ON THE CONSTRUCTIVE THEORY OF FUNCTIONS I

by

P. SZÜSZ and P. TURÁN

1. In a recent very interesting note[1] D. J. NEWMAN made the observation that the $|x|$-function, which plays an essential rôle in the theory of polynomial-approximation shows an unexpected behavior when turning to approximation by rational functions. While it is well-known that for suitable positive numerical c_1 and c_2 (and later c_3) and for all $n \geqq 1$ there exists a polynomial[2] $\pi_n^*(x)$ such that for $-1 \leqq x \leqq 1$

$$(1.1) \qquad \left| |x| - \pi_n^*(x) \right| \leqq \frac{c_1}{n}$$

but for all $\pi_n(x)$

$$(1.2) \qquad \max_{-1 \leqq x \leqq 1} \left| |x| - \pi_n(x) \right| \geqq \frac{c_2}{n},$$

he proved that for suitable $p_n^*(x)$, $q_n^*(x)$ for $n \geqq 4$, $-1 \leqq x \leqq 1$ the inequality

$$(1.3) \qquad \left| |x| - \frac{p_n^*(x)}{q_n^*(x)} \right| \leqq 3\, e^{-\sqrt{n}}$$

holds, which is much stronger than (1.1). He proved moreover that the order $e^{-\sqrt{n}}$ is „essentially" best possible. Whether or not this is only an isolated fact or there is a general theorem behind, he expresses no opinion; his words „— Now it is known that in some overall sense rational approximation is essentially not better than polynomial approximation . . ." reflect perhaps a more pessimistic than optimistic opinion about it. A more definit such opinion is formulated in a letter of Prof. NEWMAN; he shows here that for *all* $0 < \alpha < 1$ the function

$$f(x) = \sum_{m=1}^{\infty} \frac{T_{m!}\left(\frac{x}{2}\right)}{m!^{\alpha}} \qquad T_k(\cos\vartheta) = \cos k\vartheta$$

(which belongs to $\mathrm{Lip}_\alpha(-1, 1)$) cannot be approximated for all n's better than $O\left(\frac{1}{n^\alpha \log n}\right)$ by a rational function of degree[3] n, uniformly in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ say.

[1] „Rational approximation to $|x|$." *The Michigan Mathem. Journal* **11** (1964) 11—14.

[2] $\pi_n^*(x)$, $\pi_n(x)$, $p_n^*(x)$, $p_{n\nu}(x)$, $p_{n,\nu}(x, a)$, $q_n(x)$, $q_n^*(x)$ etc. stand for polynomials in x of degree $\leqq n$ (a will be a parameter).

A still more definit opinion is expressed in the paper of S. H. SHAPIRO[4] suggesting that ,, . . . if one wishes to approximate functions of class Lip_α then the nonlinear methods considered do not enable one to improve the order of magnitude of the approximation beyond what is possible by polynomial approximation having the same number of parameters". With respect to NEWMAN's result he remarks however that ,, . . . one might surmise that the main strength of rational approximation lies in the approximation of functions with special analytic properties". In what follows we are going to show that ,,not very remote" analytic properties suffice already; we are going to give a general *class* of functions (perhaps the first one in the literature) for which the approximation by rational functions of degree $\leqq n$ is *essentially* better than by polynomials of degree n. This class $K(A)$ consists of functions $f(x)$ which are convex e.g. from above in $[-1, 1]$ and satisfy here the inequality

$$\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| \leqq A \quad \text{if} \quad -1 \leqq x_1 < x_2 \leqq 1 . \tag{1.4}$$

Obviously this restriction (1.4) is rather a matter of convenience [inside of $(-1, 1)$ it is automatically satisfied]. Then we assert the

Theorem. *For the functions of the class $K(A)$ and for all $N \geqq 1$ one can find a suitable rational function $\frac{u_N(x)}{v_N(x)}$ of degree N such that for $-1 \leqq x \leqq 1$ the inequality*

$$\left| f(x) - \frac{u_N(x)}{v_N(x)} \right| < c_3 \frac{(1+A)\log^4 N}{N^2}$$

holds.

