## ON SERIES OF DILATED FUNCTIONS

ISTVÁN BERKES AND MICHEL WEBER

ABSTRACT. Given a periodic function f, we study the almost everywhere and norm convergence of series  $\sum_{k=1}^{\infty} c_k f(kx)$ . As the classical theory shows, the behavior of such series is determined by a combination of analytic and number theoretic factors, but precise results exist only in a few special cases. In this paper we use connections with orthogonal function theory and GCD sums to prove several new results, improve old ones and also to simplify and unify the theory.

## 1. Introduction and preliminary results

Let  $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1[, e(x) = \exp(2i\pi x), e_n(x) = e(nx), n \in \mathbb{Z}$ . Let  $f(x) = \sum_{\ell \in \mathbb{Z}} a_\ell e_\ell, a_0 = 0, \sum_{\ell \in \mathbb{Z}} |a_\ell|^2 < \infty$ . The convergence and asymptotic properties of sums

(1.1) 
$$\sum_{k=1}^{\infty} c_k f(kx)$$

have been studied extensively in the literature and turned out to be, in general, quite different from the trigonometric case f(x) = e(x). (For history and recent results, see e.g. Berkes and Weber [4].) For f(x) = e(x) (and consequently, if f is a trigonometric polynomial), Carleson's theorem states the almost everywhere convergence of (1.1) provided  $\mathbf{c} = \{c_k, k \ge 0\} \in \ell^2(\mathbb{N})$ . Using a simple approximation argument, Gaposhkin [7] showed that this remains valid if the Fourier series of f is absolutely convergent, i.e. if  $\sum_{\ell \in \mathbb{Z}} |a_\ell| < \infty$ . In particular, this is the case if f belongs to the  $\operatorname{Lip}_{\alpha}(\mathbb{T})$  class for some  $\alpha > 1/2$ . For  $\sum_{\ell \in \mathbb{Z}} |a_\ell| = \infty$  the convergence properties of the sum (1.1) are much more complicated and satisfactory results exist only in special situations. If the Fourier series of f is lacunary, i.e. if  $f(x) = \sum_{\ell=1}^{\infty} a_\ell e(n_\ell x)$ , where  $n_{\ell+1}/n_\ell \ge q > 1$ ,  $\ell = 1, 2, \ldots$ , then by a theorem of [7], the analogue of Carleson's theorem holds for (1.1) provided the  $L_2$  modulus of continuity  $\omega_2(f, h)$  of f satisfies

$$\sum_{k=1}^{\infty} \frac{\omega_2(f, 2^{-k})}{\sqrt{k}} < \infty$$

and this result is sharp. For general f, this criterion becomes false: if the Fourier series  $f = \sum_{p} a_{p}e(px)$   $(\sum_{p} |a_{p}| = \infty)$  contains only prime frequencies, then the analogue of Carleson's theorem is false, even though this class contains Lip (1/2) functions f, see Berkes [3]. It is also known that the asymptotic distribution of sums  $\sum_{k=1}^{N} c_k f(n_k x)$  depends sensitively on the Diophantine properties of the sequence  $(n_k)$ , see e.g. Gaposhkin [8]. These results, together with Wintner's classical criterion [28] connecting the behavior of (1.1) with boundedness properties of the Dirichet series  $\sum_{n=1}^{\infty} a_n/n^s$ , show that the convergence properties of (1.1) are determined by an interplay of analytic and number theoretic factors. Recent results of Weber [26] and Brémont [5] shed a new light on the arithmetic background of the problem and led to much improved convergence results. Weber showed that assuming a condition for f only slightly stronger

Key words and phrases. dilated functions, mean convergence, almost convergence, orthogonal systems, GCD matrix, eigenvalues, quadratic forms.

than  $f \in L^2$ , the series (1.1) converges a.e. provided

$$\sum_{r=1}^{\infty} \left( \sum_{j=2^r+1}^{2^{r+1}} c_j^2 d(j) (\log j)^2 \right)^{1/2} < \infty,$$

where  $d(k) = \sum_{d|k} 1$  is the divisor function. Brémont showed that under  $|a_n| = O(n^{-s}), 1/2 < s \leq 1$ , a.e. convergence holds if that the series  $\sum_k c_k f_k$  converges almost everywhere if for some  $\varepsilon > 0$ 

(1.2) 
$$\sum_{k} c_k^2 \exp\left\{\frac{(1+\varepsilon)(\log k)^{2(1-s)}}{2(1-s)\log\log k}\right\} < \infty,$$

when 1/2 < s < 1, and if for some  $\varepsilon > 0$ 

(1.3) 
$$\sum_{k} c_k^2 (\log k)^3 (\log \log k)^{2+\varepsilon} < \infty,$$

when s = 1.

The purpose of the present paper to give a detailed study of the series (1.1), using connections with orthogonal function theory and asymptotic estimates for GCD sums in Diophantine approximation theory. This will not only lead to an extension and improvement of earlier results, but will also simplify and unify the convergence theory.

The convergence behavior of (1.1) is naturally closely connected with estimating the quadratic form

(1.4) 
$$\left\|\sum_{k=1}^{n} c_k f_k\right\|_2^2 = \sum_{k,\ell=1}^{n} c_k c_\ell \langle f_k, f_\ell \rangle.$$

Let us first study it on a simple class of examples. We follow [18]. Consider the function

(1.5) 
$$f(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi j x}{j^s},$$

where s > 1/2. When s = 1, f(x) = x - [x] - 1/2, where [x] is the integer part of x. It is known (see [15]) that the corresponding system  $\{f_n, n \ge 1\}$  possesses properties going at the opposite of those of the trigonometrical system.

A simple calculation yields

(1.6) 
$$\langle f_k, f_\ell \rangle = \sum_{\substack{i,j=1\\ik=j\ell}}^{\infty} \frac{1}{i^s j^s} = \Big(\sum_{\nu=1}^{\infty} \frac{1}{\nu^{2s}}\Big) \frac{(k,\ell)^{2s}}{k^s \ell^s} = \zeta(2s) \frac{(k,\ell)^{2s}}{k^s \ell^s},$$

where  $\zeta$  is Riemann's zeta function, and (a, b) denotes the greatest common divisor of the positive integers a and b. It follows that

$$\left\|\sum_{k=1}^{n} c_k f_k\right\|_2^2 = \zeta(2s) \sum_{k,\ell=1}^{n} \frac{(k,\ell)^{2s}}{k^s \ell^s} c_k c_\ell$$

Subsequently, the GCD matrix

$$M_n(s) = \left(\frac{(k,\ell)^{2s}}{k^s \ell^s}\right)_{n \times k}$$

is positive definite when s > 1/2. The study of this important class of matrices is much older, and goes back to Smith's seminal paper published in 1876 (see [10] and references therein). Let  $\lambda_n(s)$  (resp.  $\Lambda_n(s)$ ) denote the smallest (resp. largest) eigenvalue of the matrix  $M_n(s)$ . We have the sharp estimate ([18], p. 152), the constants being optimal,

(1.7) 
$$\frac{\zeta(2s)}{\zeta(s)^2} \le \lambda_n(s) \le \Lambda_n(s) \le \frac{\zeta(s)^2}{\zeta(2s)},$$

when s > 1. Consequently,

(1.8) 
$$\frac{\zeta(2s)}{\zeta(s)^2} \sum_{k=1}^n c_k^2 \le \left\|\sum_{k=1}^n c_k f_k\right\|_2^2 \le \frac{\zeta(s)^2}{\zeta(2s)} \sum_{k=1}^n c_k^2$$

when s > 1. This implies that the series  $\sum_k c_k f_k$  converges in mean if  $\sum_k c_k^2 < \infty$ . In fact, it follows from Gaposhkin's theorem cited above (see also [5], p. 826) that this series converges almost everywhere. Concerning eigenvalues, Wintner ([28], p. 578) has shown that  $\limsup_{n\to\infty} \Lambda_n(s) < \infty$  if and only if s > 1. Further, when  $1/2 < s \le 1$ , it is known that ([18], p. 152)

(1.9) 
$$\liminf_{n \to \infty} \lambda_n(s) = 0, \qquad \limsup_{n \to \infty} \Lambda_n(s) = \infty.$$

Our first observation is that the quadratic form (1.4) can be for s > 0 more conveniently reformulated.

