ON SOME COMBINATORIAL RELATIONS CONCERNING THE SYMMETRIC RANDOM WALK

by

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Introduction

In one-dimensional symmetric random walk it is interesting to note how often the particle intersects a given line and also how often it remains above it. I. VINCZE and E. CSÁKI [3] have determined in connection with a statistical problem regarding the GALTON-test the distribution of the number of intersections and also the joint distribution of the number of intersections and of the positive steps in the case the particle returns at the end to the origin. In another paper E. CSÁKI [4] has given the distribution of the number of intersections without assuming to which point the particle returns at the end. In the following paper, I should like to determine the above distributions when the particle reaches at the end some fixed point other than the origin.

This problem may be interpreted as: Two players A and B play a coin tossing game in which player A wins or loses a unit amount according to whether the result of the coin tossing is "head" or "tail". Assuming that at the end of the game A leads over B by certain fixed units, we are interested in investigating how often one overtakes the other and also how often A has been leading over B.

In this paper, we shall consider the sequences $\vartheta \equiv (\vartheta_1, \vartheta_2, \ldots, \vartheta_{2n})$ of $n+k$ $(+1)$'s and $n-k$ (-1)'s, each possible array has the same probability $\binom{2n}{n-k}^{-1} = \binom{2n}{n+k}^{-1}$. The partial sum of ϑ_i's is denoted by s_i, i.e.,

$$s_i = \vartheta_1 + \vartheta_2 + \ldots + \vartheta_i\,, \quad (i = 1, 2, \ldots, 2\,n)\,, \quad s_0 = 0 \text{ and } s_{2n} = 2\,k\,.$$

We shall call the array $\{s_0, s_1, \ldots, s_{2i}, \ldots, s_{2n}\}$ the path of the particle. Thus each array $(\vartheta_1, \vartheta_2, \ldots, \vartheta_{2i}, \ldots, \vartheta_{2n})$ corresponds to a random path of the particle starting at the origin and reaching after $2\,n$ steps the point $(2\,n, 2\,k)$ $(0 < k \leq n)$. Each path has the same probability. If the points (i, s_i) are represented in a plane and each of them is connected with the next one, then we obtain a figure illustrating the path of the particle. In the following, we shall consider the distributions of λ (number of intersections) and γ (number of positive steps).

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Notations:

$E_{2n,2k}$: a path $\{s_0, s_1, \ldots, s_{2i}, \ldots, s_{2n}\}$ with $s_{2n} = 2k$, $0 < k \leq n$, $E_{2n,0} = E_{2n}$. A point $(2i, s_{2i})$ of the path $E_{2n,2k}$ for which $s_{2i} = 0$ and $s_{2i-1} \cdot s_{2i+1} = -1$ is called the intersection point or T-point.

$T^{(r)}$-point: a point $(2i, s_{2i})$ of the path $E_{2n,2k}$ for which either $(s_{2i-1} = r - 1,\ s_{2i} = r,\ s_{2i+1} = r + 1)$ or $(s_{2i-1} = r + 1,\ s_{2i} = r,\ s_{2i+1} = r - 1)$ holds. This is called the intersection point in the height r, $T^{(0)} = T$.

$\lambda_{2n}^{(r)}$: number of $T^{(r)}$-points of the path $\{s_0, s_1, \ldots, s_{2n}\}$, $\lambda_{2n}^{(0)} = \lambda_{2n}$.

$E_{2n,2k}^{l}$: an $E_{2n,2k}$-path with exactly l T-points; $E_{2n,0}^{l} = E_{2n}^{l}$.

$E_{2n,2k,r}^{l}$: an $E_{2n,2k}$-path with exactly l $T^{(r)}$-points; $E_{2n,2k,0}^{l} = E_{2n,2k}^{l}$.

$F_{2n,r}^{l}$: a path $\{s_0, s_1, \ldots, s_{2n}\}$ with exactly l $T^{(r)}$-points and without assuming where it terminates; $F_{2n,0}^{l} = F_{2n}^{l}$, F_{2n}: a path $\{s_0, s_1, \ldots, s_{2n}\}$ without knowing where it terminates.

$E_{2n,2k,r}^{(2g,l)}$: an $E_{2n,2k,r}^{l}$-path with $2g$ steps above the height r; $E_{2n,2k,0}^{(2g,l)} = E_{2n,2k}^{(2g,l)}$, $E_{2n,0,0}^{(2g,l)} = E_{2n}^{(2g,l)}$, $E_{2n,2k,0}^{(2g,0)} = E_{2n,2k}^{(2g)}$: an $E_{2n,2k}$-path with $2g$ steps above the axis.

$F_{2n,r}^{(2g,l)}$: an $F_{2n,r}^{l}$-path with $2g$ steps above the height r; $F_{2n,0}^{(2g,l)} = F_{2n}^{(2g,l)}$.

$E_{2n,2k,r}^{(2g)}$: an $E_{2n,2k}$-path with $2g$ steps above the height r.

$F_{2n,r}^{(2g)}$: a path $\{s_0, s_1, \ldots, s_{2n}\}$ with $2g$ steps above the height r and without assuming where it terminates; $F_{2n,0}^{(2g)} = F_{2n}^{(2g)}$.

$2\gamma_{2n}^{(r)}$: number of steps of the path $\{s_0, s_1, \ldots, s_{2n}\}$ above the height r, $2\gamma_{2n}^{(0)} = 2\gamma_{2n}$.

H_m^q: a path $\{s_0, s_1, \ldots, s_m\}$ starting at the origin and reaching for the first time the height q at the m-th step.

$E_{x,y}^{+}$: a path $\{s_0, s_1, \ldots, s_x\}$ from $(0, 0)$ to (x, y) such that $s_0 = 0$, $s_1 > 0$, $s_2 > 0, \ldots, s_x > 0$ $(s_x = y,\ y > 0)$.

$N(\cdot)$: number of all possible paths whose type is given in the brackets $\left(\text{e.g. } N(E_{2n,2k}) = \binom{2n}{n-k} = \binom{2n}{n+k}\right)$.

§ 1. The number of intersections

We shall give two proofs for the following

Theorem 1.1.

$$(1)\qquad N(E_{2n,2k}^{l}) = \frac{2k + 2l + 1}{2n + 1}\binom{2n + 1}{n - k - l},\quad l = 0, 1, 2, \ldots, n - k,$$

and $0 < k \leq n$.

First proof. Let $\{s_0, s_1, \ldots, s_{2i}, \ldots, s_{2n}\}$ be a path of $N(E_{2n,2k}^{l})$ with $(2i, 0)$ the last or the l-th T-point (see fig. 1). It means that the section

$\{s_{2i}, s_{2i+1}, \ldots, s_{2n}\}$ of the path is such that it does not cross the axis. Thus the total number of such paths is given by (FELLER [1] p. 71 and [3])

$$N(E^l_{2n,2k}) = N(E^l_{2n,2k} \mid s_1 = +1 \text{ or } -1) =$$
$$= \sum_{i=l}^{n-k} \frac{1}{2} N(E^{l-1}_{2i}) \cdot N(E^+_{2n-2i+1,2k+1}) =$$
$$= \sum_{i=l}^{n-k} \frac{l}{i} \binom{2\,i}{i-l} \cdot \frac{2\,k+1}{2\,n-2\,i+1} \binom{2\,n-2\,i+1}{n-i+k+1}$$

($s_1 = +1$ or -1 according to l even or odd; $l = 0, 1, \ldots, n-k$.)

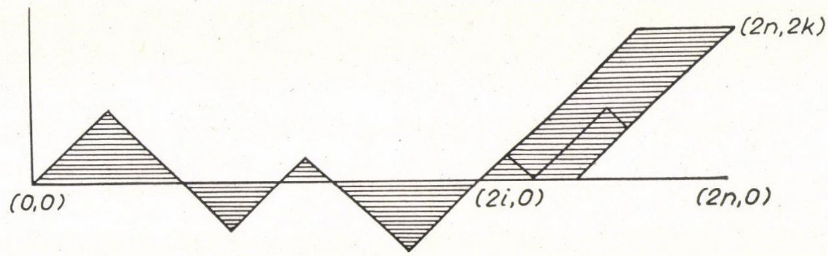

Fig. 1.

If we denote the generating function of the $N(E^l_{2n,2k})$'s by $F_l(v)$, it can be shown that

(2)
$$F_l(v) = \sum_{n-k-l=0}^{\infty} N(E^l_{2n,2k})\, v^{n-l-k} = \sum_{n-k-l=0}^{\infty} \frac{2\,k+2\,l+1}{n+k+l+1} \binom{2\,n}{n-k-l} v^{n-k-}.$$

which proves the theorem 1.1.

