ON THE THEORY OF RELATIONS

by

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Let E be a given set and suppose that to every element x of E there corresponds a subset $S(x)$ of E such that $x \notin S(x)$. For every subset M of E let

$$S(M) = \bigcup_{x \in M} S(x) .$$

A subset M of E is called independent, if

$$M \cap S(M) = \emptyset .$$

Let E be a separable topological space of the second category without isolated points. For any $x \in E$, let $S(x)$ be nowhere dense in E. P. ERDŐS has proved that there exists an independent set of power $\aleph_0$. It is not known the existence of an independent set of power $\aleph_1$.

We shall prove the following theorem which is a generalization of the theorem of P. ERDŐS.

Theorem. *Let E be a separable topological space of the second category without isolated points. Suppose that the elements of E are arranged in a given wellordering. If for every $x \in E$ the set $S(x)$ is nowhere dense in E, then there exists for every $\alpha < \omega_1$ an independent subset of the type α of E in the given wellordering.*

Proof. Let H be a subset of the second category of E and let $\eta(H) \neq \emptyset$ be a subset of H with the following property (T): For any open subset K of E satisfying $\eta(H) \cap K \neq \emptyset$ the set $\eta(H) \cap K$ is of the second category. It is known that there exists such a subset $\eta(H)$ of H. Thus we have associated to every subset H of the second category of E a non-empty subset of H with the property (T). If the set H has the property (T), then we put $\eta(H) = H$. It is clear that a subset M of $\eta(H)$ is nowhere dense or of the first category in $\eta(H)$ if and only if M is nowhere dense or of the first category in E.

Consider now the ordinal numbers of the subsets of the second category of E with respect to the given wellordering of E. Let φ be the smallest such ordinal number. It is clear that φ is not confinal to an ordinal number which is smaller than ω_1. Let B be a subset of the second category of the type φ of E and $A = \eta(B)$. It is obvious that A has the type φ.

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For each subset M of A let $W(M)$ denote the set of all elements of A which are not exceeding all the elements of M in the given wellordering of E [thus $M \subseteqq W(M)$]. It is obvious that $W(M)$ is of the first category if and only if M is not confinal to A. For example if M is a countable set, then $W(M)$ is of the first category.

Suppose now that in every set $K \subseteqq A$ of the second category there is an independent set of the type α, where $\alpha < \xi < \omega_1$. We prove that there exists in every set $P \subseteqq A$ of the second category an independent set of the type ξ. Put $Q = \eta(P)$.

We consider two cases

a) ξ is an ordinal number of the first kind, i.e. $\xi = \alpha + 1$,

b) ξ is an ordinal number of the second kind.

Ad a Let $\{Q_\vartheta\}_{\vartheta<\omega}$ be a countable base of Q. Then there exists in every Q_ϑ an independent set H_ϑ of the type α. Put $H = \bigcup\limits_{\vartheta<\omega} H_\vartheta$. It is obvious that H is a countable set and $H \subset A$ holds, thus $W(H)$ and $S(H)$ are of the first category; i.e. the set $Q^1 = Q - (W(H) \cup S(H))$ is not empty. Let $t \in Q^1$. As $S(t)$ is nowhere dense, there exists an ordinal number $\vartheta < \omega$, such that $S(t) \cap \cap Q_\vartheta = \emptyset$. It is easy to verify that $H_\vartheta \cup \{t\}$ is an independent set with the type $\alpha + 1 = \xi$.

