2-semiarcs in $PG(2, q), q \leq 13$

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Abstract

A 2-semiarc is a pointset S_2 with the property that the number of tangent lines to S_2 at each of its points is two. Using some theoretical results and computer aided search, the complete classification of 2-semiarcs in PG(2,q) is given for $q \leq 7$, the spectrum of their sizes is determined for $q \leq 9$, and some results about the existence are proven for q = 11 and q = 13. For several sizes of 2-semiarcs in PG(2,q), $q \leq 7$, classification results have been obtained by theoretical proofs.

1 Introduction

Ovals, k-arcs, and semiovals of finite projective planes are not only interesting geometric structures, but they have important applications to coding theory and cryptography, as well. For details about these objects we refer the reader to [10, 25, 26, 28].

Semiarcs are a natural generalization of arcs. Let Π_q be a projective plane of order q. A non-empty pointset $S_t \subset \Pi_q$ is called a *t-semiarc* if for every point $P \in S_t$ there exist exactly tlines $\ell_1, \ell_2, \ldots, \ell_t$ such that $S_t \cap \ell_i = \{P\}$ for $i = 1, 2, \ldots, t$. These lines are called the *tangents* to S_t at P. If a line ℓ meets S_t in 2, 3 or k points (where k > 3), then ℓ is called a bisecant, trisecant or k-secant of S_t , respectively. The classical examples of semiarcs are the semiovals (t = 1) and the subplanes (t = q - m), where m is the order of the subplane).

Semiarcs are closely connected to other combinatorial structures, too. Without the pursuit of wholeness we mention (r, 1)-designs and configurations.

Definition 1.1. A finite point-line incidence structure is called linear space if each line contains at least two points and any two distinct points are on exactly one line. If there are exactly r lines through each point, then the linear space is called (r, 1)-design.

A (v_r, b_k) -configuration is a finite point-line incidence structure with the following properties:

- There are v points and b lines.
- There are r lines through each point and there are k points on each line.
- Two distinct lines intersect each other at most once and two distinct points are connected by at most one line.

If v = b and r = k, then the configuration is called symmetric (v_k) -configuration.

^{*}The research was supported by the Italian MIUR (progetto 40% "Strutture Geometriche, Combinatoria e loro Applicazioni"), and by GNSAGA.

[†]The research was supported by the Hungarian National Foundation for Scientific Research, Grant No. K 81310, and by the Slovenian-Hungarian Intergovernmental Scientific and Technological Cooperation Project, Grant No. TÉT 10-1-2011-0606.

The following proposition gives a natural correspondence between embeddable (r, 1)-designs and semiarcs in finite planes. Its proof is straightforward.

Proposition 1.2. If S_t is a t-semiarc in Π_q , then the points of S_t and the secants of S_t form a (q+1-t, 1)-design.

If an (r, 1)-design is embeddable to Π_q , then its points form a (q + 1 - r)-semiarc.

(r, 1)-designs with small r were investigated by Gropp [22, 23]. He constructed all (r, 1)designs with at most 12 points, his list contains 974 elements, most of them are configurations. His proof is computer assisted and he has not considered the embeddability of these designs.

In the last years the interest and research on the fundamental problem of determining the spectrum of the values for which there exists a given subconfiguration of points in PG(n,q) have increased considerably (see for example [2,4–9,17,18,25,27,36,41]). In particular semiovals were investigated by several authors. Among others Lisonek [31] determined the spectrum of sizes of semiovals by exhaustive computer search for $q \leq 9$, q odd, Bartoli [3], Ranson and Dover [19,37], Kiss, Marcugini and Pambianco [29,30], and Nakagawa and Suetake [35,40] gave characterization theorems for semiovals in planes of small order.

Because of the huge diversity of semiarcs, their complete classification seems out of reach. The aim of this paper is to investigate and characterize 2-semiarcs in projective planes of order $q \leq 13$. Throughout the paper Π_q denotes an arbitrary projective plane of order q, while PG(2,q) denotes the desarguesian projective plane over the field of q elements. It is well-known, that if q = 2, 3, 4, 5, 7 or 8, then each projective plane of order q is isomorphic to PG(2,q).

The paper is organized as follows. In Section 2 we give lower and upper bounds and prove some number theoretical conditions on the sizes of 2-semiarcs in Π_q . Using these propositions and the results of Gropp, in Section 3 the complete characterization is provided for $q \leq 5$. In Section 4 we consider the 2-semiarcs in PG(2,7). A computer-free description is given for 2-semiarcs having sizes at most 12, and a computer-assisted proof shows that there are no 2semiarcs in the plane with $|S_2| \geq 13$. Section 5 is devoted to the description of the algorithm used to obtain the classification of 2-semiarcs. Finally in Section 6 results about the existence of 2-semiarcs in PG(2, q) for $q \in \{8, 9, 11, 13\}$ are given. The computer search is supported by the structural constraints proven in Section 2.

2 Some conditions on the sizes of 2-semiarcs

It follows from the definition that each t-semiarc in Π_q satisfies $t \leq q+1$. If t is close to this upper bound, then we can easily classify the t-semiarcs. The following proposition was proved by Csajbók and Kiss [16].

Proposition 2.1. Let S_t be a t-semiarc in Π_q . The following properties hold:

- if t = q + 1, then S_t is a single point,
- if t = q, then S_t is a subset of a line, and vice versa any subset of a line containing at least two points is a q-semiarc,
- if t = q 1, then S_t is a set of three non-collinear points.

A semiarc cannot contain large collinear subsets. If S_t is a *t*-semiarc in Π_q , S_t is not contained in a line and it has a *k*-secant, then $k \leq q + 1 - t$ obviously holds. Semiarcs with long secants were investigated by Csajbók. He proved the following results; see [14, Theorems 2.4 and 4.6]. **Theorem 2.2.** Let S_t be a t-semiarc in PG(2,q). Then the following properties hold.

- If t < (q-1)/2, then S_t has no (q+1-t)-secants.
- If S_t has two (q-t)-secants such that the common point of these secants is not contained in S_t and gcd(q,t) = gcd(q-1,t-1) = 1, then S_t is the union of these two (q-t)-secants.