Second proof. There holds the following

Lemma 1.1.

(3) $$N(E^l_{2n,\,2k}) = N(H^{2k+2l+1}_{2n+1}) \,.$$

For the known relation (see FELLER [1] p. 71)

$$N(H^{2k+2l+1}_{2n+1}) = \frac{2\,k+2\,l+1}{2\,n+1} \binom{2\,n+1}{n-k-l}$$

the proof of the lemma gives us the proof of the theorem 1.1.

To prove the lemma, we have to establish a one-to-one correspondence between the two types of paths $E^l_{2n,\,2k}$ and $H^{2k+2l+1}_{2n+1}$ which can be set up in the following way:

Let us consider the path $\{s_0, s_1, \ldots, s_{2i}, \ldots, s_{2n}\}$ (see fig. 1). According to a proof given in [3] the section $\{s_0, s_1, \ldots, s_{2i}\}$ corresponds to a path starting at the origin and reaching after $2\,i$ steps for the first time the height $2\,l$. Concerning the section between $(2\,i,\,0)$ and $(2\,n,\,2\,k)$, let us first alter the signs and then the direction, i.e. we replace $\vartheta_{2i}, \vartheta_{2i+1}, \ldots, \vartheta_{2n}$ by $-\vartheta_{2n}$, $-\vartheta_{2n-1}, \ldots, -\vartheta_{2i}$ and let us attach this transformed section to the end of the previous one (fig. 2).

Finally, let us now insert after ϑ_{2n} a $(+1)$. Thus we obtain a $H_{2n+1}^{2k+2l+1}$-path. By reversing this procedure it may be seen that this transformation is a one-to-one. Then our theorem 1.1 gives immediately the following

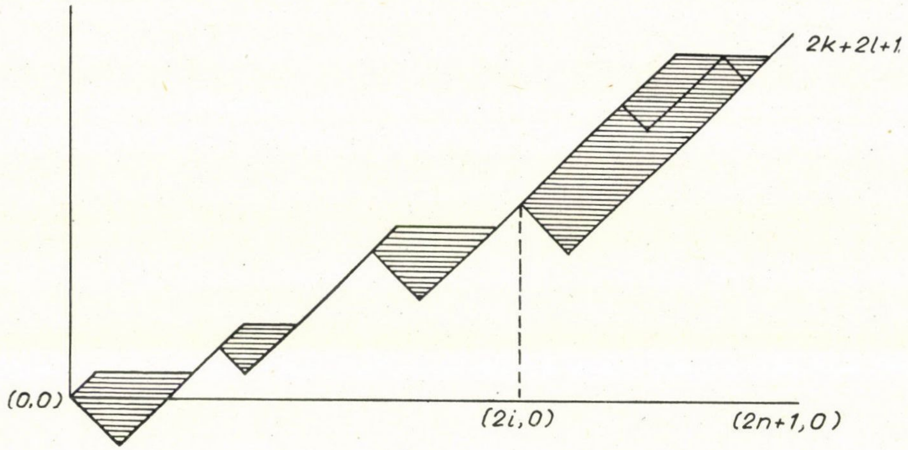

Fig. 2.

Theorem 1.1′.

$$P(\lambda_{2n} = l \mid E_{2n,2k}) = \frac{2k+2l+1}{2n+1} \cdot \frac{\binom{2n+1}{n-k-l}}{\binom{2n}{n-k}}, \quad l = 0, 1, \ldots, n-k,$$

(4) or

$$P(\lambda_{2n} < l \mid E_{2n,2k}) = 1 - \frac{\binom{2n}{n-k-l}}{\binom{2n}{n-k}}.$$

We can easily see that

$$E(\lambda_{2n} \mid E_{2n,2k}) = \frac{1}{\binom{2n}{n-k}} \sum_{r=0}^{n-k-1} \binom{2n}{r} = M,$$

(5) and

$$D^2(\lambda_{2n} \mid E_{2n,2k}) = [2(n-k) - M - 1]M - \frac{2}{\binom{2n}{n-k}} \sum_{r=1}^{n-k-1} r\binom{2n}{r}.$$

For the limiting distribution we have: for $k \sim a\sqrt{2n}$

$$\lim_{n\to\infty} P(\lambda_{2n} < y\sqrt{2n} \mid E_{2n,2k}) = 1 - e^{-2(a^2+y^2)}, \quad a \geq 0, \; y \geq 0,$$

$k = 0$ or $a = 0$ gives the result proved in [3].

Theorem 1.2.

(6) $$N(E^l_{2n+1,\,2k+1}) = \frac{2k+2l+2}{2n+2}\binom{2n+2}{n+k+l+2},\quad l = 0, 1, \ldots, n-k.$$

The proof of this theorem is similar to that of theorem 1.1 and it can easily be seen that the results corresponding to those given in theorem 1.1′ are the following

$$P(\lambda_{2n+1} = l \mid E_{2n+1,2k+1}) = \frac{2k+2l+2}{2n+2}\cdot\frac{\binom{2n+2}{n+k+l+2}}{\binom{2n+1}{n-k}};$$

(7) or

$$P(\lambda_{2n+1} < l \mid E_{2n+1,\,2k+1}) = 1 - \frac{\binom{2n+1}{n-k-l}}{\binom{2n+1}{n-k}}.$$

Also

$$E(\lambda_{2n+1} \mid E_{2n+1,2k+1}) = \frac{1}{\binom{2n+1}{n-k}}\sum_{r=0}^{n-k-1}\binom{2n+1}{r} = M_1,$$

and

$$D^2(\lambda_{2n+1} \mid E_{2n+1,2k+1}) = [2(n-k) - M_1 - 1] - \frac{2}{\binom{2n+1}{n-k}}\sum_{r=1}^{n-k-1} r\binom{2n+1}{r}.$$

For $k \sim a\sqrt{2n}$ we obtain the limiting distribution as in theorem 1.1′.

I should like to mention that by writing $|k|$ instead of k the above formulae are valid in the case of negative k as well.

When the condition that the particle reaches the point $(2n, 2k)$ at the $2n$-th step is disregarded, the number of paths with exactly l intersections on the axis is given by

$$N(F^l_{2n}) = 2\sum_{k=1}^{n} N(E^l_{2n,2k}) + N(E^l_{2n}) =$$

(8) $$= 2\sum_{k=1}^{n}\frac{2k+2l+1}{2n+1}\binom{2n+1}{n-k-l} + \frac{2(l+1)}{n}\binom{2n}{n-l-1} =$$

$$= 4\binom{2n-1}{n+l},\quad l = 0, 1, \ldots, n-1.$$

which is the same obtained in [4].

§ 2. Distribution of the number of intersections in the height r

Theorem 2.1. *For* $r < 2k$

$$P(\lambda_{2n}^{(r)} = l \mid E_{2n,2k}) = \frac{1}{\binom{2n}{n-k}} N(E_{2n,2k,r}^{l}) = \tag{9}$$

$$= \frac{k+l}{n+1} \cdot \frac{\binom{2n+2}{n-k-l+1}}{\binom{2n}{n-k}}, \quad \begin{array}{l} l = 1, 3, \ldots, n-k, \text{ if } n-k \text{ is odd,} \\ l = 1, 3, \ldots, n-k+1, \text{ if } n-k \text{ is even,} \\ r = 1, 2, \ldots, 2k-1. \end{array}$$

The proof of this theorem is trivial which follows from Lemma 1.1 by establishing the one-to-one correspondence between the two types of paths $E_{2n,2k,r}^{l}$ and H_{2n+2}^{2k+2l}. It is interesting to note that this result is independent of r.

For the limiting distribution we obtain: for $k \sim a\sqrt{2n}$

$$\lim_{n\to\infty} P\left(y \leqq \frac{\lambda_{2n}^{(r)}}{\sqrt{2n}} < y + dy \mid E_{2n,2k}\right) = 4(y+a)e^{2a^2-2(a+y)^2} dy, \quad a, y \geqq 0 \tag{10}$$

which is equal to the result obtained in theorem 2.1′ in [3] $k = 0$ or $a = 0$ gives the result (6) given in [3].

