\bar{\bar{\mathbf{L}}}'}(\bar{\bar{\Sigma}}') \,\|\, A(x)) \,|\, \bar{\bar{\mathbf{L}}}$.

Let us now consider the case of Corollary 4. Using Theorem 2 (a) we get $\overline{\overline{\mathbf{H}_{\mathbf{F}}(\mathbf{K})}} \in \mathbf{PC}_\Delta$ and by Lemma 6 of § 1 $\mathbf{H}_{\mathbf{F}}(\mathbf{K}) \in \mathbf{PC}_\Delta$ qu. e.d.

In case of Corollary 5 $\mathbf{L}'$ is a finite language and $\bar{\bar{\Sigma}}'$ is obviously a recursively enumerable set of formulae (it is irrelevant which enumeration of the symbols of $\bar{\bar{\Sigma}}'$ is chosen). Consequently by Theorem 1 $(\mathbf{M}_{\bar{\bar{\mathbf{L}}}'}(\bar{\bar{\Sigma}}'))^\infty \in \mathbf{PC}$ i.e. $(\mathbf{M}_{\bar{\bar{\mathbf{L}}}'}(\bar{\bar{\Sigma}}'))^\infty = \mathbf{M}_{\mathbf{L}''}(F) \,|\, \bar{\bar{\mathbf{L}}}'$ where F is a formula of a language $\mathbf{L}'' \supset \bar{\bar{\mathbf{L}}}'$. Further we have $(\mathbf{M}_{\mathbf{L}''}(F) \,\|\, A(x))^\infty \in \mathbf{PC}$ by Corollary 3., hence

$$\big(\overline{\overline{\mathbf{H}_{\mathbf{F}}(\mathbf{K})}}\big)^\infty = \overline{\overline{(\mathbf{H}_{\mathbf{F}}(\mathbf{K}))^\infty}} = \big(\mathbf{M}_{\bar{\bar{\mathbf{L}}}'}(\bar{\bar{\Sigma}}') \,\|\, A(x)) \,|\, \bar{\bar{\mathbf{L}}}\big)^\infty =$$

$$= \big((\mathbf{M}_{\mathbf{L}''}(F) \,\|\, A(x)) \,|\, \bar{\bar{\mathbf{L}}}\big)^\infty = \big(\mathbf{M}_{\mathbf{L}''}(F) \,\|\, A(x)\big)^\infty \,|\, \bar{\bar{\mathbf{L}}} \in \mathbf{PC}$$

and by Lemma 6 $(\mathbf{H}_{\mathbf{F}}(\mathbf{K}))^\infty \in \mathbf{PC}$ qu. e.d.

The contents of the next Theorems 6 and 7 are roughly speaking the following. For any class $\mathbf{K}$, $\mathbf{H}_{\mathbf{F}}(\mathbf{K})$ is closed under $\mathbf{F}$-homomorphisms. If moreover $\mathbf{K} \in \mathbf{EC}_\Delta$ then we know from Corollary 4 that $\mathbf{H}_{\mathbf{F}}(\mathbf{K})$ can be "axiomatized" in an enlarged language. These two facts make the question natural whether there exists an axiomatization of $\mathbf{H_F}(\mathbf{K})$ in an enlarged language whose every formula is in some normalform such that every set of sentences in this normalform is "preserved" under $\mathbf{F}$-homomorphism in a natural sense explained below (Theorem 6).

We answer this question positively by introducing a type of sentence (see $\mathbf{H_F}$-sentence below), which type has the desired property (Theorem 6) and by proving that the axiomatization in question can be given using only $\mathbf{H_F}$-sentences (Theorem 7). Thus Theorem 7 is analogous in some sense to LYNDON's theorem or to it's generalization given by KEISLER [2].

Let us consider a formula Φ of the form $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} F_{ij}$ where $F_{ij} \in \mathbf{F}$. We consider a set $\mathbf{L}_1$ of function symbols not occurring in $\mathbf{L}$. Let us consider the set T° of the terms having the form x or $f(x_1, \ldots, x_n)$ where $f \in \mathbf{L}_1$ and x, $x_1, \ldots, x_n$ are variables. We replace the variables of Φ by terms of T°[4] so we obtain a formula Ψ in the language $\mathbf{L}' = \mathbf{L} \cup \mathbf{L}_1$. We call a formula $Cl(\Psi)$ for a Ψ so obtained an *$\mathbf{H_F}$-sentence over* $\mathbf{L}$.

Theorem 6. *Every set Θ of $\mathbf{H_F}$-sentences over $\mathbf{L}$ is ,,preserved" under $\mathbf{F}$-homomorphism, i.e. if $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(\mathbf{L})$, $\mathfrak{A} \in \mathbf{H_F}(\mathfrak{B})$ and $\mathfrak{B} \in \mathbf{M}_{\mathbf{L}'}(\Theta) \mid \mathbf{L}$ then $\mathfrak{A} \in \mathbf{M}_{\mathbf{L}'}(\Theta) \mid \mathbf{L}$.*

Proof. Let $\mathfrak{B} = \mathfrak{B}' \mid \mathbf{L}$, $\mathfrak{B}' \in \mathbf{M}_{\mathbf{L}'}(\Theta)$, φ be an $\mathbf{F}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. We have to construct $\mathfrak{A}'$ such that

$$\mathfrak{A}' \in \mathbf{M}_{\mathbf{L}'}(\Theta) \tag{7}$$

and

$$\mathfrak{A} = \mathfrak{A}' \mid \mathbf{L}\,. \tag{8}$$

Let $|\mathfrak{A}| = A$, $|\mathfrak{B}| = B$. We choose an element $\bar{a} \in B$ to each element $a \in A$ such that $\varphi(\bar{a}) = a$ (by the axiom of choice) and define $f_{\mathfrak{A}'}(a_1, \ldots, a_n) = \varphi(f_{\mathfrak{B}'}(\bar{a}_1, \ldots, \bar{a}_n))$. Thus $\mathfrak{A}'$ is defined, considering also (8). From this definition it follows at once that

$$\varphi\left(\mathfrak{B}' \left| \frac{x_1, \ldots, x_m}{\bar{a}_1, \ldots, \bar{a}_m} t \right.\right) = \mathfrak{A}' \left| \frac{x_1, \ldots, x_m}{a_1, \ldots, a_m} t \right. \tag{9}$$

for an arbitrary term t of T°.