(1,1) — (1.2) shows that for polynomial approximation we have in general only the order $\frac{1}{N}$. To which extent the quantity $\frac{\log^4 N}{N^2}$ on the right is best possible, we do not know at present; possibly the log-factor can be dropped. As Prof. NEWMAN remarked in his letter, the function

$$f(x) = -Bx^2 + \sum_{m=1}^{\infty} \frac{T_{m!}\left(\frac{x}{2}\right)}{m^2\, m!^2}$$

with a suitably large positive constant B belongs to the class $K(A)$ and its approximability by a rational function of degree n cannot surpass $O\left(\frac{1}{n^2 \log n}\right)$. Hence our upper bound is ,,not far" from being best-possible.

The theorem could have been announced for the function-class $K_1(A)$ whose members satisfy (1.4) and are indefinite integrals of functions of bounded variation. Representing namely these functions as difference of two mono-

[3] A rational function $\frac{p(x)}{q(x)}$ we call of n^{th} degree, if $q(x)$ and $p(x)$ are polynomials with degree $\leqq$ n.

[4] ,,Some negative theorems of approximation theory." *Mich. Math. Journ* **11.** (1964) 211- 217.

tonic functions it means that $f(x)$ is the difference of two convex functions and the theorem applies at once.

The point of the theorem is of course the uniformity of the approximation; replacing it by L_1-metric the corresponding theorem has been proved by G. FREUD[5] even for polynomial-approximation and without the log-factor. FREUD extended his theorem to the class, whose elements are k^{th} integral of functions of bounded variation; an analogous extension of our theorem is also possible and will be given in the second paper. We shall return to study the connection between approximability by rational functions of N^{th} degree and structural properties of continuous functions. Generally speaking the approximation by rational functions seems to be advantageous over the approximation by polynomials if the function is „nasty only locally on a thin set".

This new branch of the approximation-theory seems to have a significance also for the numerical analysis, since the computational problems with polynomials are essentially identical with those of rational functions.

2. Now we turn to the proof of our theorem. Let $-1 < a < 1$ and we replace x by $\dfrac{t-a}{1-at}$ in NEWMAN's inequality (1.2). This gives for $n \geqq 4$, $-1 \leqq t \leqq 1$ the inequality

$$\left| \left|\frac{t-a}{1-at}\right| - \frac{p_n^*\left(\dfrac{t-a}{1-at}\right)}{q_n^*\left(\dfrac{t-a}{1-at}\right)} \right| \leqq 3\, e^{-\sqrt{n}},$$

which we prefer to write in the form

$$\left| \left|\frac{t-a}{1-at}\right| - 1 - \frac{p_{n,1}(t,a)}{q_{n,1}(t,a)} \right| \leqq 3\, e^{-\sqrt{n}}.$$

Since for our t's $1 - at \geqq 0$, this can be written in the form

$$\left| (1-at) - |t-a| - \frac{p_{n+1,2}(t,a)}{q_{n,1}(t,a)} \right| \leqq 6\, e^{-\sqrt{n}}. \tag{2.1}$$

Putting

$$h(t,a) \overset{\text{def}}{=} (1-at) - |t-a| \tag{2.2}$$

$h(t, a)$ represents a „roof" consisting of two straight segments with

$$h(1,a) = h(-1,a) = 0 \qquad h(a,a) = 1 - a^2 \tag{2.3}$$

and (2.1) takes the form

$$\left| h(t,a) - \frac{p_{n+1,2}(t,a)}{q_{n,1}(t,a)} \right| \leqq 6\, e^{-\sqrt{n}}, \qquad -1 \leqq t \leqq +1. \tag{2.4}$$

Here we had for the parameter a the restriction $-1 < a < 1$.