Theorem 2.2. *For* $r = 2k$

$$P(\lambda_{2n}^{(2k)} = l \mid E_{2n,2k}) = \frac{1}{\binom{2n}{n-k}} N(E_{2n,2k,2k}^{l}) =$$

$$= \frac{2k+2l+1}{2n+1} \cdot \frac{\binom{2n+1}{n-k-l}}{\binom{2n}{n-k}}, \quad l = 0, 1, \ldots, n-k. \tag{11}$$

The proof is trivial and follows from lemma 1.1 by showing a one-to-one correspondence between the paths $E_{2n,2k,2k}^{l}$ and $H_{2n+1}^{2k+2l+1}$.

For the limiting distribution

$$\lim_{n\to\infty} P\left(y \leqq \frac{\lambda_{2n}^{(2k)}}{\sqrt{2n}} < y + dy \mid E_{2n,2k}\right) = 4(y+a)\, e^{2a^2-2(y+a)^2}\, dy, \tag{12}$$

holds for $k \sim a\sqrt{2n}$, $a, y \geq 0$, which is equal to the result obtained in (10).

Theorem 2.3. *For* $r > 2\,k$

$$P(\lambda_{2n}^{(r)} = l \mid E_{2n,2k}) = \frac{1}{\binom{2\,n}{n-k}} N(E_{2n,2k,r}^{l}) = \tag{13}$$

$$= \frac{l+r-k}{n+1} \cdot \frac{\binom{2\,n+2}{n-l-r+k+1}}{\binom{2\,n}{n-k}}, \quad \begin{array}{l} r = 2\,k+1, 2\,k+2, \ldots, \\ l = 0, 2, 4, \ldots, n+k-r, \textit{ if } n+k-r \\ \quad \textit{is even or} \\ l = 0, 2, 4, \ldots, n+k-r+1, \textit{ if} \\ \quad n+k-r \textit{ is odd.} \end{array}$$

For $k = 0$, we get the result obtained in theorem 2.1′ [3] for l intersections in the height r.

The proof is similar to that of theorems 2.1 and 2.2 and can be given by establishing the one-to-one correspondence between the paths $E_{2n,2k,r}^{l}$ and $H_{2n+2}^{2l+2r-2k}$.

For the limiting distribution we obtain: for $k \sim a\sqrt{2\,n}$, and $r \sim b\sqrt{2\,n}$,

$$\lim_{n\to\infty} P\left(y \leqq \frac{\lambda_{2n}^{(r)}}{\sqrt{2\,n}} < y + dy \mid E_{2n,2k}\right) = 4(y+b-a)\, e^{2a^2-2(y+b-a)^2} dy\,, \quad a, b, y \geqq 0\,. \tag{14}$$

It is clear from (6), (9) and (13) that there is a one-to-one correspondence between the paths $E_{2n+1,2k+1}^{l-1}$, $E_{2n,2k,r}^{l}$ and $E_{2n,0,r}^{l}$.

When the condition that the particle reaches the point $(2\,n, 2\,k)$ at the $2\,n$-th step is disregarded, the number of paths with exactly l $T^{(r)}$-points is given as below according to r is even or odd:

For a fixed positive even integer r

$$N(F_{2n,r}^{l}) = \begin{cases} \displaystyle\sum_{k=\frac{r}{2}+1}^{n} N(E_{2n,2k,r}^{l}) + N(E_{2n,2k,2k}^{l})\,, & l \text{ odd}, \\ \displaystyle\sum_{k=-n}^{\frac{r}{2}-1} N(E_{2n,2k,r}^{l}) + N(E_{2n,2k,2k}^{l})\,, & l \text{ even}. \end{cases}$$

But it can easily be verified that

$$\sum_{k=-n}^{\frac{r}{2}-1} N(E_{2n,2k,r}^{l}) = \sum_{k=\frac{r}{2}+1}^{n} N(E_{2n,2k,r}^{l}) = \binom{2\,n+1}{n-l-\frac{r}{2}}.$$

Substituting the value of $N(E_{2n,2k,2k}^{l})$ from (11), we have

$$N(F_{2n,r}^{l}) = 2\binom{2\,n}{n+l+\frac{r}{2}}, \quad l = 1, 2, \ldots, n-\frac{r}{2}. \tag{15}$$

Similarly, when r is a fixed positive odd integer

$$N(F^l_{2n,r}) = \begin{cases} \sum_{k=\frac{r+1}{2}}^{n} N(E^l_{2n,2k,r}), & l \text{ odd}, \\ \sum_{k=-n}^{\frac{r-1}{2}} N(E^l_{2n,2k,r}), & l \text{ even}. \end{cases}$$

It can easily be seen that

$$\sum_{k=\frac{r+1}{2}}^{n} N(E^l_{2n,2k,r}) = \sum_{k=-n}^{\frac{r-1}{2}} N(E^l_{2n,2k,r}) = \binom{2n+1}{n+l+\frac{r+1}{2}}$$

so, for a fixed positive odd r

$$(16) \qquad N(F^l_{2n,r}) = \binom{2n+1}{n+l+\frac{r+1}{2}}, \quad l = 1, 2, \ldots, n - \frac{r-1}{2}.$$

The results in (15) and (16) lead to the following

Theorem 2.4. *For a fixed positive even r*

$$(17) \qquad P(\lambda^{(r)}_{2n} = l \mid F^l_{2n,r}) = \frac{1}{2^{2n-1}} \binom{2n}{n+l+\frac{r}{2}}, \quad l = 1, 2, \ldots, n - \frac{r}{2},$$

and for the limiting distribution

$$\lim_{n\to\infty} P\left(y \leq \frac{\lambda^{(r)}_{2n}}{\sqrt{2n}} < y + dy \mid F^l_{2n,r}\right) = 2\sqrt{\frac{2}{\pi}}\, e^{-\frac{(2y+a)^2}{2}} dy,$$

$$\textit{for } r \sim a\sqrt{2n}, \; a, y \geq 0.$$

Theorem 2.5. *For a fixed positive odd r*

$$(18) \qquad P(\lambda^{(r)}_{2n} = l \mid F^l_{2n,r}) = \frac{1}{2^{2n}} \binom{2n+1}{n+l+\frac{r+1}{2}}, \quad l = 1, 2, \ldots, n - \frac{r-1}{2},$$

and for $r \sim a\sqrt{2n}$ we get the same limiting distribution as in Theorem 2.4.

The results in (17) and (18) are same as obtained in [4].

§ 3. The joint distribution of the number of intersections and the number of positive steps

Theorem 3.1.

$$N(E_{2n,2k}^{(2g,l)}) = \begin{cases} \dfrac{l(2k+l+1)}{2(n-g)\left(g+k+\dfrac{l}{2}+1\right)}\dbinom{2g}{g-k-\dfrac{l}{2}}\dbinom{2(n-g)}{n-g-\dfrac{l}{2}}, \\ \qquad l \text{ even},\quad l+2k \leqq 2g \leqq 2n-l, \\ \dfrac{(l+1)(2k+l)}{2(n-g)\left(g+k+\dfrac{l+1}{2}\right)}\dbinom{2g}{g-k-\dfrac{l-1}{2}}\dbinom{2(n-g)}{n-g-\dfrac{l+1}{2}}, \\ \qquad l \text{ odd},\ 2k+l-1 \leqq 2g \leqq 2n-l-1. \end{cases} \tag{19}$$

Proof. Let us consider the path taken in theorem 1.1 (fig. 1) where $(2i, 0)$ is the last or the l-th intersection point. It is clear that if l is even, then the first step of the path $E_{2n,2k}^{(2g,l)}$ should be positive and if l is odd, then it should be negative. Thus, it can be seen (see [3]) that when l is even

$$N(E_{2n,2k}^{(2g,l)}) = N(E_{2n,2k}^{(2g,l)} \mid s_1 = +1) =$$

$$= \sum_{i=n-g+\frac{l}{2}}^{n-k} N(E_{2i}^{(2(g+i-n),l-1)} \mid s_1 = +1)\cdot N(E_{2n-2i+1,2k+1}^{+}) =$$

$$= \frac{1}{4}\sum_{i=n-g+\frac{l}{2}}^{n-k} \frac{l^2}{(n-g)(g+i-n)}\binom{2(g+i-n)}{g+i-n-\dfrac{l}{2}}\binom{2(n-g)}{n-g-\dfrac{l}{2}}\times$$

$$\times\frac{2k+1}{2n-2i+1}\binom{2n-2i+1}{n-i+k+1}$$

which, by using the method of generating functions, gives the required result.