**Lemma 1.1.** Let  $S = \{x_1, \ldots, x_n\}$  be a set of distinct positive integers. We assume that S is factor closed  $(d|x_i \Rightarrow d = x_j \text{ for some } j = 1, \ldots, n)$ . Let s > 0. Then for all real  $y_1, \ldots, y_n$ ,

$$\sum_{k,\ell=1}^{n} y_k y_\ell \frac{(x_k, x_\ell)^{2s}}{x_k^s x_\ell^s} = \sum_{i=1}^{n} J_{2s}(x_i) \left\{ \sum_{k=1}^{n} \mathbf{1}_{x_i \mid x_k} \frac{y_k}{x_k^s} \right\}^2,$$

where  $J_{\varepsilon} = \xi_{\varepsilon} * \mu$  is the generalized Jordan totient function,  $\xi_{\varepsilon}(k) = k^{\varepsilon}$  for all  $k \in \mathbb{Z}$ ,  $\mu$  being the Möbius function.

*Proof.* Immediate since  $(x_k, x_\ell)^{2s} = \sum_{i=1}^n J_{2s}(x_i) \mathbf{1}_{x_i | x_k} \mathbf{1}_{x_i | x_\ell}$ .

Remark 1.2. Let G and A be  $n \times n$  matrices, with entries respectively defined by  $(G)_{i,j} = (x_i, x_j)^{2s}$  and  $(A)_{i,j} = \sqrt{J_{2s}(x_i)} \mathbf{1}_{x_i|x_j}$ . By Lemma 4.1 of [13],  $G = {}^{t}AA$ . It follows that  ${}^{t}UGU = {}^{t}VV$  where V = AU, namely

$$(V)_i = \sum_{k=1}^n (A)_{i,k} u_k = \sqrt{J_{2s}(x_i)} \sum_{k=1}^n \mathbf{1}_{x_i | x_k} u_k.$$

This is another more constructive way to obtain Lemma 1.1.

The set [1, n] being factor closed, by taking  $y_1 = \ldots = y_{m-1} = 0$  in the above Lemma, we get for any s > 0 and all real  $y_m, \ldots, y_n$ ,

(1.10) 
$$\left\|\sum_{k=m}^{n} y_k f_k\right\|_2^2 = \zeta(2s) \sum_{i=1}^{n} J_{2s}(i) \left\{\sum_{k=m}^{n} \mathbf{1}_{i|k} \frac{y_k}{k^s}\right\}^2.$$

The theorem below is Brémont's recent result ([5], Theorem 1.1 (ii)) with only a slightly better formulation. Let  $\sigma_s(k) = \sum_{d|k} d^s$ .

**Theorem 1.3.** Let  $1/2 < s \le 1$ . Let  $\varphi(k) > 0$  and non decreasing. Assume that both series

$$\sum_{k} \frac{1}{k\varphi(k)}, \qquad \sum_{k} c_k^2 \varphi(k) (\log k)^2 \sigma_{1-2s}(k)$$

are convergent. Then the series  $\sum_k c_k f_k$  converges almost everywhere.

By using Gronwall's estimates ([9] p. 119–122),

(1.11) 
$$\limsup_{x \to \infty} \frac{\sigma_1(x)}{x \log \log x} = e^{\lambda}, \qquad \limsup_{x \to \infty} \frac{\log\left(\frac{\sigma_\alpha(x)}{x}\right)}{\frac{(\log x)^{1-\alpha}}{\log \log x}} = \frac{1}{1-\alpha}, \qquad (0 < \alpha < 1)$$

where  $\lambda$  is Euler's constant, and the fact that  $\sigma_{-\alpha}(x) = x^{-\alpha}\sigma_{\alpha}(x)$ , we easily recover (1.2) and (1.3). As we shall see later, the condition in Theorem 1.3 can still be weak-ened.

Brémont's proof is based on Möbius orthogonalization and an adaptation of Rademacher-Menshov's theorem. Schur's theorem and the previous lemma allow to get it shortly. *Proof.* Let  $n \ge m \ge 1$ . In the following calculation concerning the norm  $\left\|\sum_{k=m}^{n} c_k f_k\right\|_2$ , we may let  $c_k = 0$  if  $k \notin [m, n]$ . Then

$$\begin{split} \zeta(2s)^{-1} \| \sum_{k=m}^{n} c_{k} f_{k} \|_{2}^{2} &= \sum_{i=1}^{n} J_{2s}(i) \bigg\{ \sum_{k=m}^{n} \mathbf{1}_{i|k} \frac{c_{k}}{k^{s}} \bigg\}^{2} .\\ &\leq \sum_{k,\ell=1}^{n} \frac{1}{k^{s}\ell^{s}} \bigg\{ \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} |c_{ki}|| c_{\ell i}| \bigg\} \\ &\leq \sum_{k,\ell=1}^{n} \frac{1}{k^{s}\ell^{s}} \bigg( \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} c_{ki}^{2} \bigg)^{1/2} \bigg( \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} c_{\ell i}^{2} \bigg)^{1/2} \\ &= \bigg\{ \sum_{k=m}^{n} \frac{1}{k^{s}} \bigg( \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} c_{ki}^{2} \bigg)^{1/2} \bigg\}^{2} \\ &\leq \bigg\{ \sum_{k=1}^{n} \frac{1}{k^{s}\psi(k)} \bigg\} \bigg\{ \sum_{k=m}^{n} \frac{\psi(k)}{k^{s}} \bigg( \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} c_{ki}^{2} \bigg) \bigg\}. \end{split}$$

Now, choosing  $\psi(k) = k^{1-s}\varphi(k)$ , we have

$$\sum_{k=m}^{n} \frac{\psi(k)}{k^{s}} \sum_{i=1}^{n} \frac{J_{2s}(i)}{i^{2s}} c_{ki}^{2} = \sum_{1 \le \nu \le n^{2}} c_{\nu}^{2} \sum_{k \models \nu \atop m \le k \le n} \frac{\psi(\nu/k)}{(\nu/k)^{s}} \frac{J_{2s}(i)}{i^{2s}}$$

$$\leq \sum_{m \le \nu \le n} c_{\nu}^{2} \sum_{\kappa \models \nu} \varphi(\kappa) \kappa^{1-2s} \le \sum_{m \le \nu \le n} c_{\nu}^{2} \varphi(\nu) \sigma_{1-2s}(\nu).$$

By combining, we find for all  $n \ge m \ge 1$ ,

(1.12) 
$$\|\sum_{k=m}^{n} c_k f_k\|_2^2 \leq \zeta(2s) \bigg\{ \sum_{k=1}^{n} \frac{1}{k\varphi(k)} \bigg\} \bigg\{ \sum_{k=m}^{n} c_k^2 \varphi(k) \sigma_{1-2s}(k) \bigg\},$$

Assume that the series  $\sum_{k} \frac{1}{k\varphi(k)}$  converges. Let  $\tilde{f}_{k} = f_{k}/(C_{s,\varphi}\varphi(k)\sigma_{1-2s}(k))^{1/2}$ where  $C_{s,\varphi} = \zeta(2s) \sum_{k\geq 1} \frac{1}{k\varphi(k)}$ . Then  $\|\sum_{k=m}^{n} c_{k}\tilde{f}_{k}\|_{2}^{2} \leq \sum_{k=m}^{n} c_{k}^{2}$ . In view of Schur's Lemma (Lemma 3.10), this implies that these functions can be extended to an orthonormal system on a bounded interval X of the real line including [0, 1[, and endowed with the normalized Lebesgue measure. By Rademacher-Menshov's theorem, the series  $\sum_{k} c_{k}\tilde{f}_{k}$  converges almost everywhere once the series  $\sum_{k} c_{k}^{2}(\log k)^{2}$  converges. Equivalently, the series  $\sum_{k} \gamma_{k}f_{k}$  converges almost everywhere once the series  $\sum_{k} c_{k}^{2}(\log k)^{2}\sigma_{1-2s}(k)$  converges, as claimed.

Remark 1.4. The monotonicity of the  $L^2$  norm with respect to Fourier coefficients yields that Theorem 1.3 remains valid for  $f(x) = \sum_{j=1}^{\infty} a_j \sin 2\pi j x$ , assuming that  $|a_j| = \mathcal{O}(j^{-s}), 1/2 < s \leq 1$ .

*Remark* 1.5. Schur's Lemma implies much more. The series  $\sum_k c_k f_k$  converges almost everywhere for any coefficient sequence  $\{c_k, k \ge 1\}$  such that

(1.13) 
$$\left\{ c_k \big( \varphi(k) \sigma_{1-2s}(k) \big)^{1/2}, k \ge 1 \right\}$$

is universal, according to Definition 2.3.