Bounds on the sizes of t-semiarcs were also given by Csajbók and Kiss [16]. In the case t = 2 their result is the following.

Theorem 2.3. Let S_2 be a 2-semiarc in a projective plane of order q. Then

$$q \le |\mathcal{S}_2| \le 1 + \left\lfloor \frac{q(1+\sqrt{8q-7})}{4} \right\rfloor.$$

The simplest example of a 2-semiarc of size q is a q-arc, a set of q points such that no three of them are collinear. As the following proposition shows, there are no more examples of 2-semiarc of size q.

Proposition 2.4. Let S_2 be a 2-semiarc of size q in a projective plane of order q. Then S_2 is an arc.

Proof. We have to prove that no three points of S_2 are collinear. Suppose that the line ℓ is a trisecant of S_2 . If P is a point in $\ell \cap S_2$, then $|S_2| = q$ implies that there are at least (q+1) - (q-2) = 3 tangents to S_2 at P, contradiction.

Theorem 2.5. In PG(2, p^h), $p \neq 2$, there exists, up to collineations, a unique 2-semiarc S_2 of size $q = p^h$. Its stabilizer group has size hq(q-1).

Proof. S_2 is an arc of size $q = p^h$. It is known, that in PG(2, q) each q-arc is contained in a (q + 1)-arc, and if q is odd, then by the Theorem of Segre, it is contained in an irreducible conic [39]. The stabilizer of a conic is transitive on its points, hence all the q-point subsets of the conic are projectively equivalent. Since the number of conics is $q^2(q^2 + q + 1)(q - 1)$ and each has q + 1 subsets of size q, there are exactly $q^2(q^2 + q + 1)(q - 1)(q + 1)$ different 2-semiarcs of size q. Thus the stabilizer group has size $\frac{|P\Gamma L(3,q)|}{q^2(q^2+q+1)(q-1)(q+1)} = hq(q-1)$.

If Π_q contains a 2-semiarc whose size is close to the lower bound q, then the order of the plane must satisfy some number theoretical conditions.

Proposition 2.6. Let S_2 be a 2-semiarc of size q + 1 in a projective plane of order q. Then q + 1 is divisible by 3.

Proof. Let P be any point of S_2 . The total number of lines through P is q + 1, and two of them are tangents to S_2 . The remaining q points of S_2 are distributed among the q - 1 secants through P. Hence there are q - 2 bisecants and one trisecant through P. Thus each point of S_2 lies on exactly one trisecant, hence $|S_2|$ is divisible by 3.

Proposition 2.7. Let S_2 be a 2-semiarc of size q + 2 in a projective plane of order q. Then there exist integers $0 \le \alpha$ and $0 \le \beta \ne 1$ such that $q + 2 = 4\alpha + 3\beta$.

Proof. Let P be any point of S_2 . The total number of lines through P is q + 1, two of them are tangents to S_2 . The remaining q + 1 points of S_2 are distributed among the q - 1 secants through P. Hence there are either two trisecants and q - 3 bisecants, or one 4-secant and q - 2 bisecants through P. Thus each point lies on either two trisecants or one 4-secant. Let \mathcal{T}_3 be the set of points lying on two trisecants. Then it is a configuration (v_2, k_3) , where $v = |\mathcal{T}_3|$ and, by [21, Theorem 3.1], $|\mathcal{T}_3| = 3\beta$, with $\beta \neq 1$. Let \mathcal{T}_4 be the set of points lying on one 4-secant, then $|\mathcal{T}_4| = 4\alpha$. Then $q + 2 = 4\alpha + 3\beta$, with $\alpha, \beta \geq 0$ and $\beta \neq 1$.

3 The small planes

The classification of 2-semiarcs in the cases q = 2 and q = 3 follows from Proposition 2.1.

Theorem 3.1.

- In PG(2,2) each 2-semiarc S_2 consists of two or three collinear points.
- In PG(2,3) each 2-semiarc S_2 is a set of three non-collinear points.

If q = 4, then 2-semiarcs correspond to (3, 1)-designs by Proposition 1.2. Gropp [22, Table 1] proved that there are three such designs, they consist of 4, 6 and 7 points, respectively. He also gave a detailed combinatorial description of these objets. We show that each of these designs is embeddable into PG(2, 4).

Theorem 3.2. In PG(2,4) there are three projectively non-equivalent 2-semiarcs.

- $|S_2| = 4$, four points in general position.
- $|S_2| = 6$, the vertices of a complete quadrilateral.
- $|S_2| = 7$, the points of a subplane of order 2.

Proof. It is easy to verify (without applying Gropp's results), that there are only three possible sizes of a 2-semiarc. Theorem 2.3 gives $4 \le |\mathcal{S}_2| \le 7$. From Proposition 2.6 we get $|\mathcal{S}_2| \ne 5$, because q + 1 = 5 is not divisible by 3. Hence $|\mathcal{S}_2| \in \{4, 6, 7\}$.

The case $|S_2| = 4$ follows from Proposition 2.4. The combinatorial description of Gropp gives that if $|S_2| = 6$, then there are two trisecants and one bisecant through each point, hence the design corresponds to the six vertices of a complete quadrilateral and it is obviously embeddable into PG(2, 4). If $|S_2| = 7$, then according to Gropp, the design is a (7₃)-configuration. In other words this is the Fano plane PG(2, 2), which is embeddable to PG(2, 4).

$ \mathcal{S}_2 $	x_0	x_1	x_2	x_3	G
4	7	8	6	0	$\mathbb{Z}_2 \times \mathrm{S}_4$
6	2	12	3	4	$\mathbb{Z}_2 \times S_4$
7	0	14	0	7	$PSL(3,2) \times \mathbb{Z}_2$

Table 1: 2-semiarcs in PG(2,4)

Table 1 contains the non-equivalent 2-semiarcs in PG(2, 4), the number of their *i*-secants, x_i , and the description of the stabilizer groups in $P\Gamma L(3, 4)$.

If q = 5, then 2-semiarcs correspond to (4, 1)-designs by Proposition 1.2. Gropp [22, Table 1] proved that there are eight such designs with at most 12 points. We show that only three of them are embeddable into PG(2, 5).