Similarly, when l is odd, the result follows easily. Thus the results in (19) lead to the joint distribution given in the following

Theorem 3.1′.

$$P(\gamma_{2n} = g, \lambda_{2n} = l \mid E_{2n,2k}) = \tag{20}$$

$$= \begin{cases} \dfrac{1}{\dbinom{2n}{n-k}}\left[\dfrac{l(2k+l+1)}{2(n-g)\left(g+k+\dfrac{l}{2}+1\right)}\dbinom{2g}{g-k-\dfrac{l}{2}}\dbinom{2(n-g)}{n-g-\dfrac{l}{2}}\right], \; l \text{ even}, \\ \dfrac{1}{\dbinom{2n}{n-k}}\left[\dfrac{(l+1)(2k+l)}{2(n-g)\left(g+k+\dfrac{l+1}{2}\right)}\dbinom{2g}{g-k-\dfrac{l-1}{2}}\dbinom{2(n-g)}{n-g-\dfrac{l+1}{2}}\right], \; l \text{ odd}, \end{cases}$$

and for the limiting case we obtain: for $k \sim a\sqrt{2n}$

$$\lim_{n\to\infty} P\left(y \leq \frac{\lambda_{2n}}{\sqrt{2n}} < y + dy, z \leq \frac{\gamma_{2n}}{n} < z + dz \mid E_{2n,2k}\right) =$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{y(2a+y)}{\{z(1-z)\}^{3/2}} e^{\left[2a^2 - \frac{(2a+y)^2}{2z} - \frac{y^2}{2(1-z)}\right]} dy\, dz\,, \quad a, y \geq 0\,, \quad 0 \leq z \leq 1\,.$$

$k = 0$ or $a = 0$ gives the result (11) in [3].

The joint distribution of λ_{2n+1} *and* γ_{2n+1} *is given by the following*

Theorem 3.2.

(21) $P(\gamma_{2n+1} = g, \lambda_{2n+1} = l \mid E_{2n+1,2k+1}) =$

$$= \begin{cases} \dfrac{1}{\binom{2n+1}{n-k}} \left[\dfrac{l(2k+l+2)}{2(n-g)\left(g+k+\frac{l}{2}+2\right)} \binom{2g+1}{g-k-\frac{l}{2}} \binom{2(n-g)}{n-g-\frac{l}{2}} \right], \\ \qquad\qquad l \text{ even}, \ l + 2k \leq 2g \leq 2n - l\,, \\ \dfrac{1}{\binom{2n+1}{n-k}} \left[\dfrac{(l+1)(2k+l+1)}{2(n-g)\left(g+k+\frac{l+1}{2}+1\right)} \binom{2g+1}{g-k-\frac{l-1}{2}} \binom{2(n-g)}{n-g-\frac{l+1}{2}} \right], \\ \qquad\qquad l \text{ odd}, \ 2k + l - 1 \leq 2g \leq 2n - l - 1\,. \end{cases}$$

(Here the no. of positive steps is given by $2\gamma_{2n+1} + 1$.)

The proof of this theorem is quite simple which is similar to that of theorem 3.1.

For $k \sim a\sqrt{2n}$, we obtain the same limiting distribution as in theorem 3.1′.

§ 4. Joint distribution of $\lambda_{2n}^{(r)}$ and $\gamma_{2n}^{(r)}$

We shall prove the following

Theorem 4.1. *For* $r < 2k$

$$N(E_{2n,2k,r}^{(2g,l)}) = \frac{(r+l)(2k-r+l)}{\left(n-g+\frac{r}{2}+\frac{l+1}{2}\right)\left(g+k-\frac{r}{2}+\frac{l+1}{2}\right)} \times$$

$$\times \binom{2g}{g-k+\frac{r}{2}-\frac{l-1}{2}} \binom{2(n-g)}{n-g-\frac{r}{2}-\frac{l-1}{2}},$$

(22) l odd, r even,

$$l = 1, 3, \ldots, \text{ odd } (n-k,\ n-k+1)$$

$$2k - r + (l-1) \leq 2g \leq 2n - r - (l-1)\,,$$

(22) *and*

$$N(E^{(2g+1,l)}_{2n,2k,r}) = \frac{(r+l)(2k-r+l)}{2(n-g)\left(g+k-\frac{r-1}{2}+\frac{l+1}{2}\right)} \times$$

$$\times \binom{2g+1}{g-k+\frac{r+1}{2}-\frac{l-1}{2}} \binom{2(n-g)}{n-g-\frac{r-1}{2}-\frac{l+1}{2}},$$

l odd, r odd,

$$l = 1, 3, \ldots, \text{odd}\,(n-k, n-k+1)$$

$$2k - r + l - 2 \geqq 2g \geqq 2n - r - l.$$

Proof Case I: r even,

Let $\{s_0, s_1, \ldots, s_{r_1}, \ldots, s_{2n}\}$ be the path of the type $E^{(2g,l)}_{2n,2k,r}$ and let $P(r_1, r)$ be the first $T^{(r)}$-point of the path (fig. 3). Then the section between $P(r_1, r)$ and $Q(2n, 2k)$ is a path of $E^{(2g,l-1)}_{2n-r_1,2k-r}$.

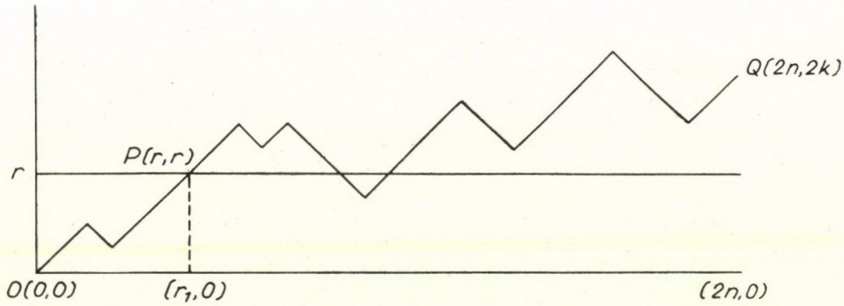

Fig. 3.

Attaching a $+1$ to the end of the section between 0 and P we obtain a path of $H^{r+1}_{r_1+1}$ type. Thus,

$$N(E^{(2g,l)}_{2n,2k,r}) = \sum_{r_1=r}^{2n-2g-(l-1)} N(H^{r+1}_{r_1+1}) \cdot N(E^{(2g,l-1)}_{2n-r_1,2k-r}) =$$

$$= \sum_{r_1=r}^{2n-2g-(l-1)} \frac{r+1}{r_1+1} \binom{r_1+1}{\frac{r_1-r}{2}} \frac{(l-1)(2k-r+l)}{(2n-r_1-2g)\left(g+k-\frac{r}{2}+\frac{l+1}{2}\right)} \times$$

$$\times \binom{2g}{g-k+\frac{r}{2}-\frac{l-1}{2}} \binom{2n-r_1-2g}{n-g-\frac{r_1}{2}-\frac{l-1}{2}}.$$

Using the method of generating functions we obtain the desired result. Similarly, when r is odd, the result follows easily. The only interesting point to note is that the number of steps above the height r will be odd.

Theorem 4.2. *When* $r = 2k$

(23) $N(E^{(2g,l)}_{2n,2k,2k}) =$

$$= \begin{cases} \dfrac{l(2k+l+1)}{2g\left(n-g+k+\dfrac{l}{2}+1\right)}\begin{pmatrix}2g\\ g-\dfrac{l}{2}\end{pmatrix}\begin{pmatrix}2(n-g)\\ n-g-k-\dfrac{l}{2}\end{pmatrix}, & l \text{ even}, \\ & l \leqq 2g \leqq 2n-2k-l, \\ \dfrac{(l+1)(2k+l)}{2g\left(n-g+k+\dfrac{l+1}{2}\right)}\begin{pmatrix}2g\\ g-\dfrac{l+1}{2}\end{pmatrix}\begin{pmatrix}2(n-g)\\ n-g-k-\dfrac{l-1}{2}\end{pmatrix}, & l \text{ odd}, \\ & l+1 \leqq 2g \leqq 2n-2k-l+1. \end{cases}$$

Proof Case I: l even.

Let $\{s_0, s_1, \ldots, s_{r_1}, \ldots, s_{2n}\}$ be the path of the type $E^{(2g,l)}_{2n,2k,2k}$ and let $P(r_1, 2k)$ be the first $T^{(r)}$-point (fig. 4). Then the section between $P(r_1, 2k)$ and $Q(2n, 2k)$ is a path of the type $(E^{(2g,l-1)}_{2n-r_1} \mid s_1 = +1)$.

Attaching a $+1$ to the end of the section between 0 and P we obtain a path of $H^{2k+1}_{r_1+1}$.