Remark 1.6. Estimate (1.12) indicates that

$$\left|\sum_{k,\ell=m}^{n} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s}\right| \leq C(\varphi) \sum_{k=m}^{n} c_k^2 \varphi(k) \sigma_{1-2s}(k),$$

assuming  $C(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k\varphi(k)} < \infty$ . Take s = 1 and let  $1 \le \kappa_1 < \kappa_2 < \ldots < \kappa_r$  be integers. Choose  $m = \kappa_1$ ,  $n = \kappa_r$  and  $c_k = 1$ , if  $k = \kappa_j$  for some  $1 \le j \le r$ , and  $c_k = 0$ 

otherwise. Letting also  $\varphi(x) = (\log x)(\log \log x)^{1+\varepsilon}$ ,  $\varepsilon > 0$ , and using (1.11), we find that

$$\sum_{i,j=1}^r \frac{(\kappa_i,\kappa_j)^2}{\kappa_i \kappa_j} \le C_{\varepsilon} \sum_{i=1}^r (\log \kappa_i) (\log \log k_i)^{2+\varepsilon}.$$

However, this is far from being optimal. Gál's estimate ([7], Theorem 2) indeed implies

(1.14) 
$$\sum_{i,j=1}^{r} \frac{(\kappa_i, \kappa_j)^2}{\kappa_i \kappa_j} \le Cr(\log \log r)^2.$$

Before going further, recall that by Wintner's fundamental theorem, the series  $\sum_{n=1}^{\infty} c_n f(nx)$  converges in the mean for all  $\mathbf{c} \in \ell^2$  iff

(1.15) 
$$\sum_{n=1}^{\infty} a_n/n^s \text{ and } \sum_{n=1}^{\infty} b_n/n^s \text{ are regular and bounded for } \Re s > 0.$$

The following result ([4], Theorem 3.1) concerns the situation when condition (1.15) fails.

**Theorem 1.7.** Let  $f \in \text{Lip}_{\alpha}(\mathbf{T})$ ,  $0 < \alpha \leq 1$ ,  $\int_{\mathbf{T}} f(t)dt = 0$  and assume that (1.15) is not valid. Then for any  $\varepsilon_k \downarrow 0$  there exists  $\mathbf{c} \in \ell^2$  and a sequence  $(n_k)$  of positive integers satisfying

$$n_{k+1}/n_k \ge 1 + \varepsilon_k \qquad (k \ge k_0)$$

such that the series  $\sum_k c_k f(n_k x)$  is a.e. divergent.

Remark 1.8. If  $(n_k)$  grows exponentially (i.e.  $n_{k+1}/n_k \ge q > 1$ ), then  $\sum_{k=1}^{\infty} c_k f(n_k x)$  converges a.e. for any  $\mathbf{c} \in \ell^2$  by Kac's theorem [16]. Thus Theorem 1.7 is sharp. It also remains true with minor modifications in the proof, if instead of  $f \in \operatorname{Lip}_{\alpha}(\mathbf{T})$  we assume only  $f \in L^2(\mathbf{T})$ . For the class of functions defined in (1.5), Brémont has recently showed a similar result in [5].

In the general case the following quadratic form appears:

$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)}{\ell\vee k}.$$

Since  $\frac{(k,\ell)}{\ell \vee k} \leq \frac{(k,\ell)}{\sqrt{k\ell}}$ , this may be regarded as a continuation of the limiting case s = 1/2. Recall some basic facts concerning quadratic forms. Let  $U_n$  denote the unit sphere of  $\mathbb{R}^n$  and let  $A = \{a_{i,j}\}_{i,j=1}^n$  be an  $n \times n$  real symmetric matrix with characteristic roots  $\lambda_1, \ldots, \lambda_n$ . It is well-known that the set of values assumed by the quadratic form  $Q(\mathbf{x}) = \sum_{i,j=1}^n x_i x_j a_{i,j}$  when  $\mathbf{x} = (x_1, \ldots, x_n) \in U_n$ , coincides with the set of values assumed by  $\sum_{i=1}^n \lambda_i y_i^2$  on  $U_n$ . See [2], p. 39 and Chapter 4.

Hence we get

(1.16) 
$$\left(\inf_{i=1}^{n}\lambda_{i}\right)\sum_{i=1}^{n}x_{i}^{2}\leq\left|Q(\mathbf{x})\right|\leq\left(\sup_{i=1}^{n}\lambda_{i}\right)\sum_{i=1}^{n}x_{i}^{2}.$$

This way to estimate  $Q(\mathbf{x})$  strongly relies on a good knewledge of the extremal eigenvalues. The classical weighted estimate below is often more convenient.

**Lemma 1.9.** For any system of complex numbers  $\{x_i\}$  and  $\{\alpha_{i,j}\}$ ,

$$\Big|\sum_{\substack{1\leq i,j\leq n\\i\neq j}} x_i x_j \alpha_{i,j}\Big| \leq \frac{1}{2} \sum_{i=1}^n x_i^2 \Big(\sum_{\substack{\ell=1\\\ell\neq i}}^n (|\alpha_{i,\ell}| + |\alpha_{\ell,i}|)\Big).$$

*Proof.* We have

$$\Big|\sum_{1 \le i < j \le n} x_i x_j \alpha_{i,j}\Big| \le \sum_{1 \le i < j \le n} \Big(\frac{x_i^2 + x_j^2}{2}\Big) |\alpha_{i,j}| \le \frac{1}{2} \sum_{i=1}^n x_i^2 \Big(\sum_{i < \ell \le n} |\alpha_{i,\ell}| + \sum_{1 \le \ell < i} |\alpha_{\ell,i}|\Big).$$

Operating similarly for the sum  $\sum_{1 \le j < i \le n} x_i x_j \alpha_{i,j}$  gives the result.

This suggests to attach to K a function  $\vartheta_K$  defined by

(1.17) 
$$\vartheta_K(k) = \sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(\ell, k)}{\ell \lor k}$$

The associated coefficient

$$\vartheta_K = \sup_{k \in K} \vartheta_K(k)$$

will serve as a measure of the arithmetical complexity of K. For instance,  $\vartheta_K$  is small if K is a set of prime numbers, and uniformly bounded over all subsets K of a given chain, as we shall see. A sequence  $\mathcal{N} = \{n_k, k \ge 1\}$  is a called a chain if  $n_k | n_{k+1}$  for all k. There are examples for which  $\vartheta_K(k) = o(1)$ , if k is large.

*Example 1.* (Prime sequence) Take  $K = \mathcal{P} \cap [N/2, N]$  where  $\mathcal{P}$  denotes the sequence of consecutive primes. Let  $\pi(n) = \#\{p \text{ prime}, p \leq n\}$  be the prime numbers function. Then

$$\sum_{\substack{N/2 \le \ell < k \\ \ell \in \mathcal{P}}} \frac{(\ell, k)}{\ell \lor k} \le \frac{\pi(k)}{k} \le \frac{C}{\log k},$$

and

$$\sum_{\substack{k < \ell \le N \\ \ell \in \mathcal{P}}} \frac{(\ell, k)}{\ell} = \sum_{\substack{k < \ell \le N \\ \ell \in \mathcal{P}}} \frac{1}{\ell} \le 2\frac{\pi(N)}{N} \le \frac{C}{\log N} \le \frac{C}{\log k}$$

So that for all  $k \in K$ ,

(1.18) 
$$\vartheta_{\mathcal{P}\cap[N/2,N]}(k) \le C/\log k$$

It is easy to extrapolate from this that  $\vartheta_K$  can be on examples as small as wished. There are also important classes of sequences for which  $\vartheta_K$  is uniformly bounded over all of its finite parts K.