Let $G \in \Theta$. For the formula G we keep the notations used in the definition of $\mathbf{H_F}$-formula.

To show (7) we must prove, that

$$\mathfrak{A}' \left| \frac{x_1, \ldots, x_m}{a_1, \ldots, a_m} \Psi \right. \tag{10}$$

for any elements $a_1, \ldots, a_m$ of A. By hypothesis we have

$$\mathfrak{B}' \left| \frac{x_1, \ldots, x_m}{\bar{a}_1, \ldots, \bar{a}_m} \Psi \right. \tag{11}$$

[4] Before the replacing we must possibly change the bound variables of Φ to avoid collisions. We shall consider that to have been peformed in all cases if necessary without any mentioning.

Ψ has the form $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} F_{ij}^*$ where F_{ij}^* arises from F_{ij} by substituting some terms of T° for the free variables of F_{ij}.

From (11) we infer that for some i_0, $1 \leq i_0 \leq n$

$$\mathfrak{B}' \left| \frac{\overline{x_1}, \ldots, \overline{x_m}}{a_1, \ldots, a_m} \right. F_{i_0 j}^* \tag{12}$$

for every j ($j = 1, \ldots, m_{i_0}$).

Let us choose an index j. By hypothesis

$$F_{i_0 j}^* = \left| \frac{y_1, \ldots, y_k}{t_1, \ldots, t_k} \right. F_{i_0 j}$$

where $t_1, \ldots, t_k \in T^\circ$.

Let

$$\tau_l = \mathfrak{B} \left| \frac{\overline{x_1}, \ldots, \overline{x_m}}{a_1, \ldots, a_m} \right. t_l \tag{13}$$

(12) can be written as

$$\mathfrak{B} \left| \frac{y_1, \ldots, y_k}{\tau_1, \ldots, \tau_k} \right. F_{i_0 j}.$$

Since φ is an $\mathbf{F}$-homomorphism and $F_{i_0 j} \in \mathbf{F}$, we have

$$\mathfrak{A} \left| \frac{y_1, \ldots, y_k}{\varphi(\tau_1), \ldots, \varphi(\tau_k)} \right. F_{i_0 j}.$$

This and (9) and (13) imply

$$\mathfrak{A}' \left| \frac{x_1, \ldots, x_m}{a_1, \ldots, a_m} \right. F_{i_0 j}^*.$$

Applying this for $j = 1, \ldots, m_{i_0}$ we obtain

$$\mathfrak{A}' \left| \frac{x_1, \ldots, x_m}{a_1, \ldots, a_m} \right. \bigwedge_{j=1}^{m_{i_0}} F_{i_0 j}^*$$

which implies (10) qu. e.d.

Theorem 7. *If* $\mathbf{K} = \mathbf{M}_{\mathbf{L}}(\Sigma)$ *for some* $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$, *and* $\mathbf{F} \subset \mathfrak{F}(\mathbf{L})$ *then there is an axiom system* Σ', *in a language* $\mathbf{L}'$, *consisting of* $\mathbf{H}_{\mathbf{F}}$*-sentences over* $\mathbf{L}$ *such that*

$$\mathbf{H}_{\mathbf{F}}(\mathbf{K}) = \mathbf{M}_{\mathbf{L}'}(\Sigma') \,|\, \mathbf{L} \tag{14}$$

Proof. We assume of each formula H of Σ to be in prenex normal form as in (4) in § 3 and we introduce the Skolem functions (function symbols) $f_1^H, \ldots, f_n^H$ and the formula H^* as in § 3. Further we bring each formula $\neg F$ for $F \in \mathbf{F}$, into prenex normal form, i.e.

$$\neg F \sim (x_1) \ldots (x_{k_1}) (\exists y_1) \ldots (\exists y_n) \ldots (x_{k_{n+1}}) \Gamma. \tag{15}$$

Let $z_1, \ldots, z_l$ be all the different free variables of F. We introduce the function symbols $g_1^F, \ldots, g_n^F$ each g_i^F having $k_i + l$ variables and we define F^{**} as the result of substituting $g_i^F(x_1, \ldots, x_{k_{n+1}}; z_1, \ldots, z_l)$ for y_i for each i in Γ.

Adjoining every f_i^H, g_i^F ($H \in \Sigma$, $F \in \mathbf{F}$) to $\mathbf{L}$ we obtain the language $\mathbf{L}_1$. A term t of $\mathbf{L}_1$ is a *free term* if (i) each variable occurs in t at only one argument place and (ii) the variables $v_0, \ldots, v_{m-1}$ occupy the argument places of t in their natural order from left to right. We state the following trivial lemma.