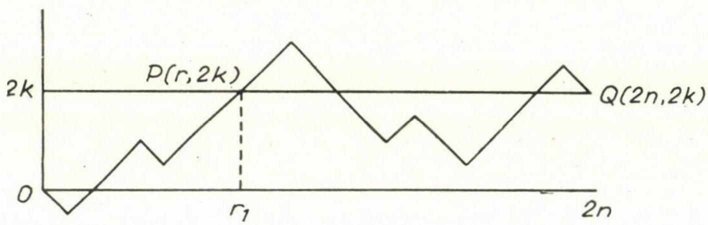

Fig. 4.

Thus

$$N(E^{(2g,l)}_{2n,2k,2k}) = \sum_{r_1=2k}^{2n-2g-l} N(H^{2k+1}_{r_1+1}) \cdot N(E^{(2g,l-1)}_{2n-r_1} \mid s_{r_1+1} = +1) =$$

$$= \frac{l}{2g}\begin{pmatrix}2g\\ g-\dfrac{l}{2}\end{pmatrix}\sum_{r_1=2k}^{2n-2g-l}\frac{2k+1}{r_1+1}\begin{pmatrix}r_1+1\\ \dfrac{r_1}{2}-k\end{pmatrix}\frac{l}{2\left(n-\dfrac{r_1}{2}-g\right)}\begin{pmatrix}2n-2g-r_1\\ n-g-\dfrac{r_1}{2}-\dfrac{l}{2}\end{pmatrix}.$$

But by using the method of generating functions we obtain the desired result.

Similarly, it is easy to show that the desired result follows for odd l.

Theorem 4.3. *For* $r > 2k$

$$N(E^{(2g,l)}_{2n,2k,r}) = \frac{l(2r-2k+l)}{2g\left(n-g+r-k+\dfrac{l}{2}+1\right)}\begin{pmatrix}2g\\ g-\dfrac{l}{2}\end{pmatrix}\begin{pmatrix}2n-2g+1\\ n-g-r+k-\dfrac{l}{2}+1\end{pmatrix}, \tag{24}$$

l *even*,

$l = 0, 2, 4, \ldots,$ *even* $(n+k-r,\ n+k-r+1)$

$r = 2k+1,\ 2k+2, \ldots$

For $k = 0$, i.e. when the particle starting at the origin returns to the origin at the $2n$-th step

$$N(E_{2n,0,r}^{(2g,l)}) = \frac{l(2r+l)}{2g\left(n-g+r+\frac{l}{2}+1\right)}\binom{2g}{g-\frac{l}{2}}\binom{2n-2g+1}{n-g-r-\frac{l}{2}+1}, \tag{25}$$

$l = 0, 2, 4, \ldots,$ even $(n-r, n-r+1)$
$r = 1, 2, \ldots$

The result (25) is the simpler form of the same result obtained in (14) of [3].

Proof. To prove this theorem we shall consider the path $\{s_0, s_1, \ldots, s_{r_1}, \ldots, s_{2n}\}$ of the type $E_{2n,2k,r}^{(2g,l)}$ and let $P(r_1, r)$ be the first $T^{(r)}$-point (fig. 5). Then the section between $P(r_1, r)$ and $Q(2n, 2k)$ is an $E_{2n-r_1,r-2k}^{(2n-2g-r_1,l-1)}$-path starting in the negative direction. The first section corresponds to an $H_{r_1+1}^{r+1}$-path.

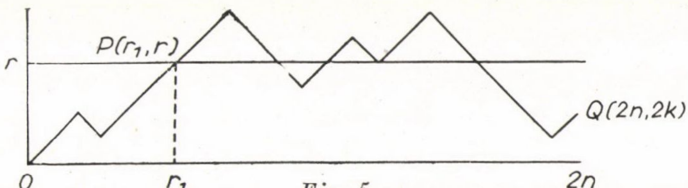

Fig. 5.

Thus

$$N(E_{2n,2k,r}^{(2g,l)}) = \sum_{r_1=r}^{2n-2g+2k-r-l+2} N(H_{r_1+1}^{r+1}) \cdot N(E_{2n-r_1,r-2k}^{(2n-2g-r_1,l-1)} \mid s_1 = -1) =$$

$$= \frac{l}{2g}\binom{2g}{g-\frac{l}{2}} \sum_{r_1=r}^{2n-2g+2k-r-l+2} \frac{r+1}{r_1+1}\binom{r_1+1}{\frac{r_1-r}{2}} \times$$

$$\times \frac{(r-2k+l-1)}{n-g-k-\frac{r_1}{2}+\frac{r}{2}+\frac{l}{2}}\binom{2n-2g-r_1}{n-g-\frac{r_1}{2}+k-\frac{r}{2}-\frac{l}{2}+1}.$$

By using the method of generating functions we obtain the desired result.

Theorems 4.1, 4.2 and 4.3 lead to the following

Theorem 4.1′. $r < 2k$, l *odd*

$P(\gamma_{2n}^{(r)} = g, \lambda_{2n}^{(r)} = l \mid E_{2n,2k}) =$

$$= \begin{cases} \dfrac{1}{\binom{2n}{n-k}}\left[\dfrac{(r+l)(2k-r+l)}{\left(n-g+\frac{r}{2}+\frac{l+1}{2}\right)\left(g+k-\frac{r}{2}+\frac{l+1}{2}\right)}\binom{2g}{g-k+\frac{r}{2}-\frac{l-1}{2}}\times \binom{2(n-g)}{n-g-\frac{r}{2}-\frac{l-1}{2}}\right], & r \text{ even}, \\[2ex] \dfrac{1}{\binom{2n}{n-k}}\left[\dfrac{(r+l)(2k-r+l)}{2(n-g)\left(g+k-\frac{r-1}{2}+\frac{l+1}{2}\right)}\binom{2g+1}{g-k+\frac{r+1}{2}-\frac{l-1}{2}}\times \binom{2(n-g)}{n-g-\frac{r-1}{2}-\frac{l+1}{2}}\right], & r \text{ odd}. \end{cases} \tag{26}$$

***For the limiting distribution we obtain:** for* $k \sim a\sqrt{2n}$, $r \sim b\sqrt{2n}$

$$\lim_{n\to\infty} P\left(y \leqq \frac{\lambda_{2n}^{(r)}}{\sqrt{2n}} < y + dy\,,\ z \leqq \frac{\gamma_{2n}^{(r)}}{n} < z + dz \mid E_{2n,2k}\right) =$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{(2a - b + y)(b + y)}{\{z(1-z)\}^{3/2}} e^{\left[2a^2 - \frac{(2a-b+y)^2}{2z} - \frac{(b+y)^2}{2(1-z)}\right]} dy\, dz,$$

$$a, b, y \geqq 0,\ 0 \leqq z \leqq 1.$$

$r = 0$ or $b = 0$ gives the result obtained in theorem 3.1′.

Theorem 4.2′. $r = 2k$

$P(\gamma_{2n}^{(r)} = g,\ \lambda_{2n}^{(r)} = l \mid E_{2n,2k}) =$

$$(27)\quad = \begin{cases} \dfrac{1}{\binom{2n}{n-k}} \left[\dfrac{l(2k+l+1)}{2g\left(n-g+k+\frac{l}{2}+1\right)} \binom{2g}{g-\frac{l}{2}} \binom{2(n-g)}{n-g-k-\frac{l}{2}} \right], & l \text{ even}, \\[2ex] \dfrac{1}{\binom{2n}{n-k}} \left[\dfrac{(l+1)(2k+l)}{2g\left(n-g+k+\frac{l+1}{2}\right)} \binom{2g}{g-\frac{l+1}{2}} \binom{2(n-g)}{n-g-k-\frac{l-1}{2}} \right], & l \text{ odd}. \end{cases}$$

For $k \sim a\sqrt{2n}$ *we obtain the same limiting form as obtained in theorem* 3.1′.

