ON SEQUENCES OF QUASI-EQUIVALENT EVENTS II

by

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Introduction

In [1] we have introduced the following:

Definition. The events A_n defined on a probability space $\{\Omega, \mathcal{S}, \mathbf{P}\}$ are called quasi-equivalent if the value of the ratio

$$\frac{\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k})}{\mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})} = a_k \quad (i_j \neq i_l \text{ if } j \neq l) \ (\mathbf{P}(A_i) > 0)$$

depends only on k and it does not depend on the indices $i_1, i_2, \ldots, i_k$ ($k = 1, 2, \ldots$). The numbers $\alpha_1, \alpha_2, \ldots$ are called the moments of the quasi-equivalent events $A_1, A_2, \ldots$.

The paper [1] contains the characterization of infinite sequences of quasi-equivalent events under the restriction

$$\liminf_{n\to\infty} \mathbf{P}(A_n) > 0 .$$

The main result of [1] can be summarized as follows:

Theorem A. *Let $A_1, A_2, \ldots$ be a sequence of quasi-equivalent events defined on the probability space $\{\Omega, \mathcal{S}, \mathbf{P}\}$ such that*

$$\liminf_{n\to\infty} \mathbf{P}(A_n) > 0 .$$

Then there exists a random variable $\lambda(\omega)$ defined on $\{\Omega, \mathcal{S}, \mathbf{P}\}$ with the following properties:

(1) $$\mathbf{P}\left\{0 \leq \lambda(\omega) \leq \inf_k \frac{1}{\mathbf{P}(A_k)}\right\} = 1$$

(2) $$\mathbf{M}(\lambda^k) = a_k \qquad (k = 1, 2, \ldots)$$

where $\alpha_1, \alpha_2, \ldots$ are the moments of the events $A_1, A_2, \ldots$.

(3) $$\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k} \mid \lambda) = \mathbf{P}(A_{i_1} \mid \lambda) \mathbf{P}(A_{i_2} \mid \lambda) \ldots \mathbf{P}(A_{i_k} \mid \lambda) = \lambda^k \mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k}) \quad \text{(with probability 1)} \quad (i_j \neq i_l \text{ if } j \neq l)$$

(4) $$\mathbf{P}\left\{\frac{1}{n} \sum_{k=1}^{n} \frac{a_k(\omega)}{\mathbf{P}(A_k)} \to \lambda(\omega)\right\} = 1$$

where $a_k(\omega)$ is the indicator function of A_k,

(5) $$\prod_{n=1}^{\infty} \mathscr{B}(A_n, A_{n+1}, \ldots) = \mathscr{B}(\lambda)$$

where $\mathscr{B}(A_n, A_{n+1}, \ldots)$ is the smallest σ-algebra which contains the events $A_n, A_{n+1}, \ldots$ and $\mathscr{B}(\lambda)$ is the smallest σ-algebra with respect to which $\lambda(\omega)$ is measurable. We say that two σ-algebras $\mathscr{F}$ and $\mathscr{G}$ are equal to each other if for every $F \in \mathscr{F}$ there exists a $G \in \mathscr{G}$ such that $\mathbf{P}(F \circ G) = 0$ and conversely for every $G \in \mathscr{G}$ there exists an $F \in \mathscr{F}$ such that $\mathbf{P}(F \circ G) = 0$.

The aim of the present paper is to obtain the characterization of the infinite sequences of quasi equivalent events substituting the condition $\liminf_{n \to \infty} \mathbf{P}(A_n) > 0$ by weaker conditions.

§ 1. The formulation of Theorems 1 and 2

In the present paper we will study the properties of an infinite sequence of quasi-equivalent events $A_1, A_2, \ldots$ under the following conditions:

Condition a:

$$\sum_{k=1}^{\infty} \frac{1}{k^2\, \mathbf{P}(A_k)} < +\infty .$$

Condition b:

$$\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{\mathbf{P}(A_k)} \to 0 \qquad (n \to \infty) .$$

Condition c:

$$\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{\mathbf{P}(A_k)} \leq K$$

where K is a positive constant which does not depend on n.

Condition d:

$$\sum_{k=1}^{\infty} \mathbf{P}(A_k) = +\infty .$$

It is easy to see that the Condition a implies the Condition b, the Condition b implies the Condition c and the Condition c implies the Condition d.