Lemma 19. *To each term t of $\mathbf{L}_1$ there is a unique free term t_0 such that t comes from t_0 by substituting certain variables for the variables of t_0. The latter substitution is also uniquely determined.*

Now we associate a new function symbol h^t with each free term t of $\mathbf{L}_1$, such that $\nu(h^t)$ is the number of the variables in t, h^t is different from each function symbol of $\mathbf{L}_1$ and for different free terms t_1, t_2 h^{t_1}, h^{t_2} are different. We define $\bar{t}$ for an arbitrary term t of $\mathbf{L}_1$. Let t_0 be as in Lemma 8, and let $t = \left| \frac{v_0, \ldots, v_{m-1}}{x_0, \ldots, x_{m-1}} \right. t_0$. Then let $\bar{t} = h^{t_0}(x_0, \ldots, x_{m-1})$. We define the language $\mathbf{L}'$ as the extension of $\mathbf{L}$ by the symbols h^t. Before defining the desired axiom system Σ' we must give some preliminary definitions.

We define the set I similarly as in § 3 except we now must take the function symbols of $\mathbf{L}$ into consideration too.

Let I be the set of the following formulae (the equality axioms for the language $\mathbf{L}$)

$$v_0 = v_0$$

$$v_0 = v_1 \rightarrow v_1 = v_0$$

$$(v_0 = v_1 \wedge v_1 = v_2) \rightarrow v_0 = v_2$$

$$(v_0 = v_n \wedge \ldots \wedge v_{n-1} = v_{2n-1}) \rightarrow \big(P(v_0, \ldots, v_{n-1}) \longleftrightarrow P(v_n, \ldots, v_{2n-1})\big)$$

$$(v_0 = v_n \wedge \ldots \wedge v_{n-1} = v_{2n-1}) \rightarrow f(v_0, \ldots, v_{n-1}) = f(v_n, \ldots, v_{2n-1})$$

for every $P, f \in \mathbf{L}$.

Let $\Sigma^* = \{H^* : H \in \Sigma\} \cup I$. If E is an open formula of $\mathbf{L}_1$ then let $\mathrm{Subst}(E)$ denote the set of all formulae $\left| \frac{x_1, \ldots, x_n}{t_1, \ldots t_n} \right. E$ where $t_1, \ldots, t_n$ are terms of $\mathbf{L}_1$. If X is a set of formulae we put $\mathrm{Subst}(X) = \bigcup_{E \in X} \mathrm{Subst}(E)$. Let $\Theta = \mathrm{Subst}(\Sigma^*)$.

Let Z denote the set of all ordered pairs (F, G) such that $F \in \mathbf{F}$ and $G \in \mathrm{Subst}(F^{**})$.

Now we define for each $(F, G) \in Z$ a formula $\Psi_{F,G} \in \mathfrak{F}(\mathbf{L}')$. By hypothesis

$$G = \left| \frac{x_1, \ldots, x_{k_{n+1}} ; z_1, \ldots, z_l}{t_1, \ldots, t_{k_{n+1}} ; u_1, \ldots, u_l} \right. F^{**} \tag{16}$$

for some terms $t_1, \ldots, t_{k_{n+1}}$; $u_1, \ldots, u_l$ of $\mathbf{L}_1$. (We suppose $\neg F$ to be of the form (15)). We put

$$\Psi_{F,G} = \left| \frac{z_1, \ldots, z_l}{\overline{u}_1, \ldots, \overline{u}_l} \right. F \tag{17}$$

where $\overline{u}_i$ was defined above.

Let $Pr(\mathbf{L}_1)$ denote the set of all prime formulae of $\mathbf{L}_1$, let U be an arbitrary subset of $Pr(\mathbf{L}_1)$. We consider functions $\varepsilon \in 2^U$ to fix valuations of the prime formulae $T \in U$, i.e. we associate the truth value $\varepsilon(T)$ with each $T \in U$. Let E be an arbitrary open formula of $\mathbf{L}_1$, and let us suppose that each prime formula occurring in E as a part is an element of U. Then the valuation ε associates with E a fix truth value if we consider the propositional connectives as operations on truth values in the well known way. This truth value will be denoted by $\tilde{\varepsilon}(E)$; for $T \in U$ $\tilde{\varepsilon}(T)$ is the same as $\varepsilon(T)$. Note that $\tilde{\varepsilon}(E)$ is defined only if E satisfies our condition. Let X_U be the subset of 2^U consisting of those functions ε for which $\tilde{\varepsilon}(E) = 1$ for every $E \in \Theta$ provided that $\tilde{\varepsilon}(E)$ is defined.

Now let U, V be arbitrary finite subsets of $Pr(\mathbf{L}_1)$ and Z respectively. We define the formula $\Phi_{U,V} \in \mathfrak{F}(\mathbf{L}')$ by

$$\Phi_{U,V} = \bigvee_{\varepsilon \in X_U} \bigwedge_{\substack{(F,G) \in V \\ \tilde{\varepsilon}(G) = 0}} \Psi'_{F,G} \tag{18}$$

Finally we define Σ' as the set of all formulae $Cl(\Phi_{U,V})$ for all U, V as before.

I. Proof of $\mathbf{H}_\mathbf{F}(\mathbf{K}) \subset \mathbf{M}_{\mathbf{L}'}(\Sigma') \mid \mathbf{L}$.

Let $\mathfrak{A} \in \mathfrak{S}(\mathbf{L})$, $\mathfrak{B} \in \mathbf{K}$, φ be a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. We must construct a system $\mathfrak{A}'$ such that

$$\mathfrak{A}' \in \mathbf{M}_{\mathbf{L}'}(\Sigma') \tag{19}$$

and

$$\mathfrak{A} = \mathfrak{A}' \mid \mathbf{L}$$

From Lemma 3 we can easily infer that there exists a system $\mathfrak{B}_1$ of $\mathbf{L}_1$ such that we have

$$\mathfrak{B}_1 \mid \mathbf{L} = \mathfrak{B} \tag{20}$$

$$\mathfrak{B}_1 \vdash Cl(H^*) \tag{21}$$

$$\mathfrak{B}_1 \vdash Cl(F^{**} \vee F) \tag{22}$$

for any $H \in \Sigma$, $F \in \mathbf{F}$.