Theorem 4.3′. *For* $r > 2k$, l *even*

$$(28)\qquad P(\gamma_{2n}^{(r)} = g,\ \lambda_{2n}^{(r)} = l \mid E_{2n,2k}) =$$

$$= \frac{1}{\binom{2n}{n-k}} \left[\frac{l(2r-2k+l)}{2g\left(n-g+r-k+\frac{l}{2}+1\right)} \binom{2g}{g-\frac{l}{2}} \binom{2n-2g+1}{n-g-r+k-\frac{l}{2}+1} \right],$$

and for the limiting joint distribution we obtain: for $k \sim a\sqrt{2n}$, $r \sim b\sqrt{2n}$

$$(29)\quad \begin{aligned} &P\left(y \leqq \frac{\lambda_{2n}^{(r)}}{\sqrt{2n}} < y + dy,\ z \leqq \frac{\gamma_{2n}^{(r)}}{n} < z + dz \mid E_{2n,2k}\right) = \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{y(2b - 2a + y)}{\{z(1-z)\}^{3/2}} e^{\left[2a^2 - \frac{y^2}{2z} - \frac{(2b-2a+y)^2}{2(1-z)}\right]} dy\, dz, \end{aligned}$$

$$a, b, y \geqq 0\,,\ 0 \leqq z \leqq 1\,,$$

$k = 0$ or $a = 0$ gives the result as obtained in theorem 2.2′ [3]. It may be mentioned that the limiting joint distributions of $(\lambda_{2n}, \gamma_{2n} \mid E_{2n,2k})$, $(\lambda_{2n}^{(2k)}, \gamma_{2n}^{(2k)} \mid E_{2n,2k})$, and $(\lambda_{2n}^{(r)}, \gamma_{2n}^{(r)} \mid E_{2n})$ are the same as seen in theorem 3.1′ and (29).

§ 5. Joint Distribution of λ_{2n} and γ_{2n} for an F_{2n}-path

We shall prove the following

Theorem 5.1.

$$(30)\qquad N(F_{2n}^{(2g,l)}) = \begin{cases} \left[\dfrac{l(2g+l+2)}{4g(n-g)}\dbinom{2g}{g-\frac{l}{2}-1}\dbinom{2(n-g)}{n-g-\frac{l}{2}} + \right. \\ \qquad \left. + \dfrac{l(2n-2g+l+2)}{4g(n-g)}\dbinom{2g}{g-\frac{l}{2}}\dbinom{2(n-g)}{n-g-\frac{l}{2}-1}\right], \\ \qquad l \text{ even}, \\ \qquad g=\dfrac{l}{2}, \dfrac{l}{2}+1, \ldots, n-\dfrac{l}{2}, \\ \dfrac{(l+1)(n+l+1)}{2g(n-g)}\dbinom{2g}{g-\frac{l+1}{2}}\dbinom{2(n-g)}{n-g-\frac{l+1}{2}}, \\ \qquad l \text{ odd}, \\ \qquad g=\dfrac{l+1}{2}, \ldots, n-\dfrac{l+1}{2}. \end{cases}$$

Proof. It can be seen from (19) and [3] that when l is even

$$N(F_{2n}^{(2g,l)}) = \sum_{k=1}^{n} N(E_{2n,2k}^{(2g,l)}) + \sum_{k=1}^{n} N(E_{2n,-2k}^{(2g,l)}) + N(E_{2n}^{(2g,l)}),$$

But it is easy to see that there is a one-to-one correspondence between the paths $E_{2n,-2k}^{(2g,l)}$ and $E_{2n,2k}^{(2n-2g,l)}$. Thus, on substituting the values of different types of paths, we get the desired result.

The proof for odd l can be given analogously. From (30) there follows the following

Theorem 5.1'. *For the random variables λ_{2n} and γ_{2n} the joint distribution law*

$$(31)\qquad P(\gamma_{2n}=g, \lambda_{2n}=l \mid F_{2n}) =$$

$$= \begin{cases} \dfrac{1}{2^{2n}}\left[\dfrac{l(2g+l+2)}{4g(n-g)}\dbinom{2g}{g-\frac{l}{2}-1}\dbinom{2(n-g)}{n-g-\frac{l}{2}} + \right. \\ \qquad \left. + \dfrac{l(2n-2g+l+2)}{4g(n-g)}\dbinom{2g}{g-\frac{l}{2}}\dbinom{2(n-g)}{n-g-\frac{l}{2}-1}\right], & l \text{ even}, \\ \dfrac{1}{2^{2n}}\left[\dfrac{(l+1)(n+l+1)}{2g(n-g)}\dbinom{2g}{g-\frac{l+1}{2}}\dbinom{2(n-g)}{n-g-\frac{l+1}{2}}\right], & l \text{ odd}, \end{cases}$$

holds and for the limiting joint distribution we obtain

$$\lim_{n\to\infty} P\left(y \leqq \frac{\lambda_{2n}}{\sqrt{2n}} < y + dy\,,\ z \leqq \frac{\gamma_{2n}}{n} < z + dz \mid F_{2n}\right) = \tag{32}$$
$$= \frac{1}{\pi} \cdot \frac{y}{\{z(1-z)\}^{3/2}} e^{-\frac{y^2}{2z(1-z)}} dy\, dz \qquad y \geqq 0\,,\ 0 \leqq z \leqq 1\,,$$

which is similar to the result obtained by E. CSÁKI and I. VINCZE (see [6]). Integration with respect to y from 0 to ∞ leads to the arc sin law [1].

§ 6. Joint distribution of $\lambda_{2n}^{(r)}$ and $\gamma_{2n}^{(r)}$ for an F_{2n}-path

Theorem 6.1. *For r even*

$$N(F_{2n,r}^{(2g,l)}) = \begin{cases} \dfrac{l}{g}\dbinom{2g}{g-\frac{l}{2}}\dbinom{2n-2g}{n-g-\frac{r}{2}-\frac{l}{2}}, & l \text{ even},\ g = \dfrac{l}{2}, \dfrac{l}{2}+1, \ldots, \\[2ex] \dfrac{(r+l)(2g+l+1)}{2g\left(n-g+\frac{r}{2}+\frac{l+1}{2}\right)}\dbinom{2g}{g-\frac{l+1}{2}}\dbinom{2(n-g)}{n-g-\frac{l-1}{2}}, & l \text{ odd}, \\[2ex] & g = \dfrac{l+1}{2}, \dfrac{l+3}{2}, \ldots, \end{cases} \tag{33}$$

and for r odd

$$N(F_{2n,r}^{(2g,l)}) = \frac{l}{2g}\binom{2g}{g-\frac{l}{2}}\binom{2n-2g+1}{n-g-\frac{r-1}{2}-\frac{l}{2}}, \quad l \text{ even},$$
$$g = \frac{l}{2}, \frac{l}{2}+1, \ldots, \tag{34}$$
$$N(F_{2n,r}^{(2g+1,l)}) = \frac{(r+l)}{2(n-g)}\binom{2g+1}{g-\frac{l-1}{2}}\binom{2(n-g)}{n-g-\frac{r-1}{2}-\frac{l+1}{2}}, \quad l \text{ odd},$$
$$g = \frac{l-1}{2}, \frac{l+1}{2}, \ldots,$$

Proof. r *even*
when l is even, it follows from (23) and (24) that

$$N(F_{2n,r}^{(2g,l)}) = \sum_{k=g-n+r+\frac{l}{2}-1}^{\frac{r}{2}-1} N(E_{2n,2k,r}^{(2g,l)}) + N(E_{2n,2k,2k}^{(2g,l)}).$$

After simplification we can get the required result. Similarly, when l is odd; it can be seen from (22) and (23) that

$$N(F_{2n,r}^{(2g,l)}) = \sum_{k=\frac{r}{2}+1}^{g+\frac{r}{2}-\frac{l-1}{2}} N(E_{2n,2k,r}^{(2g,l)}) + N(E_{2n,2k,2k}^{(2g,l)})$$

which on simplifying proves the result.