Theorem 3.3. In PG(2,5) there are three projectively non-equivalent 2-semiarcs.

- $|S_2| = 5$, five points of a conic.
- $|S_2| = 6$, the union of two trisecants.

• $|\mathcal{S}_2| = 9$, the projective triangle.

Proof. It is easy to see that there are only four possible sizes of a 2-semiarc. Theorem 2.3 gives $5 \leq |S_2| \leq 9$. From Proposition 2.7 we get $|S_2| \neq 7$, because q + 2 = 7 cannot be written as $4\alpha + 3\beta$ with $\beta \neq 1$.

First we prove that $|S_2| \neq 8$. Suppose to the contrary that S_2 is a 2-semiarc with 8 points. Gropp proved that there is only one (4, 1)-design with eight points, the symmetric (8₃)-configuration (also called Möbius-Kantor configuration). But it was proven by Abdul-Elah, Al-Dhahir and Jungnickel [1] that this configuration cannot be embedded into PG(2, 5). Hence $|S_2| \in \{5, 6, 9\}$.

The case $|\mathcal{S}_2| = 5$ follows from Proposition 2.4.

In the case $|\mathcal{S}_2| = 6$, if $P \in \mathcal{S}_2$ is a point, then there are 6 - 2 = 4 non-tangents through P, hence \mathcal{S}_2 has no 4-secants. Let a be the number of trisecants, and b be the number of bisecants through P. Then we get a + b = 4 and 2a + b = 5, hence a = 1 and b = 3. So \mathcal{S}_2 is the union of two trisecants, ℓ_1 and ℓ_2 . This is the second case of Theorem 2.2.

Finally consider the case $|S_2| = 9$. Gropp proved that there are two (4, 1)-designs with nine points. One of them is the affine plane of order 3. But AG(2, 3) cannot be embedded into PG(2, q) if $q \equiv 2 \pmod{3}$ (see e.g. [11]).

The points of the other (4, 1)-design are of two types: (i) the vertices of a triangle \mathcal{T} , (ii) the points on exactly one side of \mathcal{T} , two points on each side. If a point is of type (i), then it is on two 4-secants and on two bisecants; if a point is of type (ii), then it is on one 4-secant and hence on two trisecants and on one bisecant. Hence S_2 has three 4-secants, $6 \cdot 2/3 = 4$ trisecants and $(3 \cdot 2 + 6 \cdot 1)/2 = 6$ bisecants. S_2 also has $9 \cdot 2 = 18$ tangents, so S_2 is a blocking set because 3 + 4 + 6 + 18 = 31 equals to the total number of lines in PG(2,5). This blocking set has cardinality 3(q+1)/2, hence by a theorem of Lovász and Schrijver [32] it is a projective triangle.

A possible embedding into PG(2,q) is the following. The vertices of \mathcal{T} : {(1:0:0), (0:1:0), (0:0:1)}, the points on the sides of \mathcal{T} : {(1:1:0), (4:1:0), (1:0:1), (4:0:1), (0:1:1), (0:4:1)}.

$ \mathcal{S}_2 $	x_0	x_1	x_2	x_3	x_4	G
5	11	10	10	0	0	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
6	8	12	9	2	0	D_4
9	0	18	6	4	3	S_4

Table 2: 2-semiarcs in PG(2,5)

Table 2 contains the non-equivalent 2-semiarcs in PG(2,5), the number of their *i*-secants, x_i , and the description of the stabilizer groups in PGL(3,5).

4 2-semiarcs in PG(2,7)

The number of (6, 1)-designs with at most 12 points is 47. Instead of considering the list of Gropp [22], we give a geometric characterization of the embeddable designs and we prove that there are 25 non-equivalent 2-semiarcs in PG(2,7). First consider the long secants of the semiarcs. If q = 7 and t = 2, then Theorem 2.2 gives the following corollary. **Corollary 4.1.** Let S_2 be a 2-semiarc in PG(2,7). Then S_2 has no 6-secants. If S_2 has two 5-secants such that the common point of these secants is not contained in S_2 , then S_2 is the union of these two 5-secants.

If the common point of the long secants belongs to S_2 , then the size of the semiarc cannot be small.

Proposition 4.2. Let S_2 be a 2-semiarc in PG(2,7). If S_2 has two 5-secants such that the common point of these secants is contained in S_2 , then $|S_2| > 12$.

Proof. Let ℓ_1 and ℓ_2 be the 5-secants and let $P \in \ell_1 \cap \ell_2$. Then $P \in S_2$ implies that there are six secants of S_2 through P. Hence $S_2 \setminus (\ell_1 \cup \ell_2)$ must contain at least four points. So $|S_t| \ge 9 + 4 = 13$ holds.

Theorem 4.3. In PG(2,7) there are nine combinatorially non-equivalent 2-semiarcs (there are projectively non-equivalent subclasses in some combinatorial classes).

- $|S_2| = 7$, seven points of a conic.
- $|\mathcal{S}_2| = 9$, there are two types,
 - 1. nine vertices of a 3×3 grid,
 - 2. the six vertices of two triangles \mathcal{T}_1 and \mathcal{T}_2 , and the three points of intersections of the corresponding sides of \mathcal{T}_1 and \mathcal{T}_2 .
- $|\mathcal{S}_2| = 10$, there are two types,
 - 1. the union of two 5-secants,
 - 2. the points of a 10_3 configuration.
- $|S_2| = 11$, then the semiarc has no 5-secant. There are two types,
 - 1. four 4-secants and four trisecants,
 - 2. one 4-secant and ten trisecants.
- $|S_2| = 12$, then it has three 4-secants and these lines form a triangle \mathcal{T} . There are two types,
 - 1. two vertices of \mathcal{T} belong to \mathcal{S}_2 ,
 - 2. three vertices of \mathcal{T} belong to \mathcal{S}_2 .