We choose an element $\bar{\bar{a}}$ from $\mid \mathfrak{B} \mid = B$ for every element a of $\mid \mathfrak{A} \mid = A$ such that $a = \varphi(\bar{\bar{a}})$ (by using the axiom of choice).

Let $\mathfrak{A}'$ be the uniquely determined system of $\mathbf{L}'$ such that $\mathfrak{A}' \mid \mathbf{L} = \mathfrak{A}$ and

$$(h^t)_{\mathfrak{A}'}(a_1, \ldots, a_m) = \varphi\left(\mathfrak{B}_1 \left| \frac{v_1, \ldots, v_m}{\bar{\bar{a}}_1, \ldots, \bar{\bar{a}}_m} t \right.\right)$$

for any free term t of $\mathbf{L}_1$ and elements $a_1, \ldots, a_m$ of A. From this definition we infer easily that

$$\mathfrak{A}' \left| \frac{w_i}{a_i} \bar{u} = \varphi\left(\mathfrak{B}_1 \left| \frac{w_i}{\bar{\bar{a}}_i} u \right.\right)\right. \tag{23}$$

for arbitrary term u of $\mathbf{L}_1$.

In order to prove (19) let U, V as before (18) and $x_1, \ldots, x_m$ be all the different variables occurring in $\Phi_{U,V}$ or in some prime formula T of U. We have to show

$$\mathfrak{A}' \left| \frac{x_i}{a_i} \Phi_{U,V} \right. \tag{24}$$

for arbitrary elements $a_1, \ldots, a_m$ of A.

Let W be the set of all formulae E of Θ, each prime formula component of which is an element of U. (In other words, for which $\tilde{\varepsilon}(E)$ is defined for $\varepsilon \in 2^U$.) Let $E \in W$. Then $E \in \mathrm{Subst}(H^*)$ for some $H \in \Sigma$ or $E \in \mathrm{Subst}(I)$. In the first case we infer from (21) that

$$\mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. E . \tag{25}$$

In the second case (25) holds trivially.

We define $\varepsilon_0 \in 2^U$ by the following condition

$$\varepsilon_0(T) = 1 \sim \mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. T .$$

By (25) $\tilde{\varepsilon}_0(E) = 1$ for each $E \in W$ consequently

$$\varepsilon_0 \in X_U . \tag{26}$$

Now let $(F, G) \in V$, G be of the form (16). By (22) we have

$$\mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. \left(G \vee \left| \frac{z_1, \ldots, z_l}{u_1, \ldots, u_l} \right. F \right) . \tag{27}$$

Let us observe the definition of $\Phi_{U,V}$ under (18).

To prove (24) let us suppose $\tilde{\varepsilon}_0(G) = 0$. Then by (27)

$$\mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. \left(\left| \frac{z_1, \ldots, z_l}{u_1, \ldots, u_l} \right. F \right) . \tag{28}$$

Let $b_k = \mathfrak{B}_1 \left| \frac{x_i}{a_i} \right. u_k$ for $k = 1, \ldots, l$. Now (28) can be written in the form $\mathfrak{B}_1 \left| \frac{z_1, \ldots, z_l}{b_1, \ldots, a_l} \right. F$ or which is the same (see (20))

$$\mathfrak{B} \left| \frac{z_1, \ldots, z_l}{b_1, \ldots, b_l} \right. F .$$

Using that φ is an $\mathbf{F}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ we infer $\mathfrak{A} \left| \frac{z_1, \ldots, z_l}{\varphi(b_1), \ldots, \varphi(b_l)} \right. F$ or

$$\mathfrak{A}' \left| \frac{z_1, \ldots, z_l}{\varphi(b_1), \ldots, \varphi(b_l)} \right. F . \tag{29}$$

Using (23) we have

$$\varphi(b_k) = \mathfrak{A}' \left| \frac{x_i}{a_i} \right. \overline{u_k} . \tag{30}$$

Observing (17) we infer from (29) and (30)

$$\mathfrak{A}' \left| \frac{x_i}{a_i} \right. \Psi_{FG} . \tag{31}$$

To sum up we have proved that $\tilde{\varepsilon}_0(G) = 0$ implies (31), consequently we have

$$\mathfrak{A}' \left| \frac{x_i}{a_i} \bigwedge_{\substack{(F,G)\in V \\ \tilde{\varepsilon}_0(G)=0}} \Psi_{F,G} \right.$$

and taking (26) into account so we have shown (24) qu. e.d.

II. Proof of $\mathbf{H_F(K)} \supset \mathbf{M_{L'}}(\Sigma') \mid \mathbf{L}$.

Let $\mathfrak{A}' \in \mathbf{M_{L'}}(\Sigma')$, $\mathfrak{A} = \mathfrak{A}' \mid \mathbf{L}$. We must construct a system $\mathfrak{B}$ with

$$\mathfrak{B} \in \mathbf{K} \tag{32}$$

$$\mathfrak{A} \in \mathbf{H_F}(\mathfrak{B}) \tag{33}$$

We define new different constants c_a to every element $a \in A$, let A_1 denote the set of all c_a-s. Adjoining the elements of A_1 to $\mathbf{L}_1$ we obtain the language $\mathbf{L}_1^A = \mathbf{L}_1 \cup A_1$. We define B_1 the set of those terms of $\mathbf{L}_1^A$ which contain no variable.