[5] »Über einseitige Approximation durch Polynome I.« *Acta Litt. ac Scient. Szeged* **16** (1955) 12—28.

3. Let us consider now our $f(t)$ and let $f_1(t)$ be defined by

$$f_1(t) = f(t) - f(1) - \frac{f(1) - f(-1)}{2}(t-1). \tag{3.1}$$

Obviously $f_1(t)$ is again nonnegative, convex from above in $[-1, 1]$ and moreover

$$f_1(1) = f_1(-1) = 0. \tag{3.2}$$

Further we have for $-1 \leqq t_1 < t_2 \leqq 1$

$$\left|\frac{f_1(t_2) - f_1(t_1)}{t_2 - t_1}\right| = \left|\frac{f(t_2) - f(t_1)}{t_2 - t_1} - \frac{f(1) - f(-1)}{2}\right| \leqq 2A, \tag{3.3}$$

from (1.3). Let now $m \geqq 2$ and

$$a_0 = 1 > a_1 > a_2 > \ldots > a_{m-1} > -1 = a_m, \tag{3.4}$$

and we consider the function $g_m(t)$ which is represented by an m-gon with the vertices at the points

$$\big(a_\nu, f_1(a_\nu)\big) \qquad \nu = 0, 1, \ldots, m. \tag{3.5}$$

This is obviously nonnegative and convex from above in $[-1, 1]$ for which owing to (3.3) the inequality

$$\frac{g_m(t_2) - g_m(t_1)}{t_2 - t_1} \leqq 2A \tag{3.6}$$

holds, if only

$$-1 \leqq t_1 < t_2 \leqq 1.$$

Owing to a well-known theorem of BLASCHKE—PICK[6] $g_m(t)$ can be represented in the form

$$f_1(t) = \sum_{\nu=1}^{m-1} \lambda_\nu h(t, a_\nu) \tag{3.7}$$

with $\lambda_\nu > 0$. Then using (2.4) we get for $-1 \leqq t \leqq 1$ the inequality

$$\left|g_m(t) - \sum_{\nu=1}^{m-1} \lambda_\nu \frac{p_{n+1,2}(t, a_\nu)}{q_{n,1}(t, a_\nu)}\right| = \tag{3.8}$$

$$= \left|\sum_{\nu=1}^{m-1} \lambda_\nu \left(h(t, a_\nu) - \frac{p_{n+1,2}(t, a_\nu)}{q_{n,1}(t, a_\nu)}\right)\right| \leqq 6\, e^{-\sqrt{n}} \sum_{\nu=1}^{m-1} \lambda_\nu.$$

In order to estimate the remaining sum we replace t in (3.7) by a_μ ($\mu = 1, 2, \ldots, m-1$). This gives from (2.3) and (3.5)

$$f_1(a_\mu) = \sum_{\nu=1}^{m-1} \lambda_\nu h(a_\mu, a_\nu) > \lambda_\mu (1 - a_\mu^2).$$

[6] »Distanzabschätzungen in Funktionenräumen II.« *Math. Ann.* Bd **77** (1916) p. 277—300.

If e.g. $a_\mu \geqq 0$, this gives from (3.3)

$$\lambda_\mu < \frac{f_1(a_\mu)}{1-a_\mu^2} = \frac{f_1(a_\mu)-f_1(1)}{1-a_\mu}\,\frac{1}{1+a_\mu} = \left|\frac{f_1(a_\mu)-f(1)}{1-a_\mu}\right| \frac{1}{1+a_\mu} \leqq 2A$$

and analogously for $a_\mu < 0$. Hence from (3.8)

$$\left| g_m(t) - \sum_{\nu=1}^{m-1} \lambda_\nu \frac{p_{n+1,2}(t, a_\nu)}{q_{n,1}(t, a_\nu)} \right| \leqq 12\, Am\, e^{-\sqrt{n}}$$

or rather

$$\left| g_m(t) - \frac{p_{(m-1)n+1}(t)}{q_{(m-1)n}(t)} \right| \leqq 12\, Ame^{-\sqrt{n}} \tag{3.9}$$

for $-1 \leqq t \leqq 1$.