Proof. Theorem 2.3 gives $7 \leq |\mathcal{S}_2| \leq 15$. Let *s* be the number of points of \mathcal{S}_2 , let $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_{57}\}$ be the set of lines of PG(2,7) and let $c_i = |\mathcal{S}_2 \cap \ell_i|$ for $i = 1, 2, \ldots, 57$. If we count in two different ways the number of incident point-line pairs (P, ℓ_j) where $\ell_j \in \mathcal{L}$ and $P \in \mathcal{S}_2$, and the ordered triples (P_1, P_2, ℓ_j) where $\ell_j \in \mathcal{L}$ and the distinct points P_1 and P_2 are in $\mathcal{S}_2 \cap \ell_j$, then we get

$$\sum_{i=1}^{57} c_i = 8s \quad \text{and} \quad \sum_{i=1}^{57} c_i(c_i - 1) = s(s - 1).$$

Hence

$$\sum_{i=1}^{57} c_i^2 = s^2 + 7s.$$

We may assume without loss of generality that the lines $\ell_{58-2s}, \ell_{59-2s}, \ldots, \ell_{57}$ are the tangents to S_2 , for these lines $c_i = 1$. If we subtract these values, then we get

$$\sum_{i=1}^{57-2s} c_i = 6s \quad \text{and} \quad \sum_{i=1}^{57-2s} c_i^2 = s^2 + 5s.$$
(1)

It follows from Corollary 4.1 that if $k \ge 6$, then S_2 has no k-secant. Let x_i be the number of *i*-secants of S_2 for i = 0, 1, ..., 5. Then

$$\sum_{i=1}^{57-2s} (c_i - 2)(c_i - 3) = 6x_0 + 2x_4 + 6x_5 \quad \text{and} \quad \sum_{i=1}^{57-2s} (c_i - 3)(c_i - 4) = 12x_0 + 2x_2 + 2x_5.$$

On the other hand, Equations (1) give

$$\sum_{i=1}^{57-2s} (c_i - 2)(c_i - 3) = \sum_{i=1}^{57-2s} (c_i^2 - 5c_i + 6) = s^2 + 5s - 5 \cdot 6s + 6(57 - 2s) = s^2 - 37s + 342$$

and

$$\sum_{i=1}^{57-2s} (c_i - 3)(c_i - 4) = \sum_{i=1}^{57-2s} (c_i^2 - 7c_i + 12) = s^2 + 5s - 7 \cdot 6s + 12(57 - 2s) = s^2 - 61s + 684.$$

Hence

$$6x_0 + 2x_4 + 6x_5 = s^2 - 37s + 342$$
 and $12x_0 + 2x_2 + 2x_5 = s^2 - 61s + 684.$ (2)

First we prove the non-existence parts of the theorem. From Proposition 2.6 we get $|S_2| \neq 8$, because q + 1 = 8 is not divisible by 3.

Suppose, that s = 15. Then Equations (1) give $\sum_{i=1}^{27} c_i = 90$ and $\sum_{i=1}^{27} c_i^2 = 300$. Applying the inequality between the arithmetic and quadratic means we get

$$\frac{90}{27} = \frac{\sum_{i=1}^{27} c_i}{27} \le \sqrt{\frac{\sum_{i=1}^{27} c_i^2}{27}} = \sqrt{\frac{300}{27}} = \frac{10}{3}$$

Thus equality holds, hence $c_1 = c_2 = \ldots = c_{27}$. But 90/27 is not an integer, contradiction.

Now suppose, that s = 14. Then Equations (2) give

$$3x_0 + x_4 + 3x_5 = 10$$
 and $6x_0 + x_2 + x_5 = 13$.

Elementary counting shows that there are only nine possibilities for the numbers x_0, x_1, \ldots, x_5 . These are the following.

x_0	x_1	x_2	x_3	x_4	x_5
2	28	1	22	4	0
1	28	7	14	7	0
0	28	13	6	10	0
2	28	0	25	1	1
1	28	6	17	4	1
0	28	12	9	7	1
1	28	5	20	1	2
0	$\overline{28}$	11	12	4	2
0	$\overline{28}$	10	15	1	3

Now suppose, that s = 13. Then Equations (2) give

 $3x_0 + x_4 + 3x_5 = 15$ and $6x_0 + x_2 + x_5 = 30$.

Elementary counting shows that there are only twelve possibilities for the numbers x_0, x_1, \ldots, x_5 . These are as follows.

x_0	x_1	x_2	x_3	x_4	x_5
5	26	0	26	0	0
4	26	6	18	3	0
3	26	12	10	6	0
2	26	18	2	9	0
4	26	5	21	0	1
3	26	11	13	3	1
2	26	17	5	6	1
3	26	10	16	0	2
2	26	16	8	3	2
1	26	22	0	6	2
2	26	15	11	0	3
1	$\overline{26}$	21	3	3	3

In these cases an exhaustive computer search shows that there are no 2-semiarcs of sizes 14 and 13 in PG(2,7).

Now consider the existence parts. The case $|S_2| = 7$ follows from Proposition 2.4.

If $|S_2| = 9$ then we can apply Proposition 2.7. As $9 = 4\alpha + 3\beta$ implies $\alpha = 0$ and $\beta = 3$, we get that there is no 4-secant of S_2 and there are two trisecants through each point of S_2 . Hence the total number of trisecants is $9 \times 2/3 = 6$. There are two possibilities.

(i) There do not exist three trisecants such that they form a triangle whose three vertices are in S_2 . Then the points of S_2 are the nine vertices of a 3×3 grid, whose six lines are the trisecants of S_2 . An example for this case is the following. The points of S_2 are the points of intersections of three horizontal and three vertical lines. Their cartesian coordinates are the following: (0,0), (1,0), (3,0), (0,1), (1,1), (3,1), (0,4), (1,4) and (3,4).

The grid has two triples of lines. There are two possibilities in each triples: the lines either form a triangle or they belong to a pencil. Hence there are projectively non-isomorphic examples of this combinatorial type (see Table 3).