Let $Pr(\mathbf{L}_1^A)$ be the set of the prime formulae of $\mathbf{L}_1^A$ which contain no variable.

Let Subst$'$ X $(X \subset \mathfrak{F}(\mathbf{L}_1))$ denote the set of all formulae $\left| \frac{x_1, \ldots, x_m}{t_1', \ldots, t_m'} E \right.$ for $E \in X$, $t_1', \ldots, t_m'$ being terms of $\mathbf{L}_1^A$ containing no variables and $x_1, \ldots, x_m$ being all the free variables of E. Each element of Subst$'$ X is a closed formula of $\mathbf{L}_1^A$. Let Z' be the set of all ordered pairs (F, G') where $F \in \mathbf{F}$ and $G' \in$ $\in$ Subst$'$ F^{**} and let $\Theta' =$ Subst$'$ Θ.

The fact that $\mathfrak{A}'$ is a model of Σ' can be formulated as follows. (Let us observe our definition of Σ'.)

Lemma 20. *For any finite sets U', V' of $Pr(\mathbf{L}_1^A)$ and Z' respectively there exists a function $\varepsilon \in 2^{U'}$ such that*

(i) $\tilde{\varepsilon}(E') = 1$ *for each $E' \in \Theta'$ if $\tilde{\varepsilon}(E')$ is defined,*

(ii) *if $(F, G') \in V'$ and $\tilde{\varepsilon}(G') = 0$ then*

$$\mathfrak{A}' \left| \frac{x_i}{a_i} \left(\left| \frac{z_1, \ldots, z_l}{u_1, \ldots, u_l} F \right. \right) \right.$$

where we use the following conventions: $c_{a_1}, \ldots, c_{a_m}$ are all the distinct constants of A_1 occurring in some formula T' of U', $x_1, \ldots, x_m$ are different variables, $G = \left| \frac{c_{a_1}, \ldots, c_{a_m}}{x_1, \ldots, x_m} G' \right.$ and $\neg F$ and G have the forms (15) and (16) respectively. Further we use the notation $\tilde{\varepsilon}(E')$ analogously as before.

Let us apply Lemma 1 of § 1 with the following distribution of the roles. Let the set A be $Pr(\mathbf{L}_1^A) \cup Z'$ and call a function ε on the finite set $U' \cup V'$ $(U' \subset Pr(\mathbf{L}_1^A), V' \subset Z')$ a "good" function (i.e. $\varepsilon \in \beta(U' \cup V')$) if ε satisfies (i) and (ii) of Lemma 20. Let $c(x) = 2 = \{0, 1\}$ if $x \in Pr(\mathbf{L}_1^A)$ and $c(x) = \{0\}$ if $x \in Z'$ (We remark that $c(x)$ for $x \in Z'$ is irrelevant). One can easily see that Lemma 20 says exactly that the hypotheses of Lemma 1 hold. So we can state by Lemma 1

Lemma 21. *There exists a function* $\delta \in 2^{Pr(L_1{}^A)}$ *such that*

(i) *for each* $E' \in \Theta'$ *we have* $\tilde{\delta}(E') = 1$

(ii) *if* $F \in \mathbf{F}$, $G' \in \text{Subst}' F^{**}$ *and* $\tilde{\delta}(G') = 0$ *then*

$$\mathfrak{A}' \left| \frac{x_i}{a_i} \right(\left| \frac{z_1, \ldots, z_l}{u_1, \ldots, u_l} F \right)$$

where $c_{a_1}, \ldots, c_{a_m}$ are all the different constants of A_1 occurring in $G', x_1, \ldots, x_m$ are different variables, $G = \left| \frac{c_{a_i}}{x_i} G' \right.$ and $\neg F$ and G have the forms (15) and (16) resp.

Let us define a pseudosystem $\mathfrak{B}_1$, of the language $\mathbf{L}_1$ by the following conditions.

$$|\mathfrak{B}_1| = B_1$$

$$b_1 \equiv_{\mathfrak{B}_1} b_2 \sim \delta(b_1 \equiv b_2) = 1$$

$$P_{\mathfrak{B}_1}(b_1, \ldots, b_n) \sim \delta(P(b_1, \ldots, b_n)) = 1$$

$$f_{\mathfrak{B}_1}(b_1, \ldots, b_n) = f(b_1, \ldots, b_n)$$

for arbitrary $P, f \in \mathbf{L}_1$ and $b_1, \ldots, b_n \in B_1$. We infer easily from this definition, that

$$\mathfrak{B}_1 \left| \frac{x_i}{b_i} \Phi \sim \tilde{\delta} \left(\left| \frac{x_i}{b_i} \Phi \right) = 1 \right. \right. \tag{34}$$

for any open formula Φ of L_1.

If we take specially $\Phi \in \Sigma^*$, then $\left| \frac{x_i}{b_i} \Phi \in \Theta' \right.$ and so by Lemma 21 (i) and (34) we have

$$\mathfrak{B}_1 \left| \frac{x_i}{b_i} \Phi \right.$$

for arbitrary b_i-s from B_1, i.e.

$$\mathfrak{B}_1 \vdash Cl(\Phi) \tag{35}$$

We define $\mathfrak{B}_2 = \mathfrak{B}_1 | \mathbf{L}$. By (35) and Lemma 3 (taking $\Phi = H^*$ for $H \in \Sigma$) we see that $\mathfrak{B}_2$ satisfies all sentences of Σ. If we apply (35) to $\Phi \in I$ we can infer that $\equiv_{\mathfrak{B}_2}$ is a congruence relation on $\mathfrak{B}_2$. Let $\mathfrak{B} = \mathfrak{B}_2 / \equiv_{\mathfrak{B}_2}$. Now we have by Lemma 2 (32) as desired.

