ON THE „PARKING" PROBLEM

by

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1. Introduction. Consider the following random process in which cars of length 1 are parked on a street $[0, x]$ of length $x \geqq 1$. The first car is parked so that the position of its center is a random variable which is uniformly distributed on $\left[\frac{1}{2}, x - \frac{1}{2}\right]$. If there remains space to park another car then a second car is parked so that its center is a random variable which is uniformly distributed over the set of points in $\left[\frac{1}{2}, x - \frac{1}{2}\right]$ whose distance from the first car is $\geqq \frac{1}{2}$. If there remains an empty interval of length $\geqq 1$ on the street then a third car is parked, its center being uniformly distributed over the set of points whose distance from the cars already parked and the ends of the street is $\geqq \frac{1}{2}$. The process continues until there remains no empty interval of length $\geqq 1$. We denote by N_x the total number of cars parked and extend the definition of N_x to all $x \geqq 0$ by putting $N_x = 0$ for $0 \leqq x < 1$.

The „parking problem" is the study of the distribution of the integer-valued random variable N_x as $x \to \infty$. This problem was called to our attention by C. DERMAN and M. KLEIN in 1957. In 1958 A. RÉNYI [1] proved that the expectation $\mu(x) = \mathbf{E}(N_x)$ satisfies the relation

$$\mu(x) = \lambda_1 x + \lambda_1 - 1 + O(x^{-n}) \qquad (n \geqq 1) \tag{1.1}$$

(O and o refer throughout to the argument increasing to infinity); the constant λ_1 is given by

$$\lambda_1 = \int_0^\infty e^{-2\int_0^t \frac{1-e^{-u}}{u}\,du} \qquad \lambda_1 \cong 0.748 \tag{1.2}$$

To prove (1.1) RÉNYI employs the Laplace transform of a certain integral equation satisfied by $\mu(x)$; using similar methods P. NEY [2] has studied the higher

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moments of N_x. In the present paper we show by a direct analysis of the integral equation that (1.1) can be strengthened to

$$\mu(x) = \lambda_1 x + \lambda_1 - 1 + O\left(\left(\frac{2e}{x}\right)^{x-3/2}\right) \tag{1.3}$$

and that the variance $\sigma^2(x) = \mathbf{E}(N_x - \mu(x))^2$ satisfies

$$\sigma^2(x) = \lambda_2 x + \lambda_2 + O\left(\left(\frac{4e}{x}\right)^{x-4}\right) \tag{1.4}$$

where λ_2 is some positive constant. We show moreover that the standardized random variable $Z_x = (N_x - \mu(x))/\sigma(x)$ has the limiting normal (0,1) distribution as $x \to \infty$; this is done in two ways, the first by showing that all the moments of Z_x converge to the normal moments, and the second by a direct argument using the central limit theorem for sums of independent random variables.

In Section 2 we derive the integral equations satisfied by $\mu(x)$ and quantities related to the higher moments of N_x; these equations form the basis of our study as well as those of Rényi and Ney. Section 3 deals with the asymptotic behaviour of the solutions of these equations; our work here is somewhat similar to that of N. G. de Bruijn [3]. The results of Section 3 are applied in Section 4 to the parking problem. The second proof of the asymptotic normality of Z_x is given in Section 5. Various remarks will be found in Section 6.

2. Derivation of the integral equations. For $x \geq 0$ let $[t, t+1]$ be the random interval occupied by the first car parked on a street $[0, x+1]$ of length $x+1$. The parking process described in Section 1 is such that the number of cars which will eventually be parked to the left of the first car is independent of the number which will be parked to the right of it; moreover, the number of cars eventually parked to the left of the first car, i.e. on $[0, t]$, has the same distribution as N_t, while the number parked to the right of the first car, i.e. on $[t+1, x+1]$, has the same distribution as N_{x-t}. Hence *the conditional distribution of N_{x+1} given that the first car occupies $[t, t+1]$ is the same as the distribution of $N_t + N_{x-t} + 1$ with N_t, N_{x-t} independent*. Denoting by $|\, t$ conditioning on the event that the first car is parked at $[t, t+1]$ we therefore have

$$\mathbf{E}(N_{x+1} \mid t) = \mathbf{E}(N_t) + \mathbf{E}(N_{x-t}) + 1 \qquad (0 \leq t \leq x) \tag{2.1}$$

(here we do not use the independence of N_t and N_{x-t}). Since by hypothesis t is uniformly distributed on $[0, x]$ it follows that, setting

$$\mu(x) = \mathbf{E}(N_x), \tag{2.2}$$

we have

$$\mu(x+1) = \frac{2}{x}\int_0^x \mu(t)\,dt + 1 \qquad (x > 0). \tag{2.3}$$

Defining the function

$$f(x) = \mu(x) + 1 \tag{2.4}$$

we see that f satisfies the somewhat simpler equation

$$f(x+1) = \frac{2}{x} \int_0^x f(t)\,dt \qquad (x > 0)\,. \tag{2.5}$$

Together with the initial conditions

$$f(x) = 1 \quad (0 \leqq x < 1)\,, \quad f(1) = 2 \tag{2.6}$$

this determines $f(x)$ consecutively over the intervals $1 < x \leqq 2, 2 < x \leqq 3, \ldots$. Thus we find

$$f(x) = 2 \qquad (1 < x \leqq 2)\,, \tag{2.7}$$

$$f(x) = 4 - \frac{2}{x-1} \qquad (2 < x \leqq 3)\,, \tag{2.8}$$

$$f(x) = 8 - \frac{10}{x-1} - \frac{4}{x-1} \log(x-2) \qquad (3 < x \leqq 4)\,, \tag{2.9}$$

at which the integration of (2.5) becomes difficult.