(ii) There exist three trisecants such that they form a triangle \mathcal{T}_1 whose three vertices, say P_1, P_2 and P_3 are in \mathcal{S}_2 . In this case \mathcal{S}_2 contains three points, say Q_1, Q_2 and Q_3 from the sides of \mathcal{T}_1 , and three more points, say R_1, R_2 and R_3 . Consider the three other trisecants of \mathcal{S}_2 . If Q_iQ_j were a trisecant, then it ought to contain exactly one point from the set $\{R_1, R_2, R_3\}$, hence both of the remaining two trisecants would pass on the other two R_i , contradiction. So each of the remaining three trisecants contains one point of the set $\{Q_1, Q_2, Q_3\}$, hence two points from the set $\{R_1, R_2, R_3\}$. So the points R_1, R_2 and R_3 form a triangle \mathcal{T}_2 . An example for this case is the following. The homogeneous coordinates of the vertices of \mathcal{T}_1 are (0:0:1), (0:1:0) and (1:0:0), the coordinates of the vertices of \mathcal{T}_2 are (2:3:1), (3:4:1) and (5:5:1). The points of intersections of the corresponding sides are (1:4:0), (0:1:1) and (1:0:1).

There are projectively non-isomorphic examples of this combinatorial type, too (see Table 3).

If $|S_2| = 10$, then first we consider the largest collinear subset of S_2 . Because of Theorem 2.2 its cardinality is at most q - 2 = 5. If S_2 has a 5-secant, then Csajbók, Héger and Kiss [15, Proposition 2.3] proved that S_2 is the union of two 5-secants.

If S_2 has no 5-secants, then the points of S_2 can be partitioned into two subsets. Let $\mathcal{A} \subset S_2$ be the set of points belonging to three trisecants of S_2 and let $\mathcal{B} \subset S_2$ be the set of points belonging to one trisecant and one 4-secant of S_2 . If $|\mathcal{A}| = a$ and $|\mathcal{B}| = b$, then the total number of trisecants of S_2 is (3a + b)/3, hence 3|b. Thus if b > 0, then $b \ge 3$, and no point of S_2 lies on more than one 4-secant. Hence b > 0 implies $|\mathcal{S}_2| \ge 3 \times 4 = 12$, contradiction. So S_2 has no 4-secant, hence it is a (10₃)-configuration. An example for this case is the Desargues configuration.

It is known that there are ten projectively non-isomorphic (10_3) -configurations [21]. The embeddability of these configurations were investigated by Glynn [20], who proved that one of them is not embeddable into any pappian plane. It is also known, that the other nine can be embedded into the classical euclidean plane [12]. Our exhaustive computer search shows that these nine can also be embedded into PG(2,7).

If $|S_2| = 11$, then it is a 2-semiarc with q + 4 points. For each point $P \in S_2$ there are q - 1 secants through P, thus q+3 points of S_2 are distributed among the secants through P. It follows from Corollary 4.1 that S_2 has no 6-secant. Thus the points of S_2 can be partitioned into four subsets. Let $\mathcal{A} \subset S_2$ be the set of points belonging to four trisecants of S_2 , let $\mathcal{B} \subset S_2$ be the set of points belonging to S_2 , let $\mathcal{C} \subset S_2$ be the set of points belonging to two trisecants and one 4-secant of S_2 , let $\mathcal{C} \subset S_2$ be the set of points belonging to one trisecant of S_2 .

First we prove that $\mathcal{D} = \emptyset$. Let $|\mathcal{A}| = a$, $|\mathcal{B}| = b$, $|\mathcal{C}| = c$ and $|\mathcal{D}| = d$. Let s be the number of 5-secants. Then Corollary 4.1 and Proposition 4.2 imply that $s \leq 1$. Suppose that s = 1. Then we show that c = 0 also holds. The 4-secants cannot meet the 5-secant in a point of \mathcal{S}_2 and the union of two intersecting 4-secants contains 7 points, so if $c \neq 0$, then \mathcal{S}_2 contains at least 5 + 7 > 11 points, contradiction. So s = 1 implies a + b = 6. The number of the 4-secants of \mathcal{S}_2 is b/4, hence 4|b. There are only two possibilities, either b = 0 or b = 4. In the first case a = 6, in the second a = 2. The number of the trisecants of \mathcal{S}_2 is (5 + 4a + 2b)/3. If b = 0, then this number is $5 + 4 \cdot 6 = 29$ and it is not divisible by 3, contradiction. If b = 4 then a = 2, and \mathcal{S}_2 has one 5-secant, ℓ_5 , one 4-secant, ℓ_4 and seven trisecants. Let $\mathcal{S}_2 \setminus (\ell_5 \cup \ell_4) = \{P, R\}$. Then there are four trisecants through both P and R, hence the line PR is a trisecant. Each of the other $2 \times 3 = 6$ trisecants through P or R must contain one point of ℓ_5 , but there exists a unique trisecant at each point of ℓ_5 . This contradiction proves d = 0.

If d = 0, then a + b + c = 11. The number of the trisecants of S_2 is (4a + 2b)/3, hence

$$b \equiv a \pmod{3}$$
.

The number of the 4-secants of S_2 is (b+2c)/4 = (22-2a-b)/4, hence

$$b \equiv 2a + 2 \pmod{4}.$$

Thus the Chinese Remainder Theorem gives

$$b \equiv 10a + 6 \pmod{12}.$$

We know that $0 \le a, b \le 11$, hence if a is given, then this congruence uniquely determines b, and also $0 \le c = 11 - a - b$. We have the following possibilities.

	a	0	1	2	3	4	5	6	7	8	9	10	11
	b	6	4	2	0	10	8	6	4	4	0	10	8
	c	5	6	7	8		—		0	—	2	-	-
ĺ	Case	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}

An example for Case A_1 is the following. Let $C = \{(1:0:0), (0:1:0), (0:0:1), (1:1:0), (1:0:1)\}$. The two 4-secants through (1:0:0) contain the points (1:0:0), (0:1:0), (1:1:0), (1:0), (1:5:0) and (1:0:0), (0:0:1), (1:0:1), (1:0:4), respectively. The 4-secant through (0:0:1) and (0:1:0) contains the points $(0:1:1), (0:1:5) \in \mathcal{B}$. The 4-secant through (1:1:0) and (1:0:1) contains the points $(1:3:5), (1:2:6) \in \mathcal{B}$.