Example 2. (Hadamard gap sequences) Consider a sequence  $\mathcal{N} = \{n_k, k \ge 1\}$  satisfying the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} \ge q > 1$$

Let  $K = \{n_k, k \in \mathcal{K}\}$ . Then

(1.20) 
$$\sup_{\mathcal{K}} \sup_{\ell \in \mathcal{K}} \vartheta_{\mathcal{K}}(\ell) = \tau(q) < \infty.$$

Indeed, for  $\ell \in \mathcal{K}$ ,

$$\sum_{\substack{k \in \mathcal{K} \\ k < \ell}} \frac{(n_k, n_\ell)}{n_\ell \vee n_k} \le \sum_{k < \ell} \frac{n_k \wedge n_\ell}{n_\ell \vee n_k} = \sum_{k < \ell} \frac{n_k}{n_\ell} \le \sum_{k < \ell} q^{-(\ell-k)} \le C_q < \infty.$$

Similarly  $\sum_{\substack{k \in \mathcal{K} \\ k > \ell}} \frac{(n_k, n_\ell)}{n_\ell \lor n_k} \leq \sum_{k > \ell} \frac{n_k \land n_\ell}{n_\ell \lor n_k} = \sum_{k > \ell} \frac{n_\ell}{n_k} \leq \sum_{k > \ell} q^{-(k-\ell)} \leq C_q < \infty$ . As  $C_q$  depends on q only. This yields (1.20).

*Example 3.* (Squarefree numbers) Let  $\mathcal{G}$  be the set of squarefree numbers generated by some increasing sequence  $2 \leq p_1 < p_2 < \ldots$  of prime integers satisfying the following condition

(1.21) 
$$\mu = \sum_{i=1}^{\infty} \frac{1}{p_i} < 1.$$

Take  $K \subset \mathcal{G}$ . Writing in what follows  $\ell = \lambda d$ ,  $k = \kappa d$  with  $d = (\ell, k)$ , we easily get

$$(1.22) \qquad \vartheta_K(k) = \sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(\ell, k)}{\ell \lor k} \le \frac{1}{\kappa} \sum_{\lambda \le \kappa} 1 + \sum_{\substack{\lambda \in \mathcal{G} \\ \lambda > k}} \frac{1}{\lambda} \le C + \sum_{p_i > k} \frac{1}{p_i} + \sum_{p_i p_j > k} \frac{1}{p_i p_j} + \dots,$$

since the number of squarefree integers less than x is of order  $6x/\pi^2$ . Now

$$\sum_{p_i > k} \frac{1}{p_i} + \sum_{p_i p_j > k} \frac{1}{p_i p_j} + \dots \le \mu + \mu^2 + \dots < \infty.$$

Hence

$$\sup_{K\subset\mathcal{G}}\vartheta_K<\infty$$

A first basic mean estimate obtained in this work (the proof will be given in Section 3) is the following.

**Lemma 1.10.** For any finite sequences of reals  $\{a_j, j \in J\}, \{c_k, k \in K\},\$ 

$$\left\|\sum_{k \in K} c_k \sum_{j \in J} a_j e_{kj}\right\|_2^2 \le C|J| \sup_{j \in J} |a_j|^2 \sum_{k \in K} c_k^2 \max(1, \vartheta_K(k))$$

A general bound for  $\vartheta_K(k)$  can be provided by using Pillai's arithmetical function P(k), which is defined by  $P(k) = \sum_{d=1}^{k} (d, k)$ . Recall that we have  $P(k) = \sum_{d|k} d\phi(k/d)$ , so that the arithmetic mean of  $(1, k), \ldots, (k, k)$  is given by

(1.23) 
$$A(k) = \frac{P(k)}{k} = \sum_{\kappa|k} \frac{\phi(\kappa)}{\kappa}.$$

**Lemma 1.11.** For all finite sets K of integers and all  $k \in K$ ,

(1.24) 
$$\vartheta_K(k) \le C \log(\frac{eK_+}{k}) A(k).$$

where C is an absolute constant, and  $K_+$  (resp.  $K_-$ ) denotes the largest (resp. smallest) term of K.

This follows immediately from Lemma 3.1 below. Example 1 shows that (1.24) is not always optimal. Estimate (1.24), however, implies that if  $\mathcal{M} = \{m_k, k \geq 1\}$  is a sequence of mutually coprime integers, then

$$\sup_{N} \sup_{K \subset [\rho N, N]} \vartheta_K = C_{\rho} < \infty.$$

*Remark* 1.12. The main orders of A(k) are well known. As  $C_{\frac{k}{\log \log k}} \leq \phi(k) \leq k$ , the function A(k) always satisfies

$$\frac{d(k)}{\log\log k} \le A(k) \le d(k),$$

where d(k) denotes the number of divisors of k. As to the maximal order, we have Chidambaraswamy and Sitaramachandrarao estimate,

(1.25) 
$$\limsup_{n \to \infty} \frac{\log A(n) \log \log n}{\log n} = \log 2.$$

This is well-known for the function d(n) instead of A(n). We refer to Tóth's recent survey [25] on Pillai's function.

For the class of examples previously considered, we have the following

**Proposition 1.13.** Let K be a finite set of integers. For any  $k \in K$ ,

$$\sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^s \ell^s} \le \begin{cases} 2\left(\log \frac{K_+}{K_-}\right) \sigma_{-1}(k) & \text{if } s = 1, \\ C_s 2^s k^{s-1} \left(\int_{K_-}^{K_+} \frac{\mathrm{d}u}{u^s}\right) \sigma_{1-2s}(k) & \text{if } s < 1. \end{cases}$$

Thus 
$$\sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^s \ell^s} \leq 2^s M^{1-s} \sigma_{1-2s}(k)$$
, if  $K_+ \leq M K_-$ .

*Proof.* Let s = 1. As for  $\lambda \ge 1$ ,  $\frac{1}{\lambda} \le \min\left(\int_{\lambda-1}^{\lambda} \frac{\mathrm{d}t}{t}, 2\int_{\lambda}^{\lambda+1} \frac{\mathrm{d}t}{t}\right)$ , we have

$$\begin{split} \sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^2}{k\ell} &\leq \sum_{d|k} \frac{1}{(k/d)} \sum_{\substack{K_-/d \leq \lambda \leq K_+/d}} \frac{1}{\lambda} \\ &= \sum_{d|k} \frac{1}{(k/d)} \bigg\{ \sum_{\substack{K_-/d \leq \lambda < k/d}} \frac{1}{\lambda} + \sum_{\substack{k/d < \lambda \leq K_+/d}} \frac{1}{\lambda} \bigg\} \\ &\leq \sum_{d|k} \frac{1}{(k/d)} \bigg\{ 2 \int_{K_-/d}^{k/d} \frac{dt}{t} + \int_{k/d}^{K_+/d} \frac{dt}{t} \bigg\} \\ &= \sum_{d|k} \frac{1}{(k/d)} \bigg\{ 2 \int_{K_-}^k \frac{du}{u} + \int_k^{K_+} \frac{du}{u} \bigg\} \leq 2 \sum_{d|k} \frac{1}{(k/d)} \int_{K_-}^{K_+} \frac{du}{u}. \end{split}$$

Similarly, when 0 < s < 1,

$$\sum_{\substack{\ell \in K \\ \ell \neq k}} \frac{(k,\ell)^{2s}}{k^{s}\ell^{s}} \leq \sum_{d|k} \frac{1}{(k/d)^{s}} \left\{ \sum_{K_{-}/d \leq \lambda < k/d} \frac{1}{\lambda^{s}} + \sum_{k/d < \lambda \leq K_{+}/d} \frac{1}{\lambda^{s}} \right\}$$

$$\leq \sum_{d|k} \frac{1}{(k/d)^{s}} \left\{ 2^{s} d^{s-1} \int_{K_{-}}^{k} \frac{\mathrm{d}u}{u^{s}} + d^{s-1} \int_{k}^{K_{+}} \frac{\mathrm{d}u}{u^{s}} \right\}$$

$$\leq 2^{s} \left\{ \int_{K_{-}}^{K_{+}} \frac{\mathrm{d}u}{u^{s}} \right\} \sum_{\kappa|k} \frac{(k/\kappa)^{s-1}}{\kappa^{s}} = 2^{s} k^{s-1} \left\{ \int_{K_{-}}^{K_{+}} \frac{\mathrm{d}u}{u^{s}} \right\} \sigma_{1-2s}(k).$$

This implies when combined with Lemma 1.9, if  $K_+ \leq C K_-,$ 

(1.26) 
$$\left\|\sum_{k\in K} c_k f_k\right\|_2^2 \le C_s \sum_{k\in K} \sigma_{1-2s}(k) c_k^2,$$

when  $1/2 < s \le 1$ , which is slightly more precise than (1.12). In the case s = 1/2, not covered by the class of functions (1.5), it also gives

(1.27) 
$$\sum_{k,\ell\in K} c_k c_\ell \frac{(k,\ell)}{\sqrt{k\ell}} \le C \sum_{k\in K} d(k) c_k^2.$$