Using the independence of N_t and N_{x-t} we have for the function

$$\sigma^2(x) = \mathbf{D}^2(N_x) = \mathbf{E}(N_x - \mu(x))^2 \tag{2.10}$$

the relation

$$\mathbf{D}^2(N_{x+1} \mid t) = \sigma^2(t) + \sigma^2(x-t) \qquad (0 \leqq t \leqq x) \tag{2.11}$$

Since

$$\mathbf{D}^2(N_{x+1}) \geqq \mathbf{E}(\mathbf{D}^2(N_{x+1} \mid t))\,, \tag{2.12}$$

it follows from (2.11) that

$$\sigma^2(x+1) \geqq \frac{2}{x} \int_0^x \sigma^2(t)\,dt \qquad (x > 0)\,. \tag{2.13}$$

Let

$$L(x) = \lambda_1 x + \lambda_1 - 1\,, \tag{2.14}$$

where λ_1 is a constant to be determined later, and define for $k = 0, 1, \ldots$

$$\varphi_k(x) = \mathbf{E}((N_x - L(x))^k)\,. \tag{2.15}$$

Since

$$L(x+1) = L(t) + L(x-t) + 1\,, \tag{2.16}$$

we have

$$\mathbf{E}[(N_{x+1} - L(x+1))^k \mid t] = \\ = \mathbf{E}[\{(N_t - L(t)) + (N_{x-t} - L(x-t))\}^k]\,, \tag{2.17}$$

and on integrating we find that

$$\varphi_k(x+1) = \frac{1}{x}\sum_{i=0}^{k}\binom{k}{i}\int_0^x \varphi_i(t)\,\varphi_{k-i}(x-t)\,dt \qquad (x>0). \tag{2.18}$$

3. The integral equations. Our results on the behaviour as $x \to \infty$ of functions satisfying certain integral equations are summarized in the following two theorems.

Theorem 1. *Let $f(x)$ be defined for $x \geqq 0$ and satisfy*

$$f(x+1) = \frac{2}{x}\int_0^x f(t)\,dt + p(x+1) \qquad (x>0) \tag{3.1}$$

where $p(x)$ is continuous for $x > 1$ and is such that, setting

$$p_x = \sup_{x \leqq t \leqq x+1} |p(t)| \qquad (x>1), \tag{3.2}$$

we have

$$\sum_{i=2}^{\infty} \frac{p_i}{i} < \infty. \tag{3.3}$$

Then there exists a constant λ such that, setting

$$R_j = \frac{2j+1}{j}\,p_{j+1} + \frac{2(j+1)(j+3)}{j}\sum_{i=j+2}^{\infty}\frac{p_i}{i+1} \qquad (j=1,2,\ldots), \tag{3.4}$$

we have

$$\sup_{n+1 \leqq x \leqq n+2} |f(x) - \lambda x - \lambda| \leqq \frac{2^n}{n!}\sup_{1\leqq x \leqq 2}|f(x)-\lambda x - \lambda| + \frac{2^n}{n!}\sum_{j=1}^{n}\frac{j!}{2^j}R_j \qquad (n=1,2,\ldots). \tag{3.5}$$

Corollary. If $\alpha > 2e$ and $f(x)$ satisfies (3.1) with

$$p(x) = O\left(\left(\frac{\alpha}{x}\right)^{x+\beta}\right), \tag{3.6}$$

then

$$f(x) = \lambda x + \lambda + O\left(\left(\frac{\alpha}{x}\right)^{x+\beta-1}\right). \tag{3.7}$$

The second theorem is much less precise but easier to prove.

Theorem 2. *Let $g(x)$ be defined for $x \geqq 0$ and satisfy*

$$g(x+1) = \frac{2}{x}\int_0^x g(t)\,dt + O(x^\gamma) \qquad (x>0) \tag{3.8}$$

with $\gamma > 1$. *Then*

$$g(x) = O(x^{\gamma}) . \tag{3.9}$$

Corollary. *Let* $g(x)$ *be defined for* $x \geqq 0$ *and satisfy*

$$g(x+1) = \frac{2}{x}\int_0^x g(t)\,dt + Ax^{\beta} + O(x^{\gamma}) \qquad (x > 0) \tag{3.10}$$

with $\beta > \gamma > 1$. *Then*

$$g(x) = \frac{\beta+1}{\beta-1} Ax^{\beta} + O(x^{\max(\beta-1,\gamma)}) . \tag{3.11}$$

Proof of Theorem 1. The proof is less involved and leads to a somewhat sharper error estimate if, as in the case of $\mu(x)$, the term $p(x)$ vanishes identically. For the sake of brevity, however, we shall treat the general case directly.