An example for Case A_8 is the following. Let

$$(1:0:0) \in \mathcal{A}$$
 and $\mathcal{B} = \{(0:1:0), (0:0:1), (0:1:1), (0:1:3)\}.$

The four 3-secants through (1:0:0) contain the points

$$\{ (1:0:0), (0:1:0), (1:2:2) \}, \quad \{ (1:0:0), (0:1:1), (1:4:4) \}, \\ \{ (1:0:0), (1:1:5), (1:4:6) \}, \quad \{ (1:0:0), (1:6:1), (1:3:4) \}.$$

We prove that the other cases do not appear. Cases A_5, A_6, A_7A_9, A_{11} and A_{12} cannot appear, because they do not satisfy the condition a + b + c = 11. The number of the 4-secants of S_2 is f = (b + 2c)/4. If b = 0 and c = 2, then f = 1, but then obviously do not exist any point which is on two 4-secants. In the Cases A_2, A_3 and A_4 we have f = 4. But four lines have at most 6 points of intersections, hence c = 7 and c = 8 are impossible. If c = 6, then the four 4-secants form a complete quadrilateral, the sides of it contain the four points of the set \mathcal{B} , and \mathcal{A} consists of a single point, say P. Then each of the four trisecants through P must contain two points from \mathcal{B} . But then the pigeonhole principle implies that some of these trisecants have more than one point in common. This contradiction proves the nonexistence of this configuration.

If $|\mathcal{S}_2| = 12$, then Equations (2) give

$$6x_0 + 2x_4 + 6x_5 = 42$$
 and $6x_0 + x_2 + x_5 = 48$.

Proposition 4.2 gives that $x_5 \leq 1$, hence elementary counting shows that there are only five possibilities for the numbers x_0, x_1, \ldots, x_5 . These are the following.

x_0	x_1	x_2	x_3	x_4	x_5	Case
7	24	6	20	0	0	B_1
6	24	12	12	3	0	B_2
5	24	18	4	6	0	B_3
6	24	11	15	0	1	B_4
5	24	17	7	3	1	B_5

We show that only Case B_2 appears. For each point $P \in S_2$ there are six secants through P. We have to distribute 11 points among the secants through P. It follows from Corollary 4.1 that S_2 has no 6-secant. Thus the points of S_2 can be partitioned into four subsets. Let $\mathcal{A} \subset S_2$ be the set of points belonging to five trisecants of S_2 , let $\mathcal{B} \subset S_2$ be the set of points belonging to three trisecants of S_2 , let $\mathcal{C} \subset S_2$ be the set of points belonging to one trisecant and two 4-secant of S_2 and finally let $\mathcal{D} \subset S_2$ be the set of points belonging to two trisecants and one 5-secant of S_2 .

If S_2 has a 5-secant, ℓ , then let $\mathcal{R} = S_2 \setminus \ell$ and let $\ell \setminus S_2 = \{P, Q, R\}$.

• Case B_1 . In this case S_2 is a (12,3)-arc. In [13] the intersection sizes with lines of all the regular complete (12,3)-arcs in PG(2,7) are presented and there exist no regular complete (12,3)-arcs in PG(2,7) having 20 trisecants. An exhaustive computer search among incomplete (12,3)-arcs in PG(2,7) shows that all of them have less than 20 trisecants.

• Case B_3 . First we prove that no three of the 4-secants have a point in common. There are at most two 4-secants through any point of S_2 , and if three 4-secants would meet in a point outside S_2 , then the union of these lines would contain S_2 , so any other line could contain at most three points of S_2 , but the total number of 4-secants is six. Hence through each point of S_2 there are exactly two 4-secants and one trisecant of S_2 .

The six 4-secants have $6 \cdot 5/2 = 15$ points of intersection. Three of these points are not in S_2 , let X and Y be two of them and let O and E be two points of S_2 such that OX, OY, EX and EY are 4-secants of S_2 . There is a projectivity mapping the points of the projective frame to $\{X, Y, O, E\}$. After this projectivity the points of S_2 are in the affine plane. If we use cartesian coordinates, we get O = (0,0), E = (1,1), and the points $P = OX \cap EY = (1,0)$ and $R = OY \cap EX = (0,1)$ belong to S_2 . Let the further points of $OY \cap S_2$ and $OX \cap S_2$ be A = (0,a), B = (0,b), and C = (c,0), D = (d,0), respectively. Then $\{a, b, c, d\} \cap \{0, 1\} = \emptyset$.

Without loss of generality we may assume that AC and BD are 4-secants. The equations of these lines are X/c + Y/a = 1 and X/d + Y/b = 1, respectively. Then the remaining points of S_2 must be $PY \cap AC = K = (1, a - a/c)$, $PY \cap BD = L = (1, b - b/d)$, $RX \cap AC = M = (c - c/a, 1)$ and $RX \cap BD = N = (d - d/b, 1)$. Hence the lines OE and PR are bisecants. Consider the unique trisecant through O. It must contain one point from the set $\{K, L\}$ and one point from the set $\{M, N\}$. But none of the lines KM and LN contains O, thus without loss of generality we may assume, that the line KN is the trisecant through O. Hence

$$a - \frac{a}{c} = \frac{1}{d - \frac{d}{b}} \quad \Longleftrightarrow \quad \frac{a(c-1)}{c} = \frac{b}{d(b-1)}.$$
(3)

In the same way we get that the unique trisecants through the points P, R and E must be the lines MB, LC and DA, respectively. The equation of the line joining the points (s, 0) and (0,t) is X/s + Y/t = 1, thus from these collinearity conditions we get the following equations:

$$c - \frac{c}{a} + \frac{1}{b} = 1 \quad \Longleftrightarrow \quad \frac{c(a-1)}{a} = \frac{b-1}{b},\tag{4}$$

$$\frac{1}{c} + b - \frac{b}{d} = 1 \quad \Longleftrightarrow \quad \frac{b(d-1)}{d} = \frac{c-1}{c},\tag{5}$$

$$\frac{1}{d} + \frac{1}{a} = 1 \quad \Longleftrightarrow \quad d = \frac{a}{a-1}.$$
(6)

From the last equation we get (d-1)/d = 1/a, hence Equations (5) and (3) give b = (2a-1)/a and bc = 1. Finally from Equations (5) and (4) we get c = (a+1)/a. Hence

$$\frac{2a-1}{a} \cdot \frac{a+1}{a} = 1$$
, thus $a^2 + a - 1 = 0$.