When r is odd

It is clear from (22) and (24) that

$$N(F_{2n,r}^{(2g,l)}) = \sum_{k=g-n+r+\frac{l}{2}-1}^{\frac{r-1}{2}} N(E_{2n,2k,r}^{(2g,l)}),\ l \text{ even},$$

and

$$N(F_{2n,r}^{(2g+1,l)}) = \sum_{k=\frac{r+1}{2}}^{g+\frac{r+1}{2}-\frac{l-1}{2}} N(E_{2n,2k,r}^{(2g+1,l)}),\ l \text{ odd}.$$

After simplification we obtain the required results. These results lead to the following

Theorem 6.1′. *For r even*

$P(\gamma_{2n}^{(r)}=g,\ \lambda_{2n}^{(r)}=l \mid F_{2n}) =$

$$(35)\quad = \begin{cases} \dfrac{1}{2^{2n}}\left[\dfrac{l}{g}\binom{2g}{g-\frac{l}{2}}\binom{2n-2g}{n-g-\frac{r}{2}-\frac{l}{2}}\right], & l \text{ even}, \\[2ex] \dfrac{1}{2^{2n}}\left[\dfrac{(r+l)(2g+l+1)}{2g\left(n-g+\frac{r}{2}+\frac{l+1}{2}\right)}\binom{2g}{g-\frac{l+1}{2}}\binom{2n-2g}{n-g-\frac{r}{2}-\frac{l-1}{2}}\right], & l \text{ odd}. \end{cases}$$

From [6] it follows that

$$P(\gamma_{2n}^{(r)}=0,\ \lambda_{2n}^{(r)}=0 \mid F_{2n}) = \frac{1}{2^{2n}}\left[2\sum_{j=1}^{\frac{r}{2}}\binom{2n}{n+j}+\binom{2n}{n}\right]$$

and for the limiting joint distribution we obtain: for $r \sim b\sqrt{2n}$

$$\lim_{n\to\infty} P\left(y \leq \frac{\lambda_{2n}^{(r)}}{\sqrt{2n}} < y+dy,\ z \leq \frac{\gamma_{2n}^{(r)}}{n} < z+dz \mid F_{2n}\right) =$$

$$= \frac{1}{\pi}\cdot\frac{(y+bz)}{\{z(1-z)\}^{3/2}} e^{-\frac{b^2}{2}-\frac{(y+bz)^2}{2z(1-z)}} dy\, dz \qquad b, y \geqq 0,\ 0 \leqq z \leqq 1.$$

Integration with respect to y from 0 to ∞ leads to

$$\lim_{n\to\infty} P\left(z \leq \frac{\gamma_{2n}^{(r)}}{n} < z+dz \mid F_{2n}\right) = \frac{1}{\pi}\cdot\frac{e^{-\frac{b^2}{2(1-z)}}}{\sqrt{z(1-z)}} dz$$

which corresponds to the result obtained by E. CSÁKI and I. VINCZE (see [6]).

The result for odd values of r can be written analogously and it is easy to see that the limiting joint distribution of $\gamma_{2n}^{(r)}$ and $\lambda_{2n}^{(r)}$ is equivalent to the one just obtained above for even r.

§ 7. Distribution of γ_{2n} for an $E_{2n,2k}$-path

In 1949, Chung and Feller [7] proved that

$$P(\gamma_{2n} = g \mid E_{2n,2k}) = \frac{k}{\binom{2n}{n-k}} \sum_{i=k}^{g} \binom{2i}{i-k} \binom{2n-2i}{n-i} \cdot \frac{1}{i(n-i+1)} .$$

But we shall prove the following theorem in an alternative and easier way.

Theorem 7.1.

$$(36)\quad N(E_{2n,2k}^{(2g)}) = \sum_{i=n-g}^{n-k} \frac{1}{i+1} \binom{2i}{i} \cdot \frac{2k}{2n-2i} \binom{2n-2i}{n-i-k}, \quad g = k, k+1, \ldots, n.$$

Proof. Let us consider $\{s_0, s_1, \ldots, s_{2i}, \ldots, s_{2n}\}$ an $E_{2n,2k}^{(2g)}$-path starting at the origin and reaching the point $(2n, 2k)$ after $2n$ steps (see fig. 6).

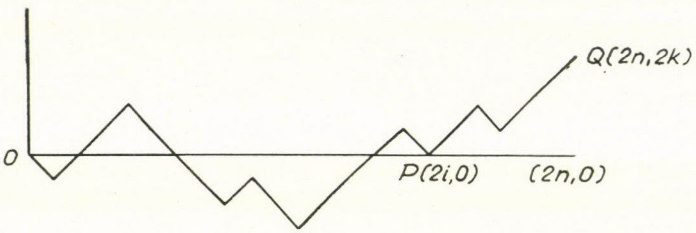

Fig. 6.

Let $P(2i, 0)$ be the last point where the path either touches or crosses the axis. Then the section from $(2i, 0)$ to $(2n, 2k)$ is a path of the type H_{2n-2i}^{2k}. The number of paths with any number of positive steps from 0 to $2i$ is given by

$$L_{2i} = \frac{1}{i+1} \binom{2i}{i}, \qquad \text{(Feller [1], p. 72).}$$

Thus

$$N(E_{2n,2k}^{(2g)}) = \sum_{i=n-g}^{n-k} L_{2i} \cdot N(H_{2n-2i}^{2k})$$

proving our theorem 7.1. This leads to the following

$$(37)\qquad P(\gamma_{2n} = g \mid E_{2n,2k}) = \frac{1}{\binom{n}{n-k}} N(E_{2n,2k}^{(2g)}) .$$

For the limiting case we obtain the following: for $k \sim a\sqrt{2n}$, $\frac{i}{n} \sim y$

$$\lim_{n\to\infty} P\left(z \leqq \frac{\gamma_{2n}}{n} < z + dz \mid E_{2n,2k}\right) = \sqrt{\frac{2}{\pi}} \int_{1-z}^{1} \frac{a}{\{y(1-y)\}^{3/2}} e^{2a^2 - \frac{2a^2}{(1-y)}} dy\, dz\,;$$

making use of the transformation

$$\frac{2a^2}{1-y} = v$$

it follows that

$$\lim_{n\to\infty} P\left(z \leqq \frac{\gamma_{2n}}{n} < z + dz \mid E_{2n,2k}\right) = \frac{1}{\sqrt{\pi}} \int_{\frac{2a^2}{z}}^{\infty} \frac{v}{\{v - 2a^2\}^{3/2}} e^{2a^2 - v} dv\,.$$

$k = 0$ or $a = 0$ gives the uniform distribution (Feller [1]).

§ 8. Distribution of $\gamma_{2n}^{(r)}$ for an $E_{2n,2k}$-path

The following

Theorem 8.1. *For $r < 2k$*

$$N(E_{2n,2k,r}^{(2g)}) = \sum_{i=0}^{g-k+\frac{r}{2}} \frac{(r+2i+1)(2k-r+2i+1)}{(2g+1)(2n-2g+1)} \binom{2g+1}{g-k+\frac{r}{2}-i} \times$$

$$\times \binom{2n-2g+1}{n-g-\frac{r}{2}-i}, \quad r \text{ even}, \quad g = k - \frac{r}{2}, \ldots, n - \frac{r}{2},$$

(38) *and*

$$N(E_{2n,2k,r}^{(2g+1)}) = \sum_{i=0}^{g-k+\frac{r+1}{2}} \frac{(r+2i+1)(2k-r+2i+1)}{4(n-g)(g+1)} \binom{2g+2}{g-k+\frac{r+1}{2}-i} \times$$

$$\times \binom{2n-2g}{n-g-\frac{r+1}{2}-i}, \quad r \text{ odd}, \quad g = k - \frac{r+1}{2}, \ldots, n - \frac{r+1}{2},$$

holds.

The proof follows from (22) and this leads to

$$P(\gamma_{2n}^{(r)} = g \mid E_{2n,2k}) = \begin{cases} \dfrac{1}{\binom{2\,n}{n-k}} \cdot N(E_{2n,2k,r}^{(2g)}) \,, & r \text{ even,} \\[2ex] \dfrac{1}{\binom{2\,n}{n-k}} \cdot N(E_{2n,2k,r}^{(2g+1)}) \,, & r \text{ odd,} \end{cases}$$

and for the limiting distribution we obtain: for $k \sim a\sqrt{2\,n}$, $r \sim b\sqrt{2\,n}$

(39)
$$\lim_{n\to\infty} P\left(z \leq \frac{\gamma_{2n}^{(r)}}{n} < z + dz \,\middle|\, E_{2n,2k}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(b+y)(2\,a-b+y)}{\{z(1-z)\}^{3/2}}\, e^{\left[2a^2 - \frac{(b+y)^2}{2(1-z)} - \frac{(2a-b+y)^2}{2z}\right]} dy\, dz\,,$$

$$a, b, y \geq 0\,,\quad 0 \leq z \leq 1\,,$$

which can also be seen from (26).