From (3.1) we have for positive x and y,

$$f(y+1) = \frac{2}{y}\int_0^x f(t)\,dt + \frac{2}{y}\int_x^y f(t)\,dt + p(y+1) =$$

$$= \frac{1}{y}[xf(x+1) - xp(x+1)] + \frac{2}{y}\int_x^y f(t)\,dt + p(y+1)$$

or

$$f(y+1) = \frac{x}{y} f(x+1) + \frac{2}{y}\int_x^y f(t)\,dt + p(y+1) - \frac{x}{y} p(x+1) . \tag{3.12}$$

Define

$$I_x = \inf_{x \leq t \leq x+1} \frac{f(t)}{t+1}, \quad S_x = \sup_{x \leq t \leq x+1} \frac{f(t)}{t+1} \qquad (x \geqq 0) . \tag{3.13}$$

Notice that $f(x) = x + 1$ satisfies (3.1) with $p \equiv 0$, and hence that

$$y + 2 = \frac{x}{y}(x+2) + \frac{2}{y}\int_x^y (t+1)\,dt . \tag{3.14}$$

Subtracting (3.14) multiplied by I_x from (3.12) we have

$$\begin{aligned} f(y+1) - I_x\cdot(y+2) &= \frac{x}{y}[f(x+1) - I_x\cdot(x+2)] + \\ &+ \frac{2}{y}\int_x^y [f(t) - I_x\cdot(t+1)]\,dt + p(y+1) - \frac{x}{y} p(x+1) . \end{aligned} \tag{3.15}$$

Hence for $x \leqq y \leqq x+1$, in view of (3.13) and (3.2),

$$f(y+1) - I_x \cdot (y+2) \geqq 0 + 0 - p_{x+1} - p_{x+1} = -2p_{x+1}. \tag{3.16}$$

It follows that

$$I_{x+1} \geqq I_x - \frac{2p_{x+1}}{x+2} \qquad (x > 0). \tag{3.17}$$

Applying (3.17) successively with x replaced by $x+1, x+2, \ldots$ we obtain

$$I_y \geqq I_x - \Delta_x, \qquad (y \geqq x > 0). \tag{3.18}$$

where by definition

$$\Delta_x = 2 \sum_{i=1}^{\infty} \frac{p_{x+i}}{x+i+1} \qquad (x > 0). \tag{3.19}$$

In exactly the same manner we obtain the inequality

$$S_y \leqq S_x + \Delta_x \qquad (y \geqq x > 0). \tag{3.20}$$

From (3.18) we have

$$\liminf_{y \to \infty} I_y \geqq I_x - \Delta_x \qquad (x > 0). \tag{3.21}$$

Since $\Delta_x = o(1)$ by (3.3) it follows that

$$\liminf_{y \to \infty} I_y \geqq \limsup_{x \to \infty} I_x. \tag{3.22}$$

From this and (3.18) with $x = 1$ we find that

$$I_\infty = \lim_{x \to \infty} I_x \text{ exists, and } I_\infty > -\infty. \tag{3.23}$$

Similarly,

$$S_\infty = \lim_{x \to \infty} S_x \text{ exists, and } S_\infty < \infty. \tag{3.24}$$

Since $I_x \leqq S_x$ it follows that

$$-\infty < I_\infty \leqq S_\infty < \infty. \tag{3.25}$$

From (3.12) we have for $x, y > 0$

$$\begin{aligned} f(y+1) - f(x+1) &= \frac{x-y}{y} f(x+1) + \\ &+ \frac{2}{y} \int_x^y f(t)\, dt + p(y+1) - \frac{x}{y} p(x+1). \end{aligned} \tag{3.26}$$

By (3.13) and (3.25), $f(x) = O(x)$, and hence by (3.26)

$$\sup_{x \leqq y \leqq x+1} |f(y+1) - f(x+1)| = O(1) + 2p_x. \tag{3.27}$$

But this implies by (3.3) that

$$S_x - I_x = o(1) \tag{3.28}$$

and therefore that

$$I_\infty = S_\infty \neq \pm\infty\,. \tag{3.29}$$

We now define λ as the common value in (3.29),

$$\lambda = \lim_{x\to\infty} I_x = \lim_{x\to\infty} S_x = \lim_{x\to\infty} \frac{f(x)}{x+1}\,. \tag{3.30}$$

By (3.18) and (3.20),

$$I_x - \Delta_x \leqq \lambda \leqq S_x + \Delta_x \qquad (x > 0). \tag{3.31}$$

Next we observe that for every $x > 1$ there exists a number x' satisfying

$$x \leqq x' \leqq x+1\,, \qquad \left|\frac{f(x')}{x'+1} - \lambda\right| \leqq \Delta_x\,. \tag{3.32}$$

Indeed, since by (3.1) $f(x)$ is continuous for $x > 1$ the non-existence of such an x' would imply that either

$$I_x > \lambda + \Delta_x \quad \text{or} \quad S_x < \lambda - \Delta_x\,, \tag{3.33}$$

contradicting (3.31). We denote by x_n a value x' satisfying (3.32) for $x = n$; thus for $n = 2, 3, \ldots$

$$|f(x_n) - \lambda(x_n+1)| \leqq (n+2)\,\Delta_n \qquad (n \leqq x_n \leqq n+1) \tag{3.34}$$

Now set

$$f^*(x) = f(x) - \lambda(x+1)\,. \tag{3.35}$$

Then f^* again satisfies (3.1), and applying (3.12) with $n \leqq y \leqq n+1$ and $x = x_{n+1} - 1$ we obtain from (3.34) for $n = 1, 2, \ldots$

$$\begin{aligned} |f^*(y+1)| \leqq {} & \frac{n+1}{n}(n+3)\,\Delta_{n+1} + \frac{2}{n}\sup_{n\leqq t\leqq n+1} |f^*(t)| + \\ & + p_{n+1} + \frac{n+1}{n}\,p_{n+1} \end{aligned} \tag{3.36}$$