But this equation has no root in GF(7), so there is no semiarc of this type in PG(2,7).

• Case B_4 . Each point of $S_2 \cap \ell$ is contained in two trisecants. Thus the number of trisecants of S_2 through the points of $\ell \cap S_2$ is 10. Let x'_2 and x'_3 be the number of bisecants and trisecants of \mathcal{R} , respectively. Then counting in two different ways the ordered triples (A, B, e) where both A and B are points in \mathcal{R} and e is a line incident with both of them, we get $2x'_2 + 6x'_3 = 42$. On the other hand, each trisecant of S_2 containing a point of ℓ corresponds to a bisecant of \mathcal{R} . Since the number of trisecants of S_2 is 15, the other 5 trisecants of S_2 must be trisecants also for \mathcal{R} , thus $x'_2 \ge 10$ and $x'_3 = 5$. Hence $42 = 2x'_2 + 6x'_3 \ge 20 + 30$, contradiction. So there is no semiarc of this type. • Case B_5 . There are no 4-secants through the points of $\ell \cap S_2$. Hence each of the three 4-secants meets \mathcal{R} in four points. But the union of the three 4-secants contains at least 4+3+2=9 distinct points and \mathcal{R} contains only seven points. So there is no semiarc of this type.

Thus only Case B_2 can appear. Now S_2 has three 4-secants, say ℓ_1, ℓ_2 and ℓ_3 . Let \mathcal{M} be the set of points of intersections of the 4-secants. The number of 4-secants through any point of S_2 is at most two, hence there are four possibilities.

- 1. $|\mathcal{M}| = 1$ and $\mathcal{M} \cap \mathcal{S}_2 = \emptyset$,
- 2. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 1$,
- 3. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 2$,
- 4. $|\mathcal{M}| = 3$ and $|\mathcal{M} \cap \mathcal{S}_2| = 3$.

An exhaustive computer search shows that there are no examples in cases 1 and 2, and there are examples in cases 3 and 4. An example of case 3 is the following. Let $S_2 \cap \ell_1 = \{(1 : 0 : 0), (1 : 0 : 4), (1 : 0 : 5), (1 : 0 : 6)\}, S_2 \cap \ell_2 = \{(1 : 0 : 0), (0 : 1 : 0), (1 : 5 : 0), (1 : 6 : 0)\}$ and $S_2 \cap \ell_3 = \{(0 : 1 : 0), (0 : 1 : 2), (0 : 1 : 3), (0 : 1 : 5)\}$, finally let $S_2 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \{(1 : 1 : 1), (1 : 5 : 1)\}$.

An example of case 4 is the following. Let x = 0, $S_2 \cap \ell_1 = \{(1:0:0), (0:0:1), (1:0:1), (1:0:5)\}$, $S_2 \cap \ell_2 = \{(1:0:0), (0:1:0), (1:2:0), (1:3:0)\}$ and $S_2 \cap \ell_3 = \{(0:0:1), (0:1:0), (0:1:4), (0:1:5)\}$, finally let $S_2 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \{(1:1:3), (1:1:6), (1:4:3)\}$. \Box

Table 3 contains the projectively non-equivalent 2-semiarcs in PG(2,7), the number of their *i*-secants, x_i , and the description of the stabilizer groups in PGL(3,7).

5 The algorithm

The algorithm used for the classification of 2-semiarcs in PG(2, q) is a modification of the one presented in [3,34]. When possible, the search is helped by the structural constraints proven in Section 2.

In this case the algorithm works on *admissible sets*, i.e. sets such that each point lies on at least two tangent lines, instead of working on partial solutions. In fact, the property of being a 2-semiarc is not an hereditary feature, i.e. a feature conserved by all the subsets, so the weaker hereditary feature of being an admissible set has been used. It is weaker in the sense that it allows to prune very few branches of the search space with respect to the cases when considering arcs and (k, 3)-arcs. This and the fact that 2-semiarcs are in general larger than arcs and (k, 3)-arcs make the problem computationally harder than the ones faced in [33, 34].

Note also that, in general, not all the admissible sets can be extended to 2-semiarcs.

The exhaustive search has been feasible because projective properties among admissible sets have been exploited to avoid obtaining too many isomorphic copies of the same 2-semiarc and to avoid searching through parts of the search space isomorphic to previously searched ones.

The algorithm starts constructing a tree structure containing a representative of each class of non-equivalent admissible sets of size less than or equal to a fixed threshold h. If the threshold h were equal to the actual size of the putative 2-semiarcs, the algorithm would be orderly, that is capable of constructing each goal configuration exactly once [38].

However, in the present case, the construction of the tree with the threshold h equal to the size of the putative 2-semiarcs would have been too space and time consuming. For this reason

a hybrid approach has been adopted. The obtained non-equivalent admissible sets of size h have been extended using a backtracking algorithm trying to determine 2-semiarcs of the desired size. In the backtracking phase, the information obtained during the classification of the admissible sets has been further exploited to prune the search tree. In fact the points that would have given admissible sets equivalent to already obtained ones have been excluded from the backtracking steps.

A simple parallelization technique, based on data distribution, has been used to divide the load of the computation in a multiprocessor computer. In our searches we used a 3.3 Ghz Intel Exacore 16 Gb of memory.

6 Results for $8 \le q \le 13$

In Table 4, the number of non-equivalent examples of 2-semiarcs in PG(2,q), $q \leq 9$, is given. The two examples of 2-semiarcs of size 8 in PG(2,8) are obtained by deleting two points from the hyperoval (two points of the conic or one point of the conic and the nucleus).

The following non-existence results are obvious corollaries of Propositions 2.6 and 2.7.

Corollary 6.1. In PG(2,9) there are no 2-semiarcs of size 10 or 11.

In Tables 5 and 6 the description of the stabilizer of the non-equivalent examples of 2-semiarcs S_2 in PG(2,8) and PG(2,9) is presented. In Table 7 (resp. 8) the 2-semiarcs in PG(2,8) (resp. PG(2,9)) having stabilizer of size larger than 16 are listed (x_i indicates the number of *i*-secants of S_2 and ω denotes an element satisfying the equation $\omega^3 + \omega^2 + 1 = 0$ (resp. $\omega^2 - 2\omega - 1 = 0$)).