# 2. Main Results

We now state the main results of this paper. We first consider mean convergence. Let  $f \in L^2$ . Define for t > 0, and any sequence  $\mathbf{c} = \{c_k, k \ge 0\}$  of reals,

$$S_t(\mathbf{c}) = \sum_{\substack{k \in \mathcal{N} \\ k \le t}} c_k f_k.$$

**Theorem 2.1.** Let  $f \sim \sum_{j} a_{j}e_{j}$  and assume that the following condition is satisfied: For some real M > 1,

(2.1) 
$$L = \sum_{v=0}^{\infty} M^{v} \Big( \sup_{M^{v} \le j < M^{v+1}} a_{j}^{2} \Big) < \infty.$$

a) Let  $\mathcal{N} = \{n_k, k \ge 1\}$  be an increasing sequence of positive integers satisfying for any  $\mu > 1$ ,

(2.2) 
$$\sup_{j\geq 0}\vartheta_{\mathcal{N}\cap[\mu^j,\mu^{j+1}[}<\infty.$$

Then there exists a constant C such that for any  $\mathbf{c} \in \ell_2$ ,

$$\left(\sum_{j\geq 0} \left\| S_{\mu^{j+1}}(\mathbf{c}) - S_{\mu^{j}}(\mathbf{c}) \right\|_{2}^{2} \right)^{1/2} \leq C \|\mathbf{c}\|_{2}.$$

b) Assume that for any  $\mu > 1$ ,

(2.3) 
$$\vartheta_{\mathcal{N}\cap[\mu^j,\mu^{j+1}]} = o(1) \qquad j \to \infty$$

If the coefficient sequences  $\mathbf{a}$ ,  $\mathbf{c}$  have each constant signs, then

$$\|\mathbf{c}\|_{2} \leq \Big(\sum_{j\geq 0} \|S_{\mu^{j+1}}(\mathbf{c}) - S_{\mu^{j}}(\mathbf{c})\|_{2}^{2}\Big)^{1/2} \leq C \|\mathbf{c}\|_{2}.$$

Remark 2.2. By (1.24), condition (2.2) is satisfied as soon as

$$\sup_{k\in\mathcal{N}}A(k)<\infty$$

We also establish new almost everywhere convergence results.

**Definition 2.3.** We say that a sequence of coefficients **c** is universal if for any orthonormal system  $\Phi$  on a bounded interval, the series  $\sum_{k=1}^{\infty} c_k \varphi_k$  converges a.e.

Typically, **c** is universal if the series  $\sum_k c_k^2 \log^2 k$  converges (Rademacher-Menshov theorem), or if the series  $\sum_k c_k^2 (\log |c_k|^{-1})^{1+h} (\log k)^{1-h}$  converges for some  $0 \le h < 1$  (Tandori's theorem [22]). And the condition  $\sum_k c_k^2 (\log |c_k|^{-1})^2 < \infty$ , with  $c_k \ne 0$ ,  $c_k \rightarrow 0$  is necessary for **c** to be universal, see [21].

**Theorem 2.4.** Assume that there exist a non-increasing sequence of positive reals  $\{\varepsilon(j), j \ge 1\}$  and an increasing sequence of positive integers  $\{j_r, r \ge 1\}$ , such that

(2.4) 
$$A = \sum_{|a_{\ell}| > \varepsilon(\ell)} |a_{\ell}| < \infty, \qquad B = \sum_{r} j_{r+1}^{1/2} \varepsilon(j_r) < \infty.$$

Let  $1 \leq k_1 < k_2 < \ldots$  be an increasing sequence of integers, which we denote by K. Then the series  $\sum_{n\geq 1} c_n f_{k_n}$  converges a.e. for any coefficient sequence  $\{c_n, n\geq 1\}$  such that

(2.5) 
$$\{c_n \max\left(1, \vartheta_K(k_n)^{1/2}\right), n \ge 1\}$$

 $is \ universal.$ 

b) In particular, the same conclusion holds if

(2.6) 
$$A = \sum_{|a_{\ell}| > \varepsilon(\ell)} |a_{\ell}| < \infty, \qquad B_1 = \sum_j \varepsilon^2(j) < \infty.$$

in place of (2.4).

Remark 2.5.

(i) If condition (2.4) is satisfied for  $j_r = M^r$ , for some M > 1, then  $B < \infty$  means  $\sum_r M^{r/2} \varepsilon(M^r) < \infty$ , which is a stronger requirement than  $\sum_r M^r \varepsilon^2(M^r) < \infty$ . And this is equivalent to  $B_1 < \infty$  in (2.6). Hence (2.4) can be replaced by the much weaker condition (2.6) when  $j_r$  is geometrically growing.

(ii) Suppose now that f satisfies assumption (2.1). Then (2.6) is fulfilled. Indeed, choose

$$\varepsilon_{\ell} = \sup_{M^r \le |j| \le M^{r+1}} |a_j|, \qquad M^r \le \ell \le M^{r+1}, \ r = 0, 1, \dots$$

The first requirement in (2.6) is trivially satisfied since the summation index is empty, whereas the second is, by (i), equivalent to (2.1).

(iii) Let  $f \in \operatorname{Lip}_{\alpha}(\mathbb{T})$ ,  $\alpha > 1/4$ . Then f satisfies condition (2.6). Indeed, it is wellknown that if  $f \in \operatorname{Lip}_{\alpha}(\mathbb{T})$ ,  $0 < \alpha \leq 1$ , then  $\sum_{2^{r} < j \leq 2^{r+1}} a_{j}^{2} \leq C2^{-2r\alpha}$ . See [29], inequalities (3·3) p. 241. Pick a real  $\beta$  such that  $2\alpha > \beta > 1/2$  and take  $\varepsilon(j) = j^{-\beta}$ , M = 2. Condition (2.6) is satisfied with this choice since  $\sum_{j} \varepsilon^{2}(j) < \infty$  and

$$\sum_{\substack{2^r < j \le 2^{r+1} \\ |a_j| > \varepsilon(j)}} |a_j| \le \sum_{2^r < j \le 2^{r+1}} \frac{|a_j|^2}{\varepsilon(j)} \le C 2^{r\beta} 2^{-2r\alpha} = C 2^{-r(2\alpha - \beta)},$$

so that  $A < \infty$ .

(iv) For any  $\alpha > 0$ , there exists  $f \in L^2(\mathbb{T})$ ,  $\int_{\mathbb{T}} f = 0$ , such that  $f \notin \operatorname{Lip}_{\alpha}(\mathbb{T})$  but f satisfies condition (2.6). Such an f can be built as follows. Let  $\psi : \mathbb{R}^+ \to \mathbb{R}^+$  be such that

$$\begin{cases} \psi(r)2^{-r/2} \downarrow 0 , \ \psi(r)2^{\alpha r} \uparrow \infty \quad \text{as } r \uparrow \infty, \\ \sum_r \psi^2(r) < \infty. \end{cases}$$

Let  $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$  be decreasing and defined by

$$\varepsilon(x) = \begin{cases} \psi(r)2^{-r/2} & \text{if } 2^r < j \le 2^{r+1}, \quad r \text{ even}, \\ \text{linear otherwise.} \end{cases}$$

We choose f such that its Fourier coefficients satisfy

$$\begin{cases} a_j = \psi(r) 2^{-r/2} & \text{if } 2^r < |j| \le 2^{r+1}, \quad r \text{ even,} \\ \sum_{r \text{ odd } 2^r < |j| \le 2^{r+1}} |a_j| < \infty. \end{cases}$$

Clearly

$$\frac{\sum_{2^r < |j| \le 2^{r+1}} |a_j|}{2^{r(\frac{1}{2} - \alpha)}} = C \frac{\psi(r) 2^{\frac{r}{2}}}{2^{r(\frac{1}{2} - \alpha)}} = \psi(r) 2^{\alpha r} \uparrow \infty.$$

Hence, in view of [29], inequality (3.4) p. 241,  $f \notin \operatorname{Lip}_{\alpha}(\mathbb{T})$ . Further,

$$\sum_{2^r < |j| \le 2^{r+1}} |a_j|^2 = 2^r \psi^2(r) 2^{-r} = \psi^2(r),$$

when r is even. Thus

$$\sum_{j \in \mathbb{Z}*} |a_j|^2 = \sum_{r \ge 0} \sum_{2^r < |j| \le 2^{r+1}} |a_j|^2 \le \sum_{r \text{ even}} \psi^2(r) + \sum_{r \text{ odd } 2^r < |j| \le 2^{r+1}} |a_j| < \infty,$$

by assumption. Moreover, by construction,

$$\sum_{\substack{2^r < |j| \le 2^{r+1} \\ |a_j| > \varepsilon(j)}} |a_j| = 0,$$

when r is even. It follows that

$$\sum_{|a_j| > \varepsilon(j)} |a_j| = \sum_{r \text{ odd } 2^r < |j| \le 2^{r+1} \atop |a_j| > \varepsilon(j)} |a_j| \le \sum_{r \text{ odd } 2^r < |j| \le 2^{r+1}} |a_j| < \infty,$$

by assumption. Now

$$\sum_{j} \varepsilon(j)^2 \le 3 \sum_{r \text{ even}} 2^r \psi^2(r) 2^{-r} = 3 \sum_{r \text{ even}} \psi^2(r) < \infty.$$

Therefore condition (2.6) is satisfied, as claimed.