Putting

$$T_x = \sup_{x\leqq t\leqq x+1} |f^*(t)| \qquad (x > 0)\,, \tag{3.37}$$

we obtain from (3.36)

$$\begin{aligned} T_{n+1} & \leqq \frac{2}{n}T_n + \frac{2n+1}{n}\,p_{n+1} + \frac{(n+1)(n+3)}{n}\,\Delta_{n+1} = \\ & = \frac{2}{n}T_n + R_n \qquad (n = 1, 2, \ldots) \end{aligned} \tag{3.38}$$

where R_n is defined by (3.4). Successive application of this inequality for $n = 1, 2, 3, \ldots$ yields the inequality

$$T_{n+1} \leq \frac{2^n}{n!} T_1 + \frac{2^n}{n!}\left[\frac{1!}{2} R_1 + \frac{2!}{2^2} R_2 + \ldots + \frac{n!}{2^n} R_n\right]. \tag{3.39}$$

In view of (3.37) this is precisely (3.5), and this completes the proof of Theorem 1.

Proof of Corollary. If (3.6) holds, then by (3.4)

$$R_j = O\left(\left(\frac{a}{j}\right)^{j+\beta+1}\right)$$

and hence (3.5), since $a > 2e$,

$$\frac{2^n}{n!} \sum_{j=1}^{n} \frac{j!}{2^j} R_j = O\left(\left(\frac{a}{n}\right)^{n+\beta+1}\right).$$

Thus by (3,5)

$$\sup_{n+1 \leq x \leq n+2} |f(x) - \lambda x - \lambda| = O\left(\left(\frac{2e}{n}\right)^{n+\frac{1}{2}}\right) + O\left(\left(\frac{a}{n}\right)^{n+\beta+1}\right) =$$

$$= O\left(\left(\frac{a}{n}\right)^{n+\beta+1}\right),$$

from which (3.7) follows.

Proof of Theorem 2. We have

$$g(x+1) = \frac{2}{x} \int_0^x g(t)\, dt + \eta(x) \qquad (x > 0),$$

where

$$\eta(x) = O(x^\gamma), \qquad \gamma > 1.$$

Choose $x_0 > 1$ and $H > 0$ such that

$$|\eta(x)| \leq Hx^\gamma \quad \text{for} \quad x \geq x_0 - 1,$$

$$\int_0^{x_0} |g(t)|\, dt \leq \frac{H}{\gamma - 1} (x_0 - 1)^{\gamma+1} = \frac{\gamma+1}{\gamma-1} H \int_0^{x_0 - 1} t^\gamma\, dt. \tag{3.40}$$

Then for $x_0 - 1 \leq x \leq x_0$ we have

$$|g(x+1)| \leq \frac{2}{x} \int_0^{x_0} |g(t)|\, dt + Hx^\gamma \leq$$

$$\leq \frac{2H}{x(\gamma-1)} (x_0 - 1)^{\gamma+1} + Hx^\gamma \leq \tag{3.41}$$

$$\leq \frac{2Hx^\gamma}{\gamma - 1} + Hx^\gamma = \frac{\gamma+1}{\gamma-1} Hx^\gamma.$$

Hence

$$\int_0^{x_0+1} |g(t)|\,dt = \int_0^{x_0} |g(t)|\,dt + \int_{x_0}^{x_0+1} |g(t)|\,dt \leqq$$

$$\leqq \frac{\gamma+1}{\gamma-1} H \int_1^{x_0} (t-1)^\gamma\,dt + \int_{x_0}^{x_0+1} \frac{\gamma+1}{\gamma-1} H(t-1)^\gamma\,dt =$$

$$= \frac{\gamma+1}{\gamma-1} H \int_0^{x_0} t^\gamma\,dt\,,$$

so that (3.40) holds with x_0 replaced by x_0+1. Hence by (3.41), for $x_0 \leqq x \leqq \leqq x_0 + 1$ we have

$$|g(x+1| \leqq \frac{\gamma+1}{\gamma-1} H x^\gamma\,. \tag{3.42}$$

By induction, (3.42) holds for all $x \geqq x_0 - 1$, which proves (3.9).

Proof of Corollary. Set

$$g^*(x) = \frac{\beta+1}{\beta-1} A x^\beta\,.$$

Then

$$g^*(x+1) = \frac{\beta+1}{\beta-1} A(x+1)^\beta = \frac{\beta+1}{\beta-1} A x^\beta + O(x^{\beta-1}) =$$

$$= \frac{2}{x} \int_0^x g^*(t)\,dt + Ax^\beta + O(x^{\beta-1})\,.$$

Hence, setting

$$\bar{g}(x) = g(x) - g^*(x)$$

we have for $x > 0$,

$$\bar{g}(x+1) = g(x+1) - g^*(x+1) = \frac{2}{x} \int_0^x \bar{g}(t)\,dt + O(x^{\max(\beta-1,\gamma)}).$$

Hence by Theorem 2,

$$\bar{g}(x) = O(x^{\max(\beta-1,\gamma)})$$

which proves (3.11).

Remarks.

1. If G is the lim sup as $x \to \infty$ of $g(x)$ in (3.8) divided by x^γ, then by taking x_1 sufficiently large we have

$$\frac{|g(x)|}{x^\gamma} \leqq G + \varepsilon \qquad \text{for all} \quad x \geqq x_1\,.$$

Then for $x \geqq x_1$,

$$|g(x+1)| \leqq \frac{2}{x_1} \int_0^{x_1} |g(t)|\,dt + \frac{2}{x} \int_0^x (G+\varepsilon)\,t^\gamma\,dt + \eta(x)\,,$$

where $\eta(x)$ denotes the error term in (3.8).