When $r = 2\,k$, it follows from (23) that

$$N(E_{2n,2k,2k}^{(2g)}) = \sum_{i=1}^{g} \frac{i(r+2\,i)}{g\left(n-g+\frac{r}{2}+i+1\right)} \binom{2\,g}{g-i} \binom{2\,n-2\,g+1}{n-g-\frac{r}{2}-i+1}\,,$$

$$g = 1, 2, \ldots, n - \frac{r}{2}\,,$$

which gives

(40)
$$P(\gamma_{2n}^{(2k)} = g \mid E_{2n,2k}) = \frac{1}{\binom{2\,n}{n-k}} \cdot N(E_{2n,2k,2k}^{(2g)})\,.$$

It is easy to see that

(41)
$$P(\gamma_{2n}^{(2k)} = 0 \mid E_{2n,2k}) = \frac{2\,k+1}{n+k+1}\,,$$

by showing the one-to-one correspondence between the paths of the types $E_{2n,2k,2k}^{(0)}$ and H_{2n+1}^{2k+1}. For the limiting distribution we obtain: for $k \sim a\sqrt{2\,n}$

$$\lim_{n\to\infty} P\left(z \leq \frac{\gamma_{2n}^{(2k)}}{n} < z + dz \,\middle|\, E_{2n,2k}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(y^2 + 2\,ay)}{\{z(1-z)\}^{3/2}}\, e^{-\frac{(y+2az)^2}{2z(1-z)}}\, dy\, dz,$$

$$a \geq 0\,,\quad 0 \leq z \leq 1\,,$$

$k = 0$ or $a = 0$ gives

$$\lim_{n\to\infty} P\left(z \leqq \frac{\gamma_{2n}}{n} < z + dz \mid E_{2n}\right) = dz$$

showing that $\frac{\gamma_{2n}}{n}$ has an uniform distribution (FELLER [1]).

When $r > 2k$, it follows from (24) that

$$N(E^{(2g)}_{2n,2k,r}) = \sum_{i=1}^{g} \frac{i(r-k+i)}{g(n-g+1)} \binom{2g}{g-i} \binom{2n-2g+2}{n-g-r+k-i+1}, \quad g = 1, 2, \ldots, n-r+k,$$

which gives

(42) $$P(\gamma^{(r)}_{2n} = g \mid E_{2n,2k}) = \frac{1}{\binom{2n}{n-k}} N(E^{(2g)}_{2n,2k,r})$$

and

$$P(\gamma^{(r)}_{2n} = 0 \mid E_{2n,2k}) = 1 - \frac{\binom{2n}{n-r+k-1}}{\binom{2n}{n-k}}$$

which can be proved by showing that

$$N(E^{(0)}_{2n,2k,r}) = \binom{2n}{n-k} - \binom{2n}{n-r+k-1}.$$

For the limiting case we obtain: if $k \sim a\sqrt{2n}$, $r \sim b\sqrt{2n}$

$$\lim_{n\to\infty} P\left(z \leqq \frac{\gamma^{(r)}_{2n}}{n} < z + dz \mid E_{2n,2k}\right) =$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{y(2b-2a+y)}{\{z(1-z)\}^{3/2}} e^{\left[2a^2 - \frac{y^2}{2z} - \frac{(2b-2a+y)^2}{2(1-z)}\right]} dy\, dz$$

$$a, b, y \geqq 0, \quad 0 \leqq z \leqq 1,$$

which can also be seen from (29).

§ 9. Distribution of $\gamma^{(r)}_{2n}$ for an F_{2n}-path

For a fixed positive even r I. VINCZE [6] has proved that

$$N(F^{(2g)}_{2n,r}) = \binom{2g}{g} \binom{2n-2g}{n-g+\frac{r}{2}}$$

which, in our case, follows easily from (33) by writing

$$N(F^{(2g)}_{2n,r}) = \sum_{l\,(\text{even})} N(F^{(2g,l)}_{2n,r}) + \sum_{l\,(\text{odd})} N(F^{(2g,l)}_{2n,r}).$$

But, for a fixed positive odd r, we shall prove the following

Theorem 9.1.

$$N(F_{2n,r}^{(2g)}) = \sum_{i=1}^{g} \frac{i}{g} \binom{2g}{g-i} \binom{2n-2g+1}{n-g-\frac{r-1}{2}-i},$$

(43) and

$$N(F_{2n,r}^{(2g+1)}) = \sum_{i=1}^{g} \frac{(r+2i-1)}{2(n-g)} \binom{2g+1}{g-i+1} \binom{2n-2g}{n-g-\frac{r-1}{2}-i}.$$

The proof is simple and follows from (34) by putting

$$N(F_{2n,r}^{(2g)}) = \sum_{l\,(\mathrm{even})} N(F_{2n,r}^{(2g,l)}),$$

and

$$N(F_{2n,r}^{(2g+1)}) = \sum_{l\,(\mathrm{odd})} N(F_{2n,r}^{(2g+1,l)}).$$

This leads to

$$P(\gamma_{2n}^{(r)} = g \mid F_{2n}) = \frac{1}{2^{2n}} \cdot N(F_{2n,r}^{(2g)}),$$

and

$$P(\gamma_{2n}^{(r)} = g \mid F_{2n}) = \frac{1}{2^{2n}} \cdot N(F_{2n,r}^{(2g+1)}).$$

In the second case there are $2\gamma_{2n}^{(r)} + 1$ steps above the height r.

For the limiting distribution we obtain: for $r \sim b\sqrt{2n}$

$$\lim_{n\to\infty} P\left(z \leq \frac{\gamma_{2n}^{(r)}}{n} < z + dz \mid F_{2n}\right) = \frac{1}{\pi} \cdot e^{-\frac{b^2}{2}} \int_0^\infty \frac{(y+bz)}{\{z(1-z)\}^{3/2}} e^{-\frac{(y+bz)^2}{2z(1-z)}} dy\, dz$$

which on putting $r = 0$ or $b = 0$ leads to the arc sin law [1].

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REFERENCES

[1] FELLER, W.: *An Introduction to Probability Theory and its Applications, Vol.* 1 (2-nd edition) New-York, 1957, John Wiley and Sons.
[2] VINCZE, I.: "On some joint distributions and joint limiting distributions in the theory of order statistics, II." *Publ. Math. Inst. Hung. Acad. Sci.* **4** (1959) 29—47.
[3] CSÁKI, E.—VINCZE, I.: "On some problems connected with the Galton test." *Publ. Math. Inst. Hung. Acad. Sci.* **6** (1961) 97—109.
[4] CSÁKI, E.: "On the number of intersections in the one-dimensional random walk." *Publ. Math. Inst. Hung. Acad. Sci.* **6** (1961) 281—286.
[5] РЫЖИК, И. М.—ГРАДШТЕЙН: *Таблицы интегралов, сумм, рядов и произведений.* Государственное издательство, Москва—Ленинград, 1951.
[6] VINCZE, I.: "On Some Combinatorial Relations Concerning the Symmetric Random Walk", *Revue de Mathématiques Pures et Appliquées (Académia de la République Populaire Roumaine)* **8** (1963) No. 2.
[7] CHUNG, K. L.—FELLER, W.: "On Fluctuations in Coin Tossing." *Proceedings of National Academy of Sciences*, Vol. **35** (1949).

О НЕКОТОРЫХ КОМБИНАТОРНЫХ ОТНОШЕНИЯХ, СВЯЗАННЫХ СИММЕТРИЧЕСКИМ СЛУЧАЙНЫМ БЛУЖДАНИЕМ

KANWAR SEN

Резюме

Пусть $\vartheta = (\vartheta_1, \vartheta_2, \ldots)$ последовательность независимых случайных величин, принимающих значения $+1$ и -1 с равными вероятностями. Пусть, далее $s_0 \equiv 0$ и $s_i = \vartheta_1 + \ldots + \vartheta_i$, $i = 1, 2, \ldots$.

Изучаются следующие величины:

$\lambda_N^{(r)}$ — число тех индексов $i \leqq N$ для которых или $s_{i-1} = r - 1$, $s_i = r$, $s_{i+1} = r + 1$ или $s_{i-1} = r + 1$, $s_i = r$, $s_{i+1} = r - 1$;

$2\,\gamma_N^{(r)}$ — число тех индексов $i \leqq N$ для которых или $s_i > 0$ или $s_i = 0$ но $s_{i-1} > 0$.

В §-ах 1, 2, 3 и 4 исследуются условные распределения этих величин, при условии $s_{2n} = 2\,k$ или $s_{2n+1} = 2\,k + 1$.

Распределение λ определяется с формулами 4, 7, 9, 11 и 13.

Совместное распределение величин λ и γ выводится из формул 20, 21, 26, 27 и 28.

Переходив к пределу в точных формулах распределений, получены предельные распределения.

Отношения 31 и 35 §-ов 5 и 6 включают себе безусловные распределения величин λ и γ. Приводятся и соответствующие предельные распределения.

В §-ах 7 и 8 исследуется условное распределение γ, а в §-е 9 ее безусловное и предельное распределение.

Одна часть результатов получаются с помощью простого комбинаторного метода и однозначного отображения, а другая часть с помощью производящей функции.