4. In order to estimate $|f_1(t) - g_m(t)|$ for $[-1, 1]$ we shall need the following

Lemma. *For the above-defined $f_1(t)$-function and $m \geqq 4$ for a suitable $g_{2m}(t) = g^*_{2m}(t)$ we have for $-1 \leqq t \leqq +1$ the inequality*

$$|f_1(t) - g^*_{2m}(t)| \leqq \frac{(1+2A)4\pi}{m^2}.$$

For the proof we remark first that $f_1(t)$ is approximable by a $g_s(t)$-polygon, convex from above and then „rounding off" the corners and adjusting slightly the sides we get a function $k(t)$ nonnegative and convex from above, everywhere derivable with strictly decreasing derivative, satisfying in $-1 \leqq \leqq t \leqq +1$ the inequality

$$|f_1(t) - k(t)| \leqq \frac{1}{2m^2} \tag{4.1}$$

moreover we have

$$|k'(t)| \leqq 2A. \tag{4.2}$$

Drawing the tangents of $y = k(t)$ which have with the positive t-axis an angle of the form $\nu\frac{2\pi}{m}$ $(\nu = 0, \pm 1, \ldots)$ they form an open polygonal line, convex from above, consisting of $l < m$ sides. Considering the projections P_ν of the tangential points on the t-axis with addition of at most m further equidistant P-points we can attain that the distance of any two consecutive P's (including the points ± 1) is

$$\leqq \frac{2}{m}. \tag{4.3}$$

Drawing the tangents also in the Q-points, corresponding to the new P-points, we obtain an r-gon with $r \leqq 2m$, tangential to the graph of $y = k(t)$ with the property (4.3); this will be $g^*_{2m}(t)$. Moreover owing to the convexity of $k(t)$ the angle of any two consecutive sides is $\geqq \pi - \frac{2\pi}{m}$. We enumerate the new and old tangential points in decreasing order again by $Q_1, \ldots, Q_r$, their projections by $P_1, \ldots, P_r$; in addition, if necessary, take also the points $Q_0(1, k(1))$

and $Q_{r+1}(-1, k(-1))$ and denote the point of intersection of the tangents in Q_ν and $Q_{\nu+1}$ by R_ν. Then the triangles $Q_\nu, Q_{\nu+1} R_\nu$ contain the whole graph of $y = k(t)$ and if the perpendicular from R_ν to the t-axis cuts $Q_\nu Q_{\nu+1}$ in the point S_ν then we have

$$(4.4) \qquad |\max_{-1 \leq t \leq 1} g^*_{2m}(t) - k(t)| \leq \max_\nu \overline{R_\nu S_\nu}.$$

Denoting the inclination of $Q_\nu Q_{\nu+1}$ by φ, so that from (4.2)

$$(4.5) \qquad |\operatorname{tg} \varphi| \leq 2A,$$

the angle $Q_\nu R_\nu Q_{\nu+1}$ by $(\pi - \beta)$, where from the construction

$$(4.6) \qquad 0 < \beta \leq \frac{2\pi}{m},$$

and the angle $Q_\nu Q_{\nu+1} R_\nu$ by α, so that

$$(4.7) \qquad 0 < \alpha < \beta,$$

simple trigonometry gives

$$\overline{R_\nu S_\nu} = \overline{P_\nu P_{\nu+1}} \frac{\sin(\beta - \alpha)\{\sin(\alpha + \varphi) - \sin\varphi\}}{\cos\varphi \sin\beta}.$$

From this one gets easily

$$\overline{R_\nu S_\nu} < \frac{2}{m} \cdot \frac{\beta}{\cos\varphi} < \frac{4\pi}{m^2} \cdot \frac{1}{\cos\varphi} \leq \frac{4\pi}{m^2}\sqrt{1 + 4A^2} \leq \frac{4\pi}{m^2}(1 + 2A).$$

This, (4.4) and (4.1) prove the lemma.