We will also obtain the following useful result, as a combination of the above Theorem with Lemma 1.11.

**Corollary 2.6.** Let  $1 \le k_1 < k_2 < \ldots$  be an increasing sequence of integers. Assume that (2.6) is satisfied and that

(2.7) 
$$\sum_{n} c_n^2 A(k_n) (\log n)^2 < \infty.$$

Then the series  $\sum_{n} c_n f_{k_n}$  converges a.e.

By Remark 2.5-ii), the same conclusions are reached if f satisfies assumption (2.1).

Remark 2.7. As  $A(k) \leq d(k)$ , (2.12) is satisfied whenever

(2.8) 
$$\sum_{n} c_n^2 d(k_n) (\log n)^2 < \infty.$$

Consequently, under condition (2.8) the series  $\sum_{n\geq 1} c_n f_n$  converges a.e. for any  $f \in \text{Lip}_{\alpha}(\mathbb{T})$ ,  $\alpha > 1/4$ . The presence of the factor  $d(k_n)$  is important. Replacing d(j) by the classical bound: for some  $c_0 > 2$ ,

(2.9) 
$$d(j) = \mathcal{O}(c_0^{\log j / \log \log j}),$$

gives rise to a much weaker result. This strictly includes a recent result obtained by Aistleitner [1] who proved by using a fine diophantine estimate due to Dyer and Harman, that the condition

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{2\log k}{\log\log k}\right) < \infty,$$

is sufficient for the convergence almost everwhere of the series  $\sum_{k=0}^{\infty} c_k f(kx)$ . It is interesting to compare the multiplicative factor of  $c_k^2$  in the above with the shape of the bound of the divisor function in (2.9). This can also be deduced from Theorem 1.1 in [26] published shortly afterward, and which was based on properties of the Erdös-Hooley function

$$\Delta(v) = \sup_{u \in \mathbb{R}} \sum_{\substack{d \mid v \\ x < d \le ex}} 1.$$

In place of condition (2.4), we assumed that f satisfies

(2.10) 
$$\sum_{\nu \ge 1} a_{\nu}^2 \Delta(\nu) < \infty.$$

This is fulfilled if  $f \in \text{Lip}_{\alpha}(\mathbb{T})$ ,  $\alpha > 1/4$ , but also if  $a_{\nu} = \mathcal{O}(\nu^{-\beta})$ ,  $\beta > 1/2$ . Conditions (2.4) and (2.10) are, however, hardly comparable. As is well known,  $\Delta$  has a slower mean behavior than d. Indeed,

$$\frac{1}{x} \sum_{v \le x} d(v) \sim x, \qquad \text{while} \qquad \frac{1}{x} \sum_{n \le x} \Delta(n) = \mathcal{O}\Big(e^{c\sqrt{\log\log x \cdot \log\log\log x}}\Big)$$

for a suitable constant c > 0; see [23]. Hence it follows by partial summation that if f has monotonic Fourier coefficient sequence, condition (2.10) can be replaced by the considerably much weaker condition

(2.11) 
$$\sum_{\nu \ge 1} a_{\nu}^2 e^{c\sqrt{\log \log \nu \cdot \log \log \log \nu}} < \infty.$$

When  $|a_j| = \mathcal{O}(j^{-s})$ , s > 1/2, the above corollary can be much improved.

**Theorem 2.8.** Let  $1 \le k_1 < k_2 < \ldots$  be an increasing sequence of integers. Let  $f(x) = \sum_{j=1}^{\infty} a_j \sin 2\pi j x$  and assume that  $|a_j| = \mathcal{O}(j^{-s})$ , s > 1/2. Assume that

(2.12) 
$$\sum_{n} c_n^2 \sigma_{1-2s}(k_n) (\log n)^2 < \infty.$$

Then the series  $\sum_{n} c_n f_{k_n}$  converges a.e.

Our paper is organized as follows. In Section 3, we collect estimates of number theoretical type, some estimates for quadratic forms and tools from the theory of orthogonal sums. The remainding sections are devoted to the proofs of the main results.

# 3. Auxiliary Results

**Lemma 3.1.** For any positive integers  $k \leq N$ ,

$$\sum_{1 \leq \ell \leq N \atop \ell \neq k} \frac{(\ell,k)}{\ell \lor k} \leq C \log(\frac{eN}{k}) \sum_{\kappa \mid k} \frac{\phi(\kappa)}{\kappa},$$

where C is an absolute constant.

*Proof.* Let  $k < \ell \leq N$ . Then,

$$\sum_{k < \ell \le N} \frac{(\ell, k)}{\ell} \le \sum_{d \mid k} \sum_{\substack{k/d < \lambda < N/d \\ (\lambda, k/d) = 1}} \frac{1}{\lambda},$$

where we write  $\ell = \lambda d$ ,  $k = \kappa d$ ,  $(\ell, k) = d$ . To estimate the inner sum, we use van Lint and Richert estimate ([17], Lemma 2): for  $x \ge 1$  and k such that  $P^+(k) \le x$ , where  $P^+(k)$  is the largest prime factor of k, we have

(3.1) 
$$\sum_{\substack{1 \le m \le x \\ (m,k)=1}} 1 \le C \frac{\phi(k)}{k} x.$$

By [24] p. 3, if  $a_n$  are complex numbers,  $A(t) = \sum_{n \leq t} a_n$  and  $b \in \mathcal{C}^1([1, x])$ ,

(3.2) 
$$\sum_{1 \le n \le x} a_n b(n) = A(x)b(x) - \int_1^x A(t)b'(t)dt,$$

Take  $a_{\lambda} = 0$  if  $1 \leq \lambda < k/d$  and  $a_{\lambda} = \chi\{(\lambda, k/d) = 1\}$  if  $\lambda \geq k/d$ ,  $b(t) = t^{-1}$ . Then A(t) = 0 if t < k/d. Now if  $k/d \leq t \leq N/d$ , obviously  $P(k/d) \leq t$ . And (3.1) applies to give

$$A(t) \le C \frac{\phi(k/d)}{(k/d)} t.$$

Therefore

$$\sum_{\substack{k/d < \lambda < N/d \\ (\lambda,k/d)=1}} \frac{1}{\lambda} = \frac{A(N/d)}{(N/d)} + s \int_{k/d}^{N/d} A(t) \frac{dt}{t^2}$$

$$\leq \frac{A(N/d)}{(N/d)} + C \frac{\phi(k/d)}{(k/d)} \int_{k/d}^{N/d} \frac{dt}{t}$$

$$\leq C \Big( \frac{1}{(N/d)} \frac{\phi(k/d)}{(k/d)} (N/d) + \frac{\phi(k/d)}{(k/d)} \log(N/k) \Big)$$

$$\leq C \frac{\phi(k/d)}{(k/d)} \log(\frac{eN}{k}).$$
(3.3)

Henceforth,

(3.4) 
$$\sum_{k<\ell\leq N} \frac{(\ell,k)}{\ell} \leq C\log(\frac{eN}{k}) \sum_{d|k} \frac{\phi(k/d)}{(k/d)} = C\log(\frac{eN}{k}) \sum_{\kappa|k} \frac{\phi(\kappa)}{\kappa}.$$

Now, similarly by writing  $\ell = \lambda d$ ,  $k = \kappa d$ ,  $(\ell, k) = d$ , we get

$$\sum_{1 \le \ell < k} \frac{(\ell, k)}{\ell \lor k} = \sum_{d \mid k} \frac{1}{(k/d)} \sum_{\substack{\frac{1}{d} \le \lambda < k/d \\ (\lambda, \kappa) = 1}} 1 \le \sum_{d \mid k} \frac{\phi(k/d)}{k/d} = \sum_{\kappa \mid k} \frac{\phi(\kappa)}{\kappa}$$

Consequently,

$$\sum_{\substack{1 \leq \ell \leq N \\ \ell \neq k}} \frac{(\ell, k)}{\ell \lor k} \leq C \log(\frac{eN}{k}) \sum_{\kappa \mid k} \frac{\phi(\kappa)}{\kappa}.$$

The proof is now complete.