We define the mapping ψ: $B_1 \to A$ by

$$\psi(b) = \mathfrak{A}' \left| \frac{x_i}{a_i} \right(\left| \frac{c_{a_i}}{x_i} b \right) \tag{36}$$

where $c_{a_1}, \ldots, c_{a_m}$ are all the different constants of A_1 occurring in b.

ψ is onto since $\psi(c_a) = a$.

We prove that

$$\mathfrak{B}_1 \left| \frac{z_1, \ldots, z_l}{b_1, \ldots, b_l} F \right. \tag{37}$$

implies

$$\mathfrak{A} \left| \frac{z_1, \ldots, z_l}{\psi(b_1). \ldots, \psi(b_l)} \right. F \tag{38}$$

for arbitrary $F \in \mathbf{F}$, $z_1, \ldots, z_l$ being all the distinct free variables of F and $b_1, \ldots, b_l$ elements of B_1.

Let us assume (37). Let $c_{a_1}, \ldots, c_{a_n}$ be all the different constants of A_1 occurring in some of $b_1, \ldots, b_l$; $x_1, \ldots, x_n$ different variables, $u_k = \left| \frac{c_{a_i}}{x_i} \right. b_k$ for $k = 1, \ldots, l$. By (36), (38) is equivalent to

$$\mathfrak{A} \left| \frac{x_i}{a_i} \right. \left(\left| \frac{z_1, \ldots, z_l}{\overline{u_1}, \ldots, \overline{u_l}} \right. F \right). \tag{39}$$

Suppose that (39) is not true.

Let G be an arbitrary formula of Subst (F^{**}) as under (16) with the stipulation that the terms $u_1, \ldots, u_l$ are the same which we have just defined. By Lemma 21 (ii) we infer from our supposition that $\widetilde{\delta}(G') = 1$ where $G' = \left| \frac{x_1, \ldots, x_m}{c_{a_1}, \ldots, c_{a_m}} \right. G$ and $x_{n+1}, \ldots, x_m$ are the additional distinct variables of G and $a_{n+1}, \ldots, a_m$ are arbitrary elements of G. This means exactly that

$$\delta \left(\left| \frac{x_1, \ldots, x_{k_{n+1}}; z_1, \ldots, z_l}{b'_1, \ldots, b'_{k_{n+1}}; b_1, \ldots, b_l} \right. F^{**} \right) = 1$$

or by (34)

$$\mathfrak{B}_1 \left| \frac{x_1, \ldots, x_{k_{n+1}}; z_1, \ldots, z_l}{b'_1, \ldots, b'_{k_{n+1}}; b_1, \ldots, b_l} \right. F^{**}$$

for arbitrary elements $b'_1, \ldots, b_{k_{n+1}}$ of B_1. This can be expressed by

$$\mathfrak{B}_1 \left| \frac{z_1, \ldots, z_l}{b_1, \ldots, b_l} \right. Cl(F^{**})$$

and by Lemma 3 we have $\mathfrak{B}_2 \left| \frac{z_1, \ldots, z_l}{b_1, \ldots, b_l} \right. \neg F$ what contradicts our hypothesis (37). So we have proved that (37) implies (38) indeed.

We need also that $b_1 \equiv_{\mathfrak{B}_1} b_2$ implies $\psi(b_1) = \psi(b_2)$. But that is contained in our last assertion because we have supposed that $v_0 = v_1$ is a formula of $\mathbf{F}$.

Now let φ be the mapping $\varphi \colon |\mathfrak{B}| = B \to A$ defined by $\varphi(b/\equiv_{\mathfrak{B}_2}) = \psi(b)$ $(b \in B_1)$. From that we have proved above it follows that the latter equality defines φ uniquely.

Finally we see that φ is an $\mathbf{F}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, consequently we have proved also (33) qu.e.d.

By I, II we have shown (14). It is trivial from the definition of Σ' that each formula of Σ' is an $\mathbf{H}_\mathbf{F}$-sentence over $\mathbf{L}$. Qu. e.d.

§ 5. Endomorphisms

If $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$ by the homomorphism φ and at the same time $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$ then we say that $\mathfrak{A}$ is an *endomorphic image of* $\mathfrak{B}$ *and* φ *is an endomorphism of* $\mathfrak{B}$ *onto* $\mathfrak{A}$. We denote the set of all endomorphic images of $\mathfrak{B}$ by $\mathbf{End}(\mathfrak{B})$, i.e. $\mathbf{End}(\mathfrak{B}) = \mathbf{H}(\mathfrak{B}) \cap \mathbf{S}(\mathfrak{B})$ and we put $\mathbf{End}(\mathbf{K}) = \bigcup_{\mathfrak{B} \in \mathbf{K}} \mathbf{End}(\mathfrak{B})$.

Corollary 8. *If* $\mathbf{K} \in \mathbf{PC}_\Delta$ *or* $\mathbf{K} \in \mathbf{EC}_\Delta$ *then* $\mathbf{End}(\mathbf{K}) \in \mathbf{PC}_\Delta$.

Corollary 9. *If* $\mathbf{K} \in \mathbf{PC}$ *or* $\mathbf{K} \in \mathbf{EC}$ *then* $(\mathbf{End}(\mathbf{K}))^\infty \in \mathbf{PC}$.