We ask: do these conditions imply the statements of Theorem A. More exactly we will prove that if $A_1, A_2, \ldots$ is a sequence of quasi-equivalent events defined on a probability space $\{\Omega, \mathscr{S}, \mathbf{P}\}$ then there exists a random variable $\lambda(\omega)$ defined on $\{\Omega, \mathscr{S}, \mathbf{P}\}$ having some of the following properties:

Property 1.

$$\mathbf{P}\left\{0 \leq \lambda(\omega) \leq \inf_k \frac{1}{\mathbf{P}(A_k)}\right\} = 1 ,$$

Property 2.

$$\mathbf{M}(\lambda^k) = a_k \qquad (k = 1, 2, \ldots)$$

where $a_1, a_2, \ldots$ are the moments of the events $A_1, A_2, \ldots$.

Property 3.

$$\mathbf{P}(A_{i_1} A_{i_2} \dots A_{i_k} \mid \lambda) = \mathbf{P}(A_{i_1} \mid \lambda) \mathbf{P}(A_{i_2} \mid \lambda) \dots \mathbf{P}(A_{i_k} \mid \lambda) =$$
$$= \lambda^k \mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \dots \mathbf{P}(A_{i_k}) \qquad \text{(with probability 1)}$$
$$(i_j \neq i_l \quad \text{if} \quad j \neq l)$$

Property 4.

$$\mathbf{P}\{\psi_n \to \lambda\} = 1$$

where $\psi_n = \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{P(A_k)}$ $(n = 1, 2, \dots)$ and $a_k(\omega)$ is the indicator function of A_k

Property 4*.

$$\mathbf{M}[(\psi_n - \lambda)^2] \to 0 \qquad (n \to \infty)$$

Property 4.**

$$\mathbf{M}(\psi_n \varphi) \to \mathbf{M}(\lambda \varphi) \qquad (n \to \infty)$$

for every random variable φ having finite variance.

Property 5.

$$\prod_{n=1}^{\infty} \mathscr{B}(A_n, A_{n+1}, \dots) = \mathscr{B}(\lambda)$$

where $\mathscr{B}(A_n, A_{n+1}, \dots)$ is the smallest σ-algebra which contains the events $A_n, A_{n+1}, \dots$ and $\mathscr{B}(\lambda)$ is the smallest σ-algebra with respect to which $\lambda(\omega)$ is measurable. (The equality of two σ-algebras was defined in the introduction).

Now our main result is the following:

Theorem 1.

	1	2	3	4	4*	4**	5
a	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$
b	$\to$	$\to$	$\to$	$\not\to$	$\to$	$\to$	$\to$
c	$\to$	$\to$	$\to$	$\not\to$	$\not\to$	$\to$	$\to$

The sign $\to$ resp. $\not\to$ in the i-th row of the k-th column of the table means that the i-th condition implies (resp. does not imply) the k-th property.

Our next theorem is the generalization of Theorem 2b of [1].

Theorem 2. *Let $A_1, A_2, \dots$ be a sequence of quasi-equivalent events for which the Condition c is valid and let $K = \inf_k \frac{1}{\mathbf{P}(A_k)}$. Then we can construct a sequence $A_1^*, A_2^*, \dots$ from the measurable subsets of the rectangle $[0, K] \times [0, 1]$ of the plane such that*

$$\mathbf{P}(A_{i_1} A_{i_2} \dots A_{i_k}) = \mu(A_{i_1}^* A_{i_2}^* \dots A_{i_k}^*)$$

$\mu = \nu \times \lambda$ where ν is a Lebesgue—Stieltjes measure on the interval $[0, K]$ and λ is the ordinary Lebesgue measure on $[0, 1]$ and if B_i is the common part of A_i^* and the line $x = x_0$ $(0 \leqq x_0 \leqq K)$ then

$$\lambda(B_{i_1} B_{i_2} \dots B_{i_k}) = \lambda(B_{i_1}) \lambda(B_{i_2}) \dots \lambda(B_{i_k}) = \left(\frac{x_0}{K}\right)^k.$$

§ 2. The proof of Theorems 1 and 2

In the proof we will apply many times the following

Lemma (see [2]) *If H is a Hilbert space and f_n is a sequence of elements of H such that*

$$\lim_{n\to\infty} (f_n, f_k) = \lambda_k \qquad (k = 1, 2, \dots)$$

and

$$\|f_n\| \leqq C$$

where C is a positive constant and λ_k is a sequence of real numbers. Then f_n converges weakly to an element f of the Hilbert space H i.e.

$$(f_n, g) \to (f, g) \qquad (n \to \infty)$$

for every element g of H.