Ad b In this case put $\xi = \sum\limits_{\lambda<\omega} \xi_\lambda$, where $\xi_\lambda < \xi$ for every $\lambda < \omega$. First prove the following

Lemma. *Let $\{Q^0_\vartheta\}_{\vartheta<\omega}$ be a countable base of $Q = Q^0$ and assume that there exists in every Q^0_ϑ an independent set H^0_ϑ of the type ξ_0. Then there exists a set Q^1 of the second category and an ordinal number $\tau_0 < \omega$, such that $\eta(Q^1) = Q^1 \subseteqq Q^0$, and $W(H^0_{\tau_0}) \cap Q^1 = H^0_{\tau_0} \cap S(Q^1) = S(H^0_{\tau_0}) \cap Q^1 = \emptyset$.*

Proof. Since H^0_ϑ is a countable set and $H^0_\vartheta \subset A$ holds, the sets $W(H^0_\vartheta)$ and $S(H^0_\vartheta)$ are of the first category; hence the set $F^0 = \bigcup\limits_{\vartheta<\omega} \left(W(H^0_\vartheta) \cup S(H^0_\vartheta)\right)$ is also of the first category. Thus the set $M^1 = Q^0 - F^0$ is of the second category. For every $\vartheta < \omega$ let

$$R_\vartheta = \{x \in M^1 : S(x) \cap Q^0_\vartheta = \emptyset\}\,.$$

Since the set $S(x)$ is nowhere dense for every $x \in E$, we get that $M^1 = \bigcup\limits_{\vartheta<\omega} R_\vartheta$. Thus there exists an ordinal number $\vartheta_0 < \omega$, such that R_{ϑ_0} is of the second category. Let $\tau_0 = \vartheta_0$ and $\eta(R_{\vartheta_0}) = Q^1$. It is easy to see that the sets $H^0_{\tau_0}$ and Q^1 satisfy the requirements of the lemma.

If we start with Q^1 instead of Q^0, then we get by the application of the lemma the set Q^2, the ordinal number τ_1 and the set $H^1_{\tau_1}$ with the corresponding properties . . ., etc. Repeat this ad infinitum we get successively the sets $Q^0, Q^1, \ldots, Q^\lambda, \ldots$; the ordinal numbers $\tau_0, \tau_1, \ldots, \tau_\lambda, \ldots$ and the sets $H^0_{\tau_0}, H^1_{\tau_1}, \ldots, H^\lambda_{\tau_\lambda} \ldots (\lambda < \omega)$, such that for every $\lambda < \omega$ the set Q^λ is of the second category, $Q^{\lambda-1} \supseteqq Q^\lambda$, $\eta(Q^\lambda) = Q^\lambda$ further $H^\lambda_{\tau_\lambda}$ is an independent set of the type ξ_λ with the properties $(\lambda < \gamma < \omega)$:

(1) $W(H^\lambda_{\tau_\lambda}) \cap H^\gamma_{\tau_\gamma} = \emptyset$

(2) $H^\lambda_{\tau_\lambda} \cap S(H^\gamma_{\tau_\gamma}) = \emptyset$

(3) $S(H^\lambda_{\tau_\lambda}) \cap H^\gamma_{\tau_\gamma} = \emptyset$.

Put $H = \bigcup\limits_{\lambda<\omega} H^\lambda_{\tau_\lambda}$. It is easy to verify that the set H is an independent set of the type $\xi = \sum\limits_{\lambda<\omega} \xi_\lambda$.

REFERENCE

ERDŐS, P.: "Some remarks on set theory III." *Michigan Math. Journ.* **2** (1953) 51—57.

ОБ ОДНОЙ ПРОБЛЕМЕ ТЕОРИИ ОТНОШЕНИЙ

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Резюме

Пусть E — топологическое пространство второй категории, не содержащее изолированные точки. Каждому $x \in E$ поставим в соответствие некоторое множество $(x \notin) S(x) \subseteqq E$, нигде не плотное в E. Пусть M — любое подмножество E и положим

$$S(M) = \bigcup_{x \in M} S(x).$$

Множество M называется независимым, если $M \cap S(M) = \emptyset$. По одной теореме P. ERDŐS-а существует независимое счетное множество. Автор доказывает, что

если дана некоторая вполне-упорядоченость множества E, то каждому порядковому числу $\alpha < \omega_1$ существует независимое множество, порядковый тип которого в вполне-упорядоченности E есть α.

Вопрос о том, что существует—ли независимое несчетное множество, пока не решен.