The proof of these statements is similar to the proof of Corollaries 4, 5.

Now we want to prove a theorem, which has the same relation to the endomorphisms as LYNDON's theorem to homomorphisms. In the proof we use LYNDON's theorem as stated in § 1 and a simple "ascending chain" construction, i.e. we get the desired relational system as the union of some sequence of systems, each of which is elementary subsystem of the next in the sequence. That is the principal tool in the proof of many model theoretic theorems. We can described this part of the proof most easily by using ultrapowers and limit ultrapowers and we shall apply some notations and well known results stated in § 1.

Theorem 10. (i) *Let* $\Sigma \subset \mathfrak{F}_0(\mathbf{L})$, $\mathbf{K} = \mathbf{M}_\mathbf{L}(\Sigma)$ *Let* Σ' *be the set of the sentences* $F_1 \vee F_2$ *such that* $F_1 \vee F_2 \in \mathbf{Cn}(\Sigma)$, F_1 *is a positive sentence and* F_2 *is a universal one. Then we have*

$$\mathbf{Th}(\mathbf{End}(\mathbf{K})) = \mathbf{Cn}(\Sigma')$$

(ii) *Moreover, if* $\mathfrak{A} \in \mathbf{M}_\mathbf{L}(\Sigma')$ *then there exists a* $\mathfrak{A}'$ *such that* $\mathfrak{A} \prec \mathfrak{A}'$ *and* $\mathfrak{A}' \in \mathbf{End}(\mathbf{K})$.[5]

Proof. To prove $\mathbf{Cn}(\Sigma') \subset \mathbf{Th}(\mathbf{End}(\mathbf{K}))$ it is sufficient to show $\Sigma' \subset \subset \mathbf{Th}(\mathbf{End}(\mathbf{K}))$. To this end let F_1 be a positive sentence, F_2 a universal one, $F_1 \vee F_2 \in \mathbf{Cn}(\Sigma)$, $\mathfrak{B} \in \mathbf{K}$, $\mathfrak{A} \in \mathbf{End}(\mathfrak{B})$. We have to show $\mathfrak{A} \vdash F_1 \vee F_2$. We have $\mathfrak{B} \vdash F_1 \vee F_2$, hence $\mathfrak{B} \vdash F_1$ or $\mathfrak{B} \vdash F_2$.

In the first case $\mathfrak{A} \in \mathbf{H}(\mathfrak{B})$ implies $\mathfrak{B} \vdash F_2$ in the second one $\mathfrak{A} \in \mathbf{S}(\mathfrak{B})$ implies $\mathfrak{B} \vdash F_2$, consequently $\mathfrak{A} \vdash F_2 \vee F_2$ at any rate.

Instead of $\mathbf{Th}(\mathbf{End}(\mathbf{K})) \subset \mathbf{Cn}(\Sigma')$ we prove the stronger assertion (ii). Let us suppose

$$\mathfrak{A} \in \mathbf{M}_\mathbf{L}(\Sigma') \tag{1}$$

We may and shall assume $\mathbf{Cn}(\Sigma) = \Sigma$. Let Θ be the set of sentences $\neg G$ for which $\neg G \in \mathbf{Th}(\mathfrak{A})$ and G is universal. Let $\Sigma_1 = \Sigma \cup \Theta$. We assert, that the positive consequence of Σ_1 are satisfied by $\mathfrak{A}$, i.e.

$$\mathbf{Pos}(\Sigma_1) \subset \mathbf{Th}(\mathfrak{A}). \tag{2}$$

To prove that, let $F_1 \in \mathbf{Pos}(\Sigma_1)$ and suppose on the contrary that $F_1 \notin \mathbf{Th}(\mathfrak{A})$.

The Compactness Theorem (Lemma 11) implies the existence of finite subsets V_1, V_2 of Σ and Θ respectively such that F_1 is a consequence of $V_1 \cup V_2$. Let the conjunction of the formulae of V_1 and V_2 be G_1 and G_2 resp. G_2 is equi-

[5] This stronger statement is a consequence of (i) and Corollary 8 by a familiar application of the Compactness Theorem; it is derivable also from the fact that $\mathbf{End}(\mathbf{K})$ is closed under ultraproduct and limit ultrapowers and from (i).

valent to a formula $\neg F_2$ where F_2 is universal. We have $G_1 \wedge \neg F_2 \vdash F_1$ i.e. $G_1 \vdash F_1 \vee F_2$, hence by $G_1 \in \Sigma$ we have $F_1 \vee F_2 \in \Sigma$ and thus $F_1 \vee F_2 \in \Sigma'$. We have $\neg F_2 \in \mathbf{Th}(\mathfrak{A})$, $F_1 \notin \mathbf{Th}(\mathfrak{A})$, consequently $F_1 \vee F_2 \notin \mathbf{Th}(\mathfrak{A})$ and that contradicts our hypothesis (1). Thus we have proved (2).