By our experimental results we are able to prove the following.

Theorem 6.2. In PG(2,11) there exist 2-semiarcs of size $k \in \{11, 12, 14 - 26\}$. In PG(2,13) there exist 2-semiarcs of size $k \in \{13, 27 - 30\}$.

Note that there exists a unique 2-semiarc of size 11 (resp. 13) in PG(2, 11) (resp. PG(2, 13)) and its stabilizer is $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$ (resp. $(\mathbb{Z}_{13} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_3$), according to Theorem 2.5. We also proved by an exhaustive computer search that there exists a unique 2-semiarc of size 12 in PG(2, 11) and its stabilizer is S₄.

Acknowledgement

The authors are grateful to the anonymous reviewer for his/her detailed and helpful comments and suggestions.

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$ \mathcal{S}_2 $	\mathcal{S}_2	x_0	x_1	x_2	x_3	x_4	x_5	G
7	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	22	14	21	0	0	0	G_{42}
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	\mathbb{Z}_2
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	\mathbb{Z}_2
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	\mathbb{Z}_3
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	\mathbb{Z}_3
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	\mathbb{Z}_6
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	15	18	18	6	0	0	S_3
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_1
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_1
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_1
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_2
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_2
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_2
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_2
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_3
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathbb{Z}_4
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	10	20	25	0	0	2	D_4
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	D_6
10	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	12	20	15	10	0	0	\mathcal{S}_4
11	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	8	22	19	4	4	0	\mathbb{Z}_1
11	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	9	22	13	12	1	0	\mathbb{Z}_1
11	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	9	22	13	12	1	0	\mathbb{Z}_1
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	6	24	12	12	3	0	\mathbb{Z}_1
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	6	24	12	12	3	0	\mathbb{Z}_3
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	6	24	12	12	3	0	\mathbb{Z}_3

Table 3: 2-semiarcs in PG(2,7)

la	ble 4:	2-semiarcs in $PG(2,q), q \leq 9$									
q	Size	# non-equivalent examples									
	4	1									
4	6	1									
	7	1									
	5	1									
5	6	1									
	9	1									
	7	1									
7	9	6									
(10	12									
	11	3									
	12	3									
	8	2									
	9	2									
	10	1									
	11	10									
8	12	26									
	13	31									
	14	29									
	15	11									
	16	2									
	9	1									
	12	30									
	13	59									
	14	360									
	15	925									
9	16	1149									
	17	655									
	18	162									
	19	19									
	20	3									

Table 4: 2-semiarcs in $PG(2, q), q \leq 9$

Table 5: 2-semiarcs in PG(2, 8)

Size	\mathbb{Z}_1	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_2^2	\mathbb{Z}_6	S_3	\mathbb{Z}_2^3	\mathbb{Z}_{12}	\mathcal{Q}_6	$\mathcal{S}_3 imes \mathbb{Z}_3$	S_4	$D_4 \times \mathbb{Z}_3$	$A_4 \times \mathbb{Z}_2$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_7 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_8)$
8														1		1
9						1				1						
10									1							
11	5	4	1													
12	8	9	1	1		1	1	1			1	1	2			
13	22	4	5													
14	14	8		6								1				
15	5	1	2		2	1										
16													1		1	

Table 6: 2-semiarcs in PG(2,9)

Size	\mathbb{Z}_1	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_4	\mathbb{Z}_2^2	\mathbb{Z}_6	\mathcal{S}_3	$\mathbb{Z}_2\times\mathbb{Z}_4$	D_4	D_6	D_8	$\mathrm{D}_4\times\mathbb{Z}_2$	$\mathbb{Z}_4\rtimes\mathbb{Z}_4$	$\mathcal{S}_3\times\mathbb{Z}_3$	\mathcal{S}_4	$\mathcal{S}_3\times\mathbb{Z}_4$	$\mathbb{Z}_6\times \mathrm{S}_3$	$\mathrm{D}_8 \times \mathrm{S}_3$	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$
9																			1
12	9	6	1		3	4	2			2					1	1		1	
13	42	11		4	2														
14	308	48		3								1							
15	836	74	3		6	2	2			1					1				
16	1054	73		6	11			2	1		1		1						
17	583	59		10	1						1		1						
18	126	22	3		3		4			1				1	1		1		
19	10	5			2				2										
20		2									1								

Table 7: 2-semiarcs in $\mathrm{PG}(2,8)$ with |G|>16

$ \mathcal{S}_2 $	\mathcal{S}_2	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	G
8	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	29	16	28	0	0	0	0	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
8	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	29	16	28	0	0	0	0	$\mathbb{Z}_7 imes (\mathbb{Z}_3 times \mathbb{Z}_8)$
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	25	18	27	3	0	0	0	$S_3 imes \mathbb{Z}_3$
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	11	24	36	0	0	0	2	S_4
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	13	24	30	0	6	0	0	$D_4\times \mathbb{Z}_3$
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	14	24	24	8	3	0	0	$A_4 \times \mathbb{Z}_2$
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	14	24	24	8	3	0	0	$A_4 \times \mathbb{Z}_2$
14	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	6	28	25	12	0	0	2	$D_4\times \mathbb{Z}_3$
16	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	5	32	0	32	4	0	0	$A_4\times\mathbb{Z}_2$
16	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	5	32	0	32	4	0	0	$((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$

$ \mathcal{S}_2 $	\mathcal{S}_2	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	G
9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	37	18	36	0	0	0	$((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8) \rtimes \mathbb{Z}_2$
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	24	24	36	4	3	0	$S_3 imes \mathbb{Z}_4$
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	25	24	30	12	0	0	S_4
12	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	24	24	36	4	3	0	$\mathrm{D}_8\times\mathrm{S}_3$
15	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	16	30	15	30	0	0	S_4
18	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	1	36	36	9	0	9	$S_3 imes \mathbb{Z}_3$
18	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	6	36	15	22	12	0	S_4
18	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 &$	4	36	27	6	18	0	$\mathbb{Z}_6\times S_3$

Table 8: 2-semiarcs in $\mathrm{PG}(2,9)$ with |G|>16