First of all we will prove that the Condition c implies the Property 4**. To prove this fact it is enough to check that the conditions of the above mentioned Lemma hold if we substitute f_n by

$$\psi_n = \frac{1}{n} \sum_{k=1}^{n} \frac{a_k(\omega)}{\mathbf{P}(A_k)},$$

I.e. we have to prove that

$$\mathbf{M}(\psi_n^2) \leqq C \qquad (n = 1, 2, \dots) \tag{6}$$

and

$$\lim_{n\to\infty} \mathbf{M}(\psi_n \psi_k) \tag{7}$$

exists for every k. (6) follows from the following formula

$$\mathbf{M}(\psi_n^2) = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{\mathbf{P}(A_k)} + \frac{2}{n^2} \sum_{k<j} \frac{\mathbf{P}(A_k A_j)}{\mathbf{P}(A_k)\mathbf{P}(A_j)} =$$

$$= \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{\mathbf{P}(A_k)} + \frac{2}{n^2} \binom{n}{2} a_2 .$$

Similarly we have

$$\lim_{n\to\infty} \mathbf{M}(\psi_n \psi_k) = \lim_{n\to\infty} \frac{1}{nk} \mathbf{M}\left[\sum_{j=1}^{n} \frac{a_j(\omega)}{\mathbf{P}(A_j)} \sum_{l=1}^{k} \frac{a_l(\omega)}{\mathbf{P}(A_l)}\right] =$$

$$= \lim_{n\to\infty} \frac{1}{nk} a_2 (n - k) k = a_2$$

which implies (7). So we have already proved that Condition c implies Property 4.** (Let the weak limit of the sequence ψ_n be $\lambda(\omega)$.)

Our next step is to prove that Condition c implies the Property 2, more exactly we prove that Property 4** implies Property 2. The Property 2 for $k = 1$ is trivial because

$$1 = \mathbf{M}(\psi_n) = \mathbf{M}(\lambda).$$

The proof for $k = 2$ is the following:

$$a_2 = \lim_{k\to\infty} \lim_{n\to\infty} \mathbf{M}(\psi_n \psi_k) = \lim_{k\to\infty} \mathbf{M}(\lambda\psi_k) = \mathbf{M}(\lambda^2).$$

Similarly for $k = 3$ we have

$$a_3 = \lim_{l\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} \mathbf{M}(\psi_n \psi_k \psi_l) = \lim_{l\to\infty} \lim_{k\to\infty} \mathbf{M}(\lambda \psi_k \psi_l) =$$

$$= \lim_{l\to\infty} \mathbf{M}(\lambda^2 \psi_l) = \mathbf{M}(\lambda^3).$$

The proof for any k is completely the same.

Now we prove that

$$a_k \leqq \left(\frac{1}{\mathbf{P}(A_l)}\right)^k \qquad (l = 1, 2, \ldots, k = 1, 2, \ldots) \tag{8}$$

(8) is trivial for $k = 1$. The proof for $k = 2$ is the following

$$a_2 = \frac{\mathbf{P}(A_l A_{l+1})}{\mathbf{P}(A_l)\,\mathbf{P}(A_{l+1})} \leqq \frac{\mathbf{P}(A_{l+1})}{\mathbf{P}(A_l)\,\mathbf{P}(A_{l+1})} = \frac{1}{\mathbf{P}(A_l)} \leqq \frac{1}{\mathbf{P}^2(A_l)}.$$

For $k = 3$ similarly we have

$$a_3 = \frac{\mathbf{P}(A_l A_{l+1} A_{l+2})}{\mathbf{P}(A_l)\,\mathbf{P}(A_{l+1})\,\mathbf{P}(A_{l+2})} \leqq \frac{\mathbf{P}(A_{l+1} A_{l+2})}{\mathbf{P}(A_l)\,\mathbf{P}(A_{l+1})\,\mathbf{P}(A_{l+2})} = \frac{a_2}{\mathbf{P}(A_l)} \leqq \frac{1}{\mathbf{P}^3(A_l)}.$$

By induction we can obtain (8) for any k.

Now we prove that Condition c implies Property 1. More exactly we prove that Property 2. and (8) imply Property 1. In fact if

$$\mathbf{M}(\lambda^k) = a_k \leqq \left(\frac{1}{\mathbf{P}(A_l)}\right)^k \quad (k = 1, 2, \ldots; l = 1, 2, \ldots)$$

then

$$\mathbf{P}\left(0 \leqq \lambda \leqq \frac{1}{\mathbf{P}(A_l)}\right) = 1 \qquad (l = 1, 2, \ldots)$$

and this relation implies the Property 1.

Now we can already prove Property 3 by exactly the same method which was used in the proof of Theorem 3 in [1], therefore we do not give in detail this part of our proof. Similarly we do not give in detail the proof of our Theorem 2 because its proof is exactly the same as the proof of Theorem 2b in [1].

Using Theorem 2 and the well known zero-one law we obtain that Condition c implies Property 5.

The fact that Condition b implies Property 4* can be proven by a simple calculation. Property 4 follows from Condition a using Theorem 2 and the well known KOLMOGOROV's strong law of large numbers.