Hence

$$G = \limsup_{x\to\infty} \frac{g(x+1)}{(x+1)^\gamma} \leqq 0 + \frac{2(G+\varepsilon)}{\gamma+1} + \limsup_{x\to\infty} \frac{\eta(x)}{(x+1)^\gamma}.$$

Suppose now that (3.8) holds with O replaced by o. Since $\varepsilon > 0$ was arbitrary it follows that

$$G \leqq \frac{2G}{\gamma+1},$$

and since $\gamma > 1$ it follows that $G = 0$. Hence Theorem 2 holds if O is replaced by o in both (3.8) and (3.9).

2. Theorem 1 continues to hold if (3.1) is replaced by

$$(3.1)' \qquad f(x+1) = \frac{2}{x}\int_0^x f(t)\,dt + \frac{C}{x} + p(x+1) \qquad (x > 0)$$

where C is any constant; this follows from the fact that the fundamental relation (3.12) follows from (3.1)′. Thus e.g. if $p(x+1)$ in (3.1) is of the form

$$\frac{C}{x} + O\left(\left(\frac{a}{x}\right)^{x+\beta}\right) \qquad (a > 2e)$$

then (3.7) still holds.

4. Application to the parking problem. Since $f(x) = \mu(x) + 1$ satisfies, by (2.5), the equation (3.1) with $p \equiv 0$, we have by Theorem 1 that

$$(4.1) \qquad \lim_{x\to\infty} \frac{\mu(x)}{x} = \lambda_1$$

exists, and by (3.31) for every $x > 0$,

$$(4.2) \qquad \inf_{x\leqq t\leqq x+1} \frac{\mu(t)+1}{t+1} = I_x \leqq \lambda_1 \leqq S_x = \sup_{x\leqq t\leqq x+1} \frac{\mu(t)+1}{t+1}.$$

Taking $x = 2$ we obtain easily from (2.8) that

$$(4.3) \qquad 0.66\ldots = \frac{2}{3} \leqq \lambda_1 \leqq 3 - \sqrt{5} = 0.76\ldots,$$

and (2.9) yields much narrower bounds. Since I_x and S_x approach λ_1 very rapidly it is easy to obtain extremely good approximations from (4.2) (cf. (1.2)). Since $\mu(x) = 1$ for $1 \leqq x \leqq 2$, even the crude approximation $1/2 < \lambda_1 < 1$ yields

$$(4.4) \qquad \begin{aligned} \sup_{1\leqq x\leqq 2} |\mu(x) + 1 - \lambda_1 x - \lambda_1| &= \max_{1\leqq x\leqq 2} |2 - \lambda_1 x - \lambda_1| = \\ &= \max(|2 - 2\lambda_1|, |2 - 3\lambda_1|) < 1. \end{aligned}$$

Hence from Theorem 1 with $p \equiv 0$ we have

Theorem 3. *There exists a constant* $\lambda_1 \left(\frac{1}{2} < \lambda_1 < 1\right)$ *such that the expectation* $\mu(x)$ *of* N_x *satisfies the relation*

$$\sup_{n+1 \leqq x \leqq n+2} |\mu(x) + 1 - \lambda_1 x - \lambda_1| < \frac{2^n}{n!} \qquad (n = 0, 1, \ldots). \tag{4.5}$$

By Stirling's formula it follows that

$$\mu(x) - \lambda_1 x - \lambda_1 + 1 = O\left(\left(\frac{2e}{x}\right)^{x-3/2}\right). \tag{4.6}$$

We now define $L(x)$ and $\varphi_k(x)$ by (2.14) and (2.15) with λ_1 given by (4.1) Then by (2.18) with $k = 2$ we have

$$\varphi_2(x+1) = \frac{2}{x}\int_0^x \varphi_2(t)\,dt + \frac{2}{x}\int_0^x \varphi_1(t)\,\varphi_1(x-t)\,dt \qquad (x > 0) \tag{4.7}$$

But $\varphi_1(x)$ is precisely the left hand member of (4.6), and therefore

$$\sup_{o<t<x} |\varphi_1(t)\,\varphi_1(x-t)| = O\left(\left(\frac{4e}{x}\right)^{x-3}\right). \tag{4.8}$$

Thus $f(x) = \varphi_2(x)$ satisfies (3.1) with $p(x)$ estimated by (4.8). From this we deduce

Theorem 4. *There exists a constant* $\lambda_2 > 0$ *such that the variance* $\sigma^2(x)$ *of* N_x *satisfies the relation*

$$\sigma^2(x) = \lambda_2 x + \lambda_2 + O\left(\left(\frac{4e}{x}\right)^{x-4}\right). \tag{4.9}$$

Proof. $\varphi_2(x)$ satisfies (4.9) by the Corollary to Theorem 1, and

$$\sigma^2(x) - \varphi_2(x) = -(\varphi_1(x))^2,$$

which, by (4.6), is absorbed into the error term. It remains to show that $\lambda_2 > 0$. This may be done numerically from estimates obtained in the course of the proof of Theorem 1, but it is much simpler to deduce it as follows. Since $\sigma^2(x) \neq$ $\neq 0$ for $2 < x < 3$ it follows from (2.13) that $\sigma^2(x) > \frac{\delta}{x}$ for some $\delta > 0$. But this contradicts (4.9) unless $\lambda_2 > 0$.

We now prove a result on the central moments of N_x.