We pass to mean estimates. Lemma 1.9 implies

$$\Big|\sum_{i,j=1}^{n} x_i x_j \alpha_{i,j} - \sum_{i=1}^{n} x_i^2 \alpha_{i,i}\Big| \le \frac{1}{2} \sum_{i=1}^{n} x_i^2 \Big(\sum_{\substack{\ell=1\\\ell \neq i}}^{n} (|\alpha_{i,\ell}| + |\alpha_{\ell,i}|)\Big),$$

which is extremely useful. Another simple consequence concerns Riesz sequences.

**Definition 3.2.** A sequence of vectors  $\{v_i, i \ge 1\}$  in a Hilbert space H is called a Riesz sequence if there exist positive constants  $C_1, C_2$  such that

$$C_1\left(\sum_{i=1}^n |x_i|^2\right) \le \left\|\sum_{i=1}^n x_i v_i\right\|^2 \le C_2\left(\sum_{i=1}^n |x_i|^2\right),$$

for all sequences of scalars  $\{x_i, 1 \leq i \leq n\}$ .

**Theorem 3.3.** Let  $\mathbf{v} = \{v_i, i \ge 1\}$  be a sequence of vectors in a Hilbert space H such that

(3.5) 
$$\sup_{i \ge 1} \sum_{j \ne i} |\langle v_i, v_j \rangle| < \inf_{i \ge 1} ||v_i||^2.$$

Then  $\{v_i, i \geq 1\}$  is a Riesz sequence.

Proof. Put

$$b(\mathbf{v}) = \sup_{i \ge 1} \sum_{\substack{j \ge 1 \\ j \neq i}} |\langle v_i, v_j \rangle|.$$

By taking  $\alpha_{i,j} = \langle v_i, v_j \rangle$  in Lemma 1.9, we get

$$\left| \left\| \sum_{i=1}^{n} x_{i} v_{i} \right\|^{2} - \sum_{i=1}^{n} x_{i}^{2} \|v_{i}\|^{2} \right| \leq \frac{1}{2} \sum_{i=1}^{n} x_{i}^{2} \left( \sum_{\ell=1 \ \ell \neq i}^{n} (|\alpha_{i,\ell}| + |\alpha_{\ell,i}|) \right) \leq b(\mathbf{v}) \sum_{i=1}^{n} x_{i}^{2}.$$

Hence,

$$\left(\inf_{i\geq 1} \|v_i\|^2 - b(\mathbf{v})\right) \sum_{i=1}^n x_i^2 \le \left\|\sum_{i=1}^n x_i v_i\right\|^2 \le \left(\sup_{i\geq 1} \|v_i\|^2 + b(\mathbf{v})\right) \sum_{i=1}^n x_i^2.$$

Hedenmalm, Lindquist and Seip [11], [12] proved that if  $g(t) \sim \sum_{k=1}^{\infty} \varphi_k \cos 2\pi kt$ ,  $g \in L^2(\mathbb{T})$ , then  $\{g_n, n \geq 1\}$  (recall that  $g_n(x) = g(nx)$ ) is a Riesz sequence in  $L^2(\mathbb{T})$  if and only if the Dirichlet series  $\sum_{n=1}^{\infty} \varphi_n n^{-s}$  is analytic and bounded away from 0 and  $\infty$  in the whole right half-plane  $\Re z > 0$ , i.e.

(3.6) 
$$\delta \le \left| \sum_{n=1}^{\infty} \varphi_n n^{-\sigma - it} \right| \le \Delta, \quad \text{for } \sigma > 0,$$

with some positive constants  $\delta$  and  $\Delta$ .

In view of Theorem 3.3, we deduce that a sufficient condition for (3.6) to be satisfied is

(3.7) 
$$\sup_{i\geq 1}\sum_{\substack{j\geq 1\\j\neq i}} |\langle g_i, g_j\rangle| < \|g\|^2$$

Concerning the class of examples considered in the Introduction, we deduce

**Corollary 3.4.** Let f be defined as in (1.5) with  $1/2 < s \le 1$ . Let  $\{n_i, i \ge 1\}$  be increasing and satisfying

(3.8) 
$$\sup_{i \ge 1} \sum_{\substack{j \ge 1 \\ j \ne i}} \frac{(n_i, n_j)^{2s}}{n_i^s n_j^s} < 1.$$

Then  $\{f_{n_i}, i \geq 1\}$  is a Riesz sequence in  $L^2(\mathbb{T})$ .

Remark 3.5. Brémont ([5] Theorem 1.2-i)) showed, using Möbius orthogonalization, that the sequence  $\{f_{n_k}, k \ge 1\}$  is a Riesz sequence in  $L^2(\mathbb{T})$  whenever

$$n_{k+1}/n_k \ge c > 1$$

If c > 3, this follows immediately from Corollary 3.4 since

$$\sum_{\substack{j \ge 1 \\ j \ne i}} \frac{(n_i, n_j)^{2s}}{n_i^s n_j^s} = 2 \sum_{j > i} \frac{(n_i, n_j)^{2s}}{n_i^s n_j^s} \le 2 \sum_{j > i} \left(\frac{n_i}{n_j}\right)^s \le 2 \sum_{j > i} c^{-(j-i)} = \frac{2}{c-1} < 1.$$

For c > 1, this is however a special case of Kac's result [16] later extended by Gaposhkin, since the square modulus of continuity of f

$$\omega_2(\delta, f) = \sup_{0 < h \le \delta} \left\{ \int_0^1 |f(x+h) - f(x)|^2 \mathrm{d}x \right\}^{1/2}$$

satisfies  $\omega_2(\delta, f) = \mathcal{O}(\delta^{\varepsilon})$  for some  $\varepsilon > 0$ . Let indeed  $f(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m^s}$ , where s > 1/2. Using formula (3.2) in [29] p.241, gives

$$\int_0^1 |f(x+h) - f(x)|^2 \mathrm{d}x = C \sum_{m=1}^\infty \frac{\sin^2(\pi m h)}{m^{2s}} \le C \sum_{m=1}^\infty \frac{(mh \wedge 1)}{m^{2s}} \le C h^{2s-1}.$$

We now give the

Proof of Lemma 1.10. Putting  $Z_j = \sum_{k \in K} c_k e_{jk}$ , we have

(3.9) 
$$\begin{aligned} \|\sum_{j\in J} a_j \sum_{k\in K} c_k e_{jk}\|_2^2 &= \|\sum_{j\in J} a_j Z_j\|_2^2 = \sum_{j\in J} a_j^2 \|Z_j\|_2^2 + \sum_{\substack{i\neq j\\i,j\in J}} a_i a_j \langle Z_i, Z_j \rangle \\ &= \sum_{j\in J} \sum_{k\in K} a_j^2 c_k^2 + \sum_{k,\ell\in K} c_k c_\ell \sum_{\substack{i\neq j\\i,j\in J}} a_i a_j \mathbf{1}_{\{jk=i\ell\}}. \end{aligned}$$

Let  $a = J_{-}, b = J_{+}$ . Fix  $k, \ell \in K$ . The equation  $jk = i\ell, i \neq j, i, j \in J$ , being impossible for  $k = \ell$ , let  $k < \ell$ . Writing  $d = (k, \ell), k = k'd, \ell = \ell'd$ , the equation becomes  $jn'_{k} = i\ell'$ . General solutions are  $j = u\ell', i = uk'$ . Then

$$a \le j \le b$$
  $\Rightarrow$   $\frac{ad}{\ell} = \frac{a}{\ell'} \le u \le \frac{b}{\ell'} = \frac{bd}{\ell}.$ 