By LYNDON's theorem (Lemma 7) we infer from (2) that there exist systems $\mathfrak{A}_0^1$, $\mathfrak{B}_0$ and a homomorphism φ_0 of $\mathfrak{B}_0$ onto $\mathfrak{A}_0^1$ such that $\mathfrak{A} \prec \mathfrak{A}_0^1$, $\mathfrak{B}_0 \in \mathbf{M}_\mathbf{L}(\Sigma_1)$. We assert that every universal formula G satisfied by $\mathfrak{B}_0$ is satisfied by $\mathfrak{A}_0^1$ too. In the contrary case $\neg G$ would be satisfied by $\mathfrak{A}$ hence $\neg G \in \Theta$ and so by $\mathfrak{B}_0 \in \mathbf{M}_\mathbf{L}(\Theta)$ $\mathfrak{B}_0 \vdash \neg G$ what is contradiction. Now by Lemma 8 there exists a non empty set I and an ultrafilter D on I so that $\mathfrak{A}_0^1$ is isomorphic to a subsystem $\mathfrak{A}_0$ of $\mathfrak{B}_0^{[I/D]}$. Let us define by induction

$$\mathfrak{A}_{n+1}^1 = (\mathfrak{A}_n^1)^{[I/D]} \qquad (n = 0, 1, \ldots)$$

$$\mathfrak{A}_{n+1} = \mathfrak{A}_n^{[I/D]} \qquad (n = 0, 1, \ldots)$$

$$\mathfrak{B}_{n+1} = \mathfrak{B}_n^{[I/D]} \qquad (n = 0, 1, \ldots)$$

$$\varphi_{n+1} = \varphi_n^{[I/D]} \qquad (n = 0, 1, \ldots$$

and we put

$$\mathfrak{A}'' = \bigcup_{n<\omega} \mathfrak{A}_n^1$$

$$\mathfrak{A}' = \bigcup_{n<\omega} \mathfrak{A}_n$$

$$\mathfrak{B}' = \bigcup_{n<\omega} \mathfrak{B}_n$$

$$\varphi' = \bigcup_{n<\omega} \varphi_n$$

Then by Lemma 10 φ' is a homomorphism of $\mathfrak{B}'$ onto $\mathfrak{A}''$, by Lemma 9 $\mathfrak{A} \prec \mathfrak{A}''$ and $\mathfrak{B}' \in \mathbf{M}_\mathbf{L}(\Sigma)$. $\mathfrak{A}'$ is trivially a subsystem of $\mathfrak{B}'$ and $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}''$. Consequently there exists a system $\mathfrak{B}''$ isomorphic to $\mathfrak{B}'$ for which $\mathfrak{A}'' \in \mathbf{H}(\mathfrak{B}'')$ and the same time $\mathfrak{A}'' \subset \mathfrak{B}''$. Thus we have $\mathfrak{A}'' \in \mathbf{End}(\mathfrak{B}'')$ and $\mathfrak{B}'' \in \mathbf{M}_\mathbf{L}(\Sigma)$ and $\mathfrak{A} \prec \mathfrak{A}''$ qu. e.d.

Corollary 11. *If F is a sentence which is preserved under endomorphism (that is $\mathfrak{B} \vdash F$ and $\mathfrak{A} \in \mathbf{End}(\mathfrak{B})$ imply $\mathfrak{A} \vdash F$), then F is equivalent to a sentence*

$$\bigwedge_{i=1}^{n} F_1^i \vee F_2^i$$

where F_1^i is positive and F_2^i is universal for each $i = 1, \ldots, n$. The proof proceeds in a well known way by Theorem 10.

We remark that the notion of endomorphism can be generalized analogously as we did with the notion of homomorphism by introducing the **F**-homomorphism. The analogon of Theorem 10 for the generalized case can be proved in a similar way, using results of KEISLER [2].

(Received December 6, 1963)

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О КЛАССАХ $\mathbf{PC}_\Delta$ ТЕОРИИ МОДЕЛЕЙ

M. MAKKAI

Резюме

Содержание теоремы 1 статьи: бесконечные системы отношений класса $\mathbf{PC}_\Delta$ над конечным языком, где этот класс удовлетворяет определенным (очевидным) «конструктивным» условиям, образуют также класс $\mathbf{PC}$. Далее определяется операция, определяющая для системы отношений $\mathfrak{A}$ и формулы $F(x)$ содержащей единственного свободного переменного, систему отношений $\mathfrak{A} \| F(x)$, являющейся частью от $\mathfrak{A}$ и основное множество которой состоит точно из тех элементов $|\mathfrak{A}|$, которые удовлетворяют формуле $F(x)$ на $\mathfrak{A}$. Доказывается, что системы отношений $\mathfrak{A} \| F(x)$, полученные для систем отношений $\mathfrak{A}$ класса $\mathbf{K} \in \mathbf{EC}_\Delta$ и для фиксированной формулы $F(x)$, образуют класс $\mathbf{PC}_\Delta$ (при условии, что $(\exists x)\, F(x)$ справедливо в $\mathbf{K}$) (Теорема 2*a*). Далее, если $\mathbf{K} \in \mathbf{EC}$, тогда бесконечные системы отношений только, что определенного класса образуют класс $\mathbf{PC}$ (следствие 3). В качестве применения доказывается, что если $\mathbf{K} \in \mathbf{PC}_\Delta$, тогда гомоморфные образы системы $\mathbf{K}$ образуют класс $\mathbf{PC}_\Delta$ (следствие 4), кроме того если $\mathbf{K} \in \mathbf{PC}$, то бесконечные системы этого класса $\mathbf{PC}_\Delta$ образуют класс $\mathbf{PC}$ (Следствие 5). Согласно одному варианту следствии 4′ $\mathbf{H}_\mathbf{F}(\mathbf{K})$ получается некоторым специальным образом в качестве класса $\mathbf{PC}_\Delta$ если $\mathbf{K} \in \mathbf{EC}_\Delta$ (Теорема 7). ($\mathbf{H}_\mathbf{F}(\mathbf{K})$ — класс $\mathbf{F}$ — гомоморфных образов систем из $\mathbf{K}$, см. например [2]). Наконец, относительно эндоморфизмов доказывается аналог теоремы Lyndon-а [6] с подобными следствиями (Теорема 10, следствие 11).