Theorem 5. For every $k = 1, 2, \ldots$ and $\varepsilon > 0$,

$$\mathbf{E}\left((N_x - \mu(x))^k\right) = c_k x^{\left[\frac{k}{2}\right]} + O\left(x^{\left[\frac{k}{2}\right]-1+\varepsilon}\right) \tag{4.10}$$

($[x]$ denotes the greatest integer $\leqq x$), where the c_k are constants and

$$c_{2k} = \frac{(2k)!}{2^k\,k!}\lambda_2^k. \tag{4.11}$$

Proof. Since by (2.15) for $k = 1$

$$N_x - \mu(x) = N_x - L(x) - (\mu(x) - L(x)) = N_x - L(x) - \varphi_1(x),$$

it follows from (4.6) that (4.10) is equivalent to

$$\varphi_k(x) = c_k x^{\left[\frac{k}{2}\right]} + O(x^{\left[\frac{k}{2}\right]-1+\varepsilon}). \tag{4.12}$$

By (4.6) and (4.9), (4.10) holds for $k = 1,2$ and (4.11) holds for $k = 1$. By (2.18)

$$\varphi_3(x+1) = \frac{2}{x}\int_0^x \varphi_3(t)\,dt + \frac{6}{x}\int_0^x \varphi_1(t)\,\varphi_2(x-t)\,dt,$$

and by (4.6) and (4.9) the second integrand is $O\left(\left(\frac{C}{x}\right)^x\right)$ with a suitable C. Hence φ_3 satisfies (3.1) with p estimated as in (3.6). It follows from (3.7) that $\varphi_3(x) = c_3 x + O\ (1)$ and thus (4.12) holds for $k \leq 3$.

Now let $m > 3$ and assume that (4.12) holds for $k < m$. Then by (2.18),

$$\varphi_m(x+1) = \frac{2}{x}\int_0^x \varphi_m(t)\,dt + \sum_{i=1}^{m-1}\binom{m}{i}\frac{1}{x}\int_0^x \varphi_i(t)\,\varphi_{m-i}(x-t)\,dt \quad (x > 0). \tag{4.13}$$

By the induction assumption

$$\varphi_i(t)\,\varphi_{m-i}(x-t) = c_i c_{m-i} t^{\left[\frac{i}{2}\right]}(x-t)^{\left[\frac{(m-i)}{2}\right]} + O\left(x^{\left[\frac{i}{2}\right]+\left[\frac{m-i}{2}\right]-1+\varepsilon}\right). \tag{4.14}$$

Since

$$\frac{1}{x}\int_0^x t^{\left[\frac{i}{2}\right]}(x-t)^{\left[\frac{m-i}{2}\right]}dt = \frac{\left[\frac{i}{2}\right]!\left[\frac{m-i}{2}\right]!}{\left(\left[\frac{i}{2}\right]+\left[\frac{m-i}{2}\right]+1\right)!}\,x^{\left[\frac{i}{2}\right]+\left[\frac{m-i}{2}\right]} \tag{4.15}$$

and since

$$\max_{1\leq i\leq m-1}\left(\left[\frac{i}{2}\right]+\left[\frac{m-i}{2}\right]\right) = \left[\frac{m}{2}\right] \qquad \text{for } m \geq 3,$$

the sum on the right hand side of (4.13) is

$$\text{Const.}\ x^{\left[\frac{m}{2}\right]} + O(x^{\left[\frac{m}{2}\right]-1+\varepsilon}). \tag{4.16}$$

Since $\left[\frac{m}{2}\right] \geq 2$ for $m > 3$, (4.12) for $k = m$ follows from (4.13) by the Corollary of Theorem 2. Thus (4.12) holds for all $k = 1, 2, \ldots,$

By (4.13), (4.14), and (4.15) the constant in (4.16) for $m = 2k$ is

$$\sum_{j=1}^{k-1} \binom{2k}{2j} j! \frac{(k-j)!}{(k+1)!} c_{2j} c_{2k-2j} . \tag{4.17}$$

Assume that (4.11) holds for $c_2, c_4, \ldots, c_{2k-2}$. By (4.17) the coefficient of x^k in the equation

$$\varphi_{2k}(x+1) = \frac{2}{x} \int_0^x \varphi_{2k}(t)\, dt + cx^k + O(x^{k-1+\varepsilon})$$

is

$$\frac{(k-1)(2k)!}{(k+1)!\, 2^k} \lambda_2^k ,$$

so that by the Corollary of Theorem 2

$$\varphi_{2k}(x) = \frac{k+1}{k-1} \frac{(k-1)(2k)!}{(k+1)!\, 2^k} \lambda_2^k x^k + O(x^{k-1+\varepsilon}) ,$$

and hence

$$c_{2k} = \frac{(2k)!}{k!\, 2^k} \lambda_2^k ,$$

so that (4.11) holds for all $k = 1, 2, \ldots$. This completes the proof of Theorem 5.

Theorem 6. *The random variable*

$$Z_x = \frac{N_x - \mu(x)}{\sigma(x)}$$

is asymptotically normal (0,1) *as* $x \to \infty$.

