

## A GENERALIZATION OF A THEOREM OF E. VINCZE

by

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The functional equation  $\varphi(x) = \varphi(ax)\varphi(bx)$  ( $a, b > 0, a^2 + b^2 = 1$ ) was solved by E. VINCZE [1] under the assumption that  $\varphi(x)$  is a complex valued function of the real variable  $x$  which can be differentiated twice at the origin. This equation occurs in certain problems in probability theory and was therefore studied by a number of authors under the restriction that  $\varphi(x)$  is a positive definite function.

In the present note we prove the following generalization of VINCZE's result:

**Theorem.** *Let  $\varphi(x)$  be a complex valued function of the real variable  $x$  and let  $\{a_j\}$  be a sequence of nonnegative real numbers such that  $\sum_{j=1}^{\infty} a_j^2 = 1$  and  $0 < a_1 < 1$ . Suppose that there exist complex constants  $A$  and  $B$  such that*

$$(1) \quad \lim_{x \rightarrow 0} \frac{\varphi(x) - \varphi(0) - Ax}{x^2} = B.$$

Assume further that  $\varphi(x)$  satisfies for all real  $x$  the functional equation

$$(2) \quad \varphi(x) = \prod_{j=1}^{\infty} \varphi(a_j x)$$

where the infinite product converges.<sup>2</sup> Then  $A = 0$  and  $\varphi(x) = e^{Bx^2}$ .

**Proof.** It follows from (1) that  $\varphi(x)$  is continuous at  $x = 0$ . It follows further from the convergence of the infinite product (2) that  $\lim_{j \rightarrow \infty} \varphi(a_j x) = 1$ , and as clearly  $\lim_{j \rightarrow \infty} a_j = 0$  we obtain  $\varphi(0) = 1$ . Thus it follows from (1) that

$$(3) \quad \varphi(x) = 1 + Ax + O(x^2) \quad \text{for } x \rightarrow 0.$$

Thus we have

$$\prod_{j=1}^N \varphi(a_j x) = e^{A(\sum_{j=1}^N a_j)x + O(x^2)}.$$

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<sup>2</sup> As usually the convergence of an infinite product  $\prod_{n=1}^{\infty} z_n$  is understood in the sense that only a finite number of factors may be equal to 0 and if  $z_n \neq 0$  for  $n \geq n_0$  then  $\lim_{N \rightarrow \infty} \prod_{n=n_0}^N z_n$  exists and is different from 0.

As the product (2) is convergent, it follows that the series  $\sum_{j=1}^{\infty} a_j$  is convergent too. Let us put

$$(4) \quad \sum_{j=1}^{\infty} a_j = C.$$

We have evidently  $C > 1$ . It follows that  $\varphi(x) = e^{ACx + O(x^2)}$  and thus

$$(5) \quad \frac{\varphi(x) - 1 - Ax}{x^2} = \frac{A(C - 1)}{x} + O(1) \quad \text{for } x \rightarrow 0.$$

Clearly (5) is compatible with (1) only if  $A = 0$ . Condition (1) now reduces to

$$(6) \quad \lim_{x \rightarrow 0} \frac{\varphi(x) - 1}{x^2} = B$$

which yields

$$(7) \quad \lim_{x \rightarrow 0} \frac{\log \varphi(x)}{x^2} = B.$$

Clearly (6) implies that there exist positive numbers  $d$  and  $D$  such that

$$(8) \quad |\varphi(x) - 1| < Dx^2 \quad \text{for } |x| \leq d.$$

As for  $|z - 1| \leq \frac{1}{2}$  we have  $|\arg z| \leq \frac{\pi}{3}|z - 1|$ , it follows that for  $|x| \leq \Delta = \min\left(d, \frac{1}{\sqrt{2D}}\right)$  we have

$$\sum_{j=1}^{\infty} |\arg \varphi(a_j x)| < \pi.$$

Thus we obtain from (2)

$$(9) \quad \log \varphi(x) = \sum_{j=1}^{\infty} \log \varphi(a_j x) \quad \text{for } |x| \leq D.$$

We obtain from (9) by iteration for any natural number  $k$

$$(10) \quad \log \varphi(x) = \sum_{j_1=1}^{\infty} \dots \sum_{j_k=1}^{\infty} \log \varphi(a_{j_1} \dots a_{j_k} x).$$

Since  $\max a_j = \alpha < 1$  we see that

$$(11) \quad \max a_{j_1} \dots a_{j_k} \leq \alpha^k.$$

In view of (11) and (7) for any  $\varepsilon > 0$  we can choose the value of  $k$  so large that

$$(12) \quad |\log \varphi(a_{j_1} \dots a_{j_k} x) - Bx^2 a_{j_1}^2 \dots a_{j_k}^2| \leq \varepsilon a_{j_1}^2 \dots a_{j_k}^2$$

for all  $|x| \leq \Delta$ . It follows from (10) and (12) that

$$(13) \quad |\log \varphi(x) - Bx^2| \leq \varepsilon.$$

As  $\varepsilon$  can be chosen arbitrarily small we obtain that  $\varphi(x) = e^{Bx^2}$  for  $|x| \leq \Delta$ .

Let us put  $\psi(x) = \varphi(x) e^{-Bx^2}$ . Then clearly  $\psi(x)$  satisfies for all  $x$  the equation

$$(14) \quad \psi(x) = \prod_{j=1}^{\infty} \psi(a_j x)$$

further  $\psi(x) = 1$  for  $|x| \leq \Delta$ . Let  $x_0$  denote the upper bound of those positive numbers  $\beta$  for which  $\psi(x) = 1$  for  $|x| \leq \beta$ . We have already shown that  $x_0 \geq \Delta > 0$ . Suppose that  $x_0$  is finite; we shall prove that this leads to a contradiction. Clearly  $\psi(\pm x_0) = 1$  because  $\max a_j = \alpha < 1$ . Let now  $\eta$  be an arbitrary real number such that  $1 > \eta > \alpha$ ; then  $\frac{a_j}{\eta} < \frac{a_j}{\alpha} \leq 1$  so that  $\psi\left(\pm \frac{a_j}{\eta} x_0\right) = 1$ . It follows that  $\psi\left(\pm \frac{x_0}{\eta}\right) = 1$ . This however contradicts the definition of  $x_0$ . Thus  $x_0 = +\infty$  and  $\varphi(x) = e^{Bx^2}$  for all  $x$ .

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#### REFERENCE

- [1] VINCZE, E.: »Bemerkung zur charakterisierung des Gausschen Fehlergesetzes.« *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* 7 (1962) 357–361.

### ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ Е. VINCZE

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#### Резюме

Доказывается следующее обобщение теоремы Е. VINCZE [1]:

**Теорема.** Пусть  $\varphi(x)$  — функция вещественной переменной  $x$  с комплексными значениями и  $\{a_j\}$  — последовательность неотрицательных чисел, для которых  $\sum_{j=1}^{\infty} a_j^2 = 1$  и  $0 < a_1 < 1$ . Предполагаем, что существуют комплексные константы  $A$  и  $B$  такие, что

$$\lim_{x \rightarrow 0} \frac{\varphi(x) - \varphi(0) - Ax}{x^2} = B,$$

и что  $\varphi(x)$  удовлетворяет функциональному уравнению

$$\varphi(x) = \prod_{j=1}^{\infty} \varphi(a_j x),$$

где бесконечное произведение сходится. В этом случае  $A = 0$  и  $\varphi(x) = e^{Bx^2}$ .