Our last step is to prove that Condition c does not imply Property 4* and Condition b does not imply Property 4. We also do not detail these statements because these statements are well known for independent random variables (Cf [2] pp. 204).

So the proof of our Theorems 1 and 2 are complete.

Remark. It is easy to see that if Condition d does not hold then in general there does not exist a random variable $\lambda(\omega)$ having any of the mentioned Properties. This fact is shown by the following example: Let $A_1, A_2, \ldots$ be a sequence of events defined in the interval $[0, 1]$ such that

$$\mathbf{P}(A_i) = \frac{1}{2^i}$$

$$\mathbf{P}(A_i A_j) = \mathbf{P}(A_i)\,\mathbf{P}(A_j) \qquad \text{if } i \neq j$$

$$\mathbf{P}(A_i A_j A_k) = 0 \qquad \text{if } i \neq j \;\; i \neq k \text{ and } j \neq k\,.$$

It is easy to see that this sequence of events can be constructed.

We do not know what happens if Condition d holds but Condition does not hold.

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О КВАЗИЭКВИВАЛЕНТНЫХ ПОСЛЕДОВАТЕЛЬНОСТЯХ СОБЫТИЙ II.

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Резюме

Последовательность событий $A_1, A_2, \ldots$ называется квазиэквивалентной, если значение дроби

$$a_k = \frac{\mathbf{P}(A_{i_1}, A_{i_2}, \ldots A_{i_k})}{\mathbf{P}(A_{i_1})\,\mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})} \qquad (i_j \neq i_l, \text{ если } j = l)$$

зависит лишь от k и не зависит от индекхов $i_1, i_2, \ldots, i_k$.

В настоящей рабoте изучаются свойства последовательности квазиэквивалентных событий $A_1, A_2, \ldots$ при следующих условиях:

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2\,\mathbf{P}(A_k)} < +\infty,$

b) $\displaystyle\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{\mathbf{P}(A_k)} \to 0 \qquad (n \to \infty)\,,$

c) $\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\mathbf{P}(A_k)} \leqq K$, где K положительная постоянная, независящая от n,

d) $\sum_{k=1}^{\infty}\mathbf{P}(A_k) = +\infty$.

Доказывается, что если выполняются некоторые из этих условий, то существует случайная величина λ со следующими свойствами:

1) $\mathbf{P}\left\{0 \leqq \lambda(\omega) \leqq \inf_k \frac{1}{\mathbf{P}(A_k)}\right\} = 1$,

2) $\mathbf{M}(\lambda^k) = a_k \qquad (k = 1, 2, \ldots)$,

3) $\mathbf{P}(A_{i_1} A_{i_2} \ldots A_{i_k} \mid \lambda) = \mathbf{P}(A_{i_1} \mid \lambda) P(A_{i_2} \mid \lambda) \ldots \mathbf{P}(A_{i_k} \mid \lambda) =$

$= \lambda^k \cdot \mathbf{P}(A_{i_1}) \mathbf{P}(A_{i_2}) \ldots \mathbf{P}(A_{i_k})$ с вероятностью 1 $(i_1 < i_2 < \ldots < i_k)$

4) $\mathbf{P}\{\psi_n \to \lambda\} = 1$, где $\psi_n = \frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{\mathbf{P}(A_k)} \qquad (n = 1, 2, \ldots)$,

и $a_k(\omega)$ — индикаторная функция события A_k.

4*) $\mathbf{M}[(\psi_n - \lambda)^2] \to 0 \qquad (n \to \infty)$

4**) для любой интегрируемой с квадратом случайной величины φ

$$\mathbf{M}(\psi_n \varphi) \to \mathbf{M}(\lambda \varphi) \qquad (n \to \infty)$$

5) $\prod_{n=1}^{\infty} \mathscr{B}(A_n, A_{n+1}, \ldots) = \mathscr{B}(\lambda)$,

где $\mathscr{B}(A_n, A_{n+1}, \ldots)$ обозначает σ - алгебру, порожденную событиями $A_n, A_{n+1} \ldots$, а $\mathscr{B}(\lambda)$ обозначает σ - алгебру, порожденную случайной величиной $\lambda(\omega)$. Две σ - алгебры считаются равными, если любой элемент одной из них отличается от некоторого элемента другой лишь на множестве меры нуль и наоборот.

Точную связь между нашими условиями и упомянутыми свойствами случайной величины λ выражает теорема 1.

Легко видеть, что если не выполняется условие d), то, вообще говоря, не существует случайной величины λ, соответствующей любому из свойств 2), 3), 4), 4*), 4**).

Еще не решена проблема относительно того, что произойдет, если условие d) выполнено, а условие c) нет.