Operating similarly for i, it follows that

$$\frac{(k,\ell)}{k}a \le u \le \frac{(k,\ell)}{\ell}b.$$

Thus solutions exist only if  $\ell$  and k are such that

$$\frac{\ell}{k} \le \frac{b}{a}.$$

And in that case, their number is bounded by

$$(k,\ell)\Big(\frac{b}{\ell}-\frac{a}{k}\Big).$$

Thus

$$\Big|\sum_{\substack{i\neq j\\i,j\in J}}a_ia_j\mathbf{1}_{\{jk=i\ell\}}\Big| \le \sup_{j\in J}a_j^2(k,\ell)\Big(\frac{b}{\ell}-\frac{a}{k}\Big) \le (b-a)\sup_{j\in J}a_j^2\frac{(k,\ell)}{\ell}.$$

But this bound remains trivially valid if  $\frac{\ell}{k} > \frac{b}{a}$ , since the sum in the left term is empty. The case  $\ell < k$  being identical, it follows that

$$\Big|\sum_{\substack{i\neq j\\i,j\in J}}a_ia_j\mathbf{1}_{\{jk=i\ell\}}\Big| \le \sup_{j\in J}a_j^2\Big(\frac{(k,\ell)}{\ell\vee k}b - \frac{(k,\ell)}{\ell\wedge k}a\Big) \le (b-a)\sup_{j\in J}a_j^2\frac{(k,\ell)}{\ell\vee k}$$

By reporting in (3.9), next using Lemma 1.9, we get

$$\left\| \left\| \sum_{j \in J} a_j \sum_{k \in K} c_k e_{jk} \right\|_2^2 - \sum_{j \in J} \sum_{k \in K} a_j^2 c_k^2 \right\| \leq (b-a) \sup_{j \in J} a_j^2 \sum_{k, \ell \in K} |c_k| |c_\ell| \frac{(k,\ell)}{\ell \lor k} \\ \leq (b-a) \sup_{j \in J} a_j^2 \sum_{k \in K} c_k^2 \max(1, \vartheta_K(k)). \\ \square$$

By combining Lemma 1.11 with estimate b) of Lemma 1.10, we immediately get

Corollary 3.6. Under assumptions of Lemma 1.10,

$$\left\| \left\| \sum_{k \in K} c_k \left( \sum_{j \in J} a_j e_{jk} \right) \right\|_2^2 - \sum_{j \in J} a_j^2 \sum_{k \in K} c_k^2 \right\| \le C |J| \left( \sup_{j \in J} a_j^2 \right) \sum_{k \in K} c_k^2 \log(\frac{eK_+}{k}) A(k).$$

*Remark* 3.7. The factor  $\log(\frac{eK_+}{k})$  appearing in Lemma 3.1 and in Corollary 3.6 is very restrictive, but seems unavoidable. However, when the coefficients  $c_k$ ,  $k \in K$  are commensurable, it can be removed. We indeed also have,

$$\left\| \left\| \sum_{k \in K} c_k \left( \sum_{j \in J} a_j e_{jk} \right) \right\|_2^2 - \sum_{j \in J} a_j^2 \sum_{k \in K} c_k^2 \right\| \le \sup_{k \in K} c_k^2 \left( \sum_{k \in K} A(k) \right) |J| \sup_{j \in J} a_j^2.$$

We omit the proof.

We pass to orthogonality results. Let  $M \ge \mu > 1$ . Let K, L, I, J be sets of positive integers such that: For some integers  $B, u, v \ge 0$  with |v - u| > 1,

(3.11) 
$$K \cup L \subset [\mu^B \mu^{B+1}]$$
 and  $I \subset [M^u, M^{u+1}], \ J \subset [M^v, M^{v+1}].$ 

Put

$$T_H(G) = \sum_{k \in H} c_k \sum_{j \in G} a_j e_{kj}, \qquad H \in \{K, L\}, \ G \in \{I, J\}.$$

**Lemma 3.8.** Under assumption (3.11),  $\langle T_K(J), T_L(I) \rangle = 0$ .

*Proof.* First notice that for any  $k \in K, \ell \in L$ , the ratio  $\ell/k$  satisfies  $1/\mu < \ell/k < \mu$ . Now plainly,

$$\langle T_K(J), T_L(I) \rangle = \sum_{\substack{k \in K \\ \ell \in L}} c_k c_\ell \sum_{\substack{|i| \in I \\ |j| \in J}} a_j a_i \delta_{jk=i\ell}.$$

Suppose v > u + 1. Then  $\frac{|j|}{|i|} \ge M^{v-(u+1)} \ge M$ . The equation  $jk = i\ell$  is impossible. Indeed,

$$\frac{j}{i} = \frac{\ell}{k} \qquad \Rightarrow \qquad M \le \frac{\ell}{k} < \mu.$$

Hence a contradiction since we assumed  $M \ge \mu$ . If u > v+1, then  $\frac{|i|}{|j|} \ge M^{u-(v+1)} \ge M$ , and we arrive similarly to  $M \le \frac{k}{\ell} < \mu$ .

Put for any finite set K of integers,

$$T_K(v) := T_K([M^v, M^{v+1}]).$$

Corollary 3.9.

$$\left\|\sum_{k\in K} c_k f_k\right\|_2^2 \le 3\sum_{u=0}^{\infty} \|T_K(u)\|_2^2.$$

When further the coefficients  $\mathbf{a}$ ,  $\mathbf{c}$  have each constant signs, we also have

$$\sum_{u=0}^{\infty} \|T_K(u)\|_2^2 \le \|\sum_{k \in K} c_k f_k\|_2^2 \le 3 \sum_{u=0}^{\infty} \|T_K(u)\|_2^2.$$

*Proof.* Set  $\Delta(v) = \sum_{M^v \le |j| < M^{v+1}} a_j e_j, v \ge 0$ . As  $f = \sum_{u=0}^{\infty} \Delta(u)$ , Lemma 3.8 implies

(3.12) 
$$\begin{aligned} \left\| \sum_{k \in K} c_k f_k \right\|_2^2 &= \left\| \sum_{u=0}^{\infty} \sum_{k \in K} c_k \Delta_k(u) \right\|_2^2 = \left\| \sum_{u=0}^{\infty} T_K(u) \right\|_2^2 \\ &= \sum_{u=0}^{\infty} \left\| T_K(u) \right\|_2^2 + 2 \sum_{u=0}^{\infty} \langle T_K(u), T_K(u+1) \rangle, \end{aligned}$$

which easily allows to conclude.

Now recall Schur's Theorem ([19], p. 56).

**Lemma 3.10.** Let X be a bounded interval of the real line endowed with the normalized Lebesgue measure. Let  $\{f_k, 1 \leq k \leq n\}$  be measurable functions on a measurable set  $E \subset X$ ,  $\lambda(X \setminus E) > 0$ . These functions can be extended to an orthonormal system on X if and only if the following condition is satisfied

(3.13) 
$$\left\|\sum_{k=1}^{n} c_k f_k\right\|_2^2 \le \sum_{k=1}^{n} c_k^2 \qquad (\forall c_1, \dots, c_n)$$

It is true by induction for infinite sequences. The main argument of the proof is that I - G, where G is the Gram matrix of the system i.e.  $G = (\gamma_{k,\ell}), \ \gamma_{k,\ell} = \int_E f_k f_\ell dx$ , is nonnegative definite. Hence, it is possible to construct on  $E^c$  a system of functions having I - G as Gram matrix.

*Remark* 3.11. More generally, if for positive reals  $\{\delta_k, 1 \leq k \leq n\}$  we have

(3.14) 
$$\left\| \sum_{k=1}^{n} c_k f_k \right\|_2^2 \le \sum_{k=1}^{n} \delta_k c_k^2 \qquad (\forall c_1, \dots, c_n),$$

then  $\{f_k, 1 \leq k \leq n\}$  can be extended to an orthogonal system  $\{\xi_k, 1 \leq k \leq n\}$  on X satisfying  $\|\xi_k\|_2 = \sqrt{\delta_k}$  for all k. Therefore the series  $\sum_k c_k f_k$  converges almost everywhere for all sequences  $\{c_k, k \geq 1\}$  such that  $\{c_k\sqrt{\delta_k}, k \geq 1\}$  is universal.

A complete characterization of universal coefficient sequences has been recently obtained in [20] by Paszkiewicz, solving a long standing open problem. Let

$$A^{\infty} = \left\{ \sum_{k \ge m} c_k^2; m = 1, 2, \dots \right\}$$

**Theorem 3.12.** A sequence of coefficients  $\{c_k, k \ge 1\}