Proof. By (4.10), (4.11) and (4.9) for $\varepsilon = 1/2$,

$$\mathbf{E}(Z_x^k) = \frac{c_k x^{\left[\frac{k}{2}\right]} + o\left(x^{\left[\frac{k}{2}\right]}\right)}{(\lambda_2 x + o(x))^{\frac{k}{2}}}$$

where $\lambda_2 > 0$ and

$$c_{2k} = \frac{(2k)!}{2^k k!} \lambda_2^k \qquad (k = 1, 2, \ldots).$$

Hence

$$\lim_{x\to\infty} \mathbf{E}(Z_x^k) = \begin{cases} \dfrac{k!}{2^{\frac{k}{2}} \left(\dfrac{k}{2}\right)!} & (k \text{ even}), \\ 0 & (k \text{ odd}). \end{cases}$$

Sihce these are the moments of the normal (0,1) distribution which is uniquely determined by its moments, the theorem follows from the moment convergence theorem.

5. Another proof of the asymptotic normality. This proof will use very much less information about the moments of N_x than that of the preceding section. In fact it will be based entirely on the relation

$$\sigma^2(x) = \lambda_2 x + o(x)\,, \qquad (\lambda_2 > 0)\,. \tag{5.1}$$

We shall need two simple lemmas.

Lemma 1. *Let $\psi(x)$ be a non-negative function defined for $x \geqq 0$, bounded over finite intervals and satisfying $\psi(x) = o(x)$. Then $n = o(x)$ implies*

$$\sup \sum_{i=1}^{n} \psi(x_i) = o(x)\,, \tag{5.2}$$

the sup being taken over all sets of non-negative $x_1, \ldots, x_n$ with $x_1 + \ldots + x_n = x$.

Indeed, $\psi(x) < H + Hx$ for all $x \geqq 0$ with a suitable H. Let $\delta > 0$ be given and choose $a = a(\delta)$ so that $\psi(x) < \delta\, x$ for $x > a$. Divide the sum in (5.2) into two parts, one over the i with $x_i \leqq a$, the other over the remaining i. Then the first sum is $\leqq n(H + H\,a)$ while the second is $< \delta x$. Hence the left side of (5.2) is bounded by $2\delta x$ for large x and the lemma is established.

Lemma 2. *Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $Y_0, Y_1, \ldots, Y_n$ are independent random variables satisfying*

$$\left| \sum_{i=0}^{n} \mathbf{E}(Y_i) \right| \leqq \delta\,, \tag{5.3}$$

$$\left| \sum_{i=0}^{n} \mathbf{D}^2(Y_i) - 1 \right| \leqq \delta\,, \tag{5.4}$$

$$| Y_i - \mathbf{E}(Y_i) | \leqq \delta\,, \qquad (i = 0, 1, \ldots, n), \tag{5.5}$$

then the distribution function of $\sum_{i=0}^{n} Y_i$ approximates uniformly to within ε the normal distribution with zero mean and unit variance.

It is clearly sufficient to establish the lemma with (5.3) replaced by $\mathbf{E}(Y_i) = 0$ $(i = 0, 1, \ldots, n)$ and δ replaced by 0 in (5.4). But then the lemma follows at once from the 'triangular' version of Liapounov's theorem.

We now proceed to the proof of the asymptotic normality of

$$Z_x = \frac{N_x - \mu(x)}{\sigma(x)}\,. \tag{5.6}$$

Let $n = n_x$ be a fixed non-negative integer-valued function of x defined for $x > 2$ and satisfying

$$0 \leqq n_x \leqq x/2\,, \quad n_x = o(x)\,. \tag{5.7}$$

(Eventually it will be specified further.) Consider the first $n - 1 \geqq 1$ cars parked on $[0, x]$. Denote by y_1 the distance between 0 and the leftmost car, by y_2 that between this car and the one parked second from the left, etc., by y_n the distance between the car parked to the extreme right and x. Then (see the derivation of the italicized statement in Section 2) the conditional distribution of N_x given $\underline{y} = (y_1, y_2 \dots, y_n)$ is the same as the distribution of $n - 1 + N_{y_1} + N_{y_2} + \dots + N_{y_n}$ with $N_{y_1}, N_{y_2}, \dots, N_{y_n}$ independent. Therefore, the conditional distribution of Z_x given $\underline{y}$ is equal to the distribution of $\sum_{i=0}^{n} Y_i$, with the Y_i independent and defined by

$$(5.8) \qquad Y_i = \frac{N_{y_i}}{\sigma(x)} \ (i = 1, \dots, n), \quad Y_0 = \frac{n - 1 - \mu(x)}{\sigma(x)}.$$

Applying Lemma 1 with $\psi(x) = | \sigma^2(x) - \lambda_2 x |$ we deduce from (5.1) and (5.7) that

$$\sum_{i=1}^{n} | \sigma^2(y_i) - \lambda_2 y_i | = o(x), \quad \text{or} \quad \sum_{i=1}^{n} \sigma^2(y_i) = \lambda_2 x + o(x)$$

for every $\underline{y}$. Hence we obtain

$$(5.9) \qquad \mathbf{D}^2(Z_x | \underline{y}) = \sum_{i=0}^{n} \mathbf{D}^2(Y_i | \underline{y}) = 1 + o(1)$$

for the conditional variance of Z_x. Thus (5.4) holds for $Y_i = Y_i(\underline{y})$ for all sufficiently large x and all random vectors $\underline{y}$.

From

$$1 = \mathbf{E}(Z_x^2) = \mathbf{E}\{\mathbf{D}^2(Z_x | \underline{y}) + \mathbf{E}^2(Z_x | \underline{y})\}$$

and (5.9) we see that

$$(5.10) \qquad \mathbf{E}(\mathbf{E}^2(Z_x | \underline{y})) = o(1)$$

Let A_x be the event: $\underline{y}$ is such that $| \mathbf{E}(Z_x | \underline{y}) | \leq \delta$; then it follows from (5.10) that for any fixed $\delta > 0$,

$$(5.11) \qquad \lim_{x \to \infty} \mathbf{P}(A_x) = 1 ,$$

and $\underline{y} \in A_x$ implies that $Y_i = Y_i(\underline{y})$ satisfy (5.3).