5. Now we can complete the proof of our theorem. Applying (3.9) with $2m$ instead of m and also the lemma we get for $-1 \leq t \leq 1$

$$\left| f_1(t) - \frac{p_{(2m-1)n+1}(t)}{q_{(2m-1)n}(t)} \right| \leq 24\, mAe^{-\sqrt{n}} + \frac{(1 + 2A)\, 4\pi}{m^2}.$$

Using (3.1) we get for $-1 \leq t \leq +1$ the inequality

$$(5.1) \qquad \left| f(t) - \frac{p_{(2m-1)n+1}(t)}{q_{(2m-1)n}(t)} \right| \leq 24\, Ame^{-\sqrt{n}} + \frac{(1 + 2A)\, 4\pi}{m^2}.$$

So far m and n were arbitrary integers ≥ 4. Now we want approximate $f(t)$ by rational functions of degree $\leq N$. We choose for $N \geq 100$

$$(5.2) \qquad m = \left[\frac{N}{20 \log^2 N}\right]$$

$$n = [9 \log^2 N].$$

Then we have

$$(2m-1)n+1<2mn<N,$$

further

$$me^{-\sqrt{n}}<\frac{1}{N^2\log^2 N}$$

and

$$\frac{1}{m^2}<\frac{10^4\log^4 N}{N^2}.$$

This and (5.1) prove our theorem.

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О КОНСТРУКТИВНОЙ ТЕОРИИ ФУНКЦИЙ I

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Резюме

Пусть $\pi_n(x)$ многочлен степени n, $r_n(x)=\dfrac{\pi_n^*(x)}{\pi_n^{**}(x)}$ рациональная функция степени n, $E_n(f)$ соотв. $R_n(f)$ наилучшее приближение, в смысле Чебышева, непрерывной функции $f(x)$ в $[-1,1]$, с помощью многочленов степени n, соотв. рациональными функциями степени n, и если C какой-нибудь подкласс непрерывных функций, то

$$\sup_{f\in C} E_n(f)=E_n(C)$$

$$\sup_{f\in C} R_n(f)=R_n(C)\,.$$

Распространено, что $R_n(C)$ не может быть «существенно» лучше, чем $E_{2n+1}(C)$, количественной формой этого утверждения является тот факт, сообщенный Newman -ом, что если $0<\alpha<1$, то с одной стороны

$$E_n(\mathrm{Lip}\ \alpha)<\frac{c_1(\alpha)}{n^\alpha} \qquad \text{(С. Бернштейн)}$$

с другой стороны

$$R_n(\mathrm{Lip}\ \alpha)>\frac{c_2(\alpha)}{n^\alpha\log n}$$

В первой заметке наших сообщений мы покажем, что если K является в $[-1,+1]$ выпуклым и ε любое положительное число, тогда в $[-1+\varepsilon, 1-\varepsilon]$ для $f\in K$

$$R_n(f)<\frac{c_3(\varepsilon,f)\log^4 n}{n^2}$$

когда для подходящего $f^* \in K$ в $[-1/2, 1/2]$

$$R_n(f^*) > \frac{1}{n^2 \log^2 n}.$$

Для ориентации заметим, что $|x| \in K$ и в $[-1/2, 1/2]$

$$E_n(|x|) > \frac{c_4}{n}$$

по С. Бернштейну, значит в $[-1+\varepsilon, 1-\varepsilon]$ для класса K приближение с помощью рациональных функций в *порядке* лучше чем с многочленами. В дальнейших статьях мы показываем подобное явление на других известных классах функций, у некоторых классов приближение намного лучше, чем у K.