We now specify the function $n = n_x$ by putting

$$(5.12) \qquad n = [x^{1/2} \log^2 x]$$

and let $B_x = B_x(\eta)$ denote the event

$$(5.13) \qquad \max_{i=1,\dots,n} y_i < \eta x^{1/2}, \qquad (\eta > 0)$$

Take $k = \left[\frac{2x^{1/2}}{\eta}\right] + 1$ and divide $[0, x]$ into k equi-long intervals $I_1, \dots, I_k$. Then if (5.13) were false it would imply that at least one of the intervals I_j $(j = 1, \dots, k)$ is disjoint from the first $n - 1$ cars parked. The probability of this is smaller than

$$k\left(1 - \frac{1}{k}\right)^{n-1} < \left(\frac{2x^{1/2}}{\eta} + 1\right)\left(1 - \frac{\eta}{2x^{1/2}}\right)^{x^{1/2}\log^2 x - 2}$$

and, thus, tends to zero as $x \to \infty$. Hence

$$\lim_{x \to \infty} \mathbf{P}(B_x) = 1 . \tag{5.14}$$

But $Y_0(\underline{y})$ is a constant and, taking $\eta < \frac{\delta}{2} \lambda_2^{1/2}$, (5.13) implies $|Y_i(\underline{y})| < \delta$ $(i = 1, \ldots, n)$ for large x and hence that $Y_i = Y_i(\underline{y})$ satisfy (5.5).

From the above and Lemma 2 we conclude that the conditional distribution of Z_x given $A_x \cap B_x$ is asymptotically normal with zero mean and unit variance. It then follows from (5.11) and (5.14) that the same holds for the distribution of Z_x itself, and the proof is complete.

6. Remarks. 1. The parking process described in Section 1 may also be described as the process of taking independent observations on a rectangular random variable, but rejecting all those observations which differ by less than unity from any previously observed and not rejected observation. The retained observations form a finite dependent stochastic sequence and we have studied the asymptotic behaviour of the length of this sequence. It would be interesting to extend the results to other kinds of dependence, and the preceding section indicates such possibilities; however, one would have to prove some relations like (5.1) and (5.2) and we do not know how to do this under reasonably general assumptions (see, however, the next remark).

2. Returning to the parking problem, we may equivalently consider a street of unit length and cars of length $1/x$ with x tending to infinity. This suggests at once generalizing the problem by replacing the rectangular density by other probability densities. Assume e.g. that the position of the center of each parked car is a random variable whose density is constant on each half of the street but that the constants in the two halves are different. Even in this simple case it is not quite trivial to prove rigorously that the expected total number of cars parked will be approximately $\lambda_1 x$, of those parked in the left half approximately $\lambda_1 x/2$ etc. However, the technique of the end of Section 5 can be used here. This makes it possible to treat densities which are step functions etc.; we expect to study in a future paper the case of continuous densities.

3. In the uniform density case the distribution of the lengths of the empty spaces between the parked cars has been considered by G. Bánkövi [4].

4. It is natural to consider the parking problem in more dimensions. No functional equation similar to the one derived here is available, and a rigorous treatment becomes extremely difficult. In the plane one would consider, say, placing unit squares, with sides parallel to the axes, uniformly in a convex region. Such curiosities occur as lowering the expected total of squares placed while increasing the region (consider, in the (u, v) plane the regions

$$-5/4 \leqq u \leqq 5/4,\ 0 \leqq v \leqq 1 \text{ and } v \geqq 0,\ u + v \leqq 9/4,\ v - u \leqq 9/4) .$$

In the one-dimensional case it is clear from the functional equation (transformed as in (3.12)) that $\mu(x)$ is monotone, but the analogous result for homothetic regions in the plane is not evident, even if we confine ourselves to regions which are squares. Some numerical studies of the problem of placing squares in the plane have been carried out by Mrs. I. PALÁSTI, [5].

5. Differentiating (2.3) we have $x\mu'(x+1) + \mu(x+1) = 2\mu(x) + 1$. In view of (4.6) $\mu'(x)$ is approximated extremely closely by $\lambda_1(x+1)/(x-1)$. Higher derivatives may be treated similarly ($\mu(x)$ is, of course, n times differentiable for $x > n$). The same remarks apply to $\sigma^2(x)$ etc.

6. The estimates of the error involved in Theorem 1 can be somewhat sharpened, but this necessitates much work and we seem to have reached the point of diminishing returns. It may be more interesting to study other functional equations by the same method.

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О ЗАДАЧЕ »ПАРКИРОВАНИЯ«

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Резюме

В работе [1] A. RÉNYI исследовал одномреную задачу о случайном. заполнении пространства (модель «паркирования»). Процедура состоит в последовательном расположении на отрезке $(0, x)$ случайным образом непересекающихся единичных отрезков. Число расположимых отрезков N_x — случайная величина.

Авторы исследуют асимптотическое поведение моментов величины N_x ((4.6), (4.9), (4.10)). Доказывается двумя способами, что величина Z_x (нормированная величина N_x) имеет асимптотически нормальное распределение с параметрами (0,1) при $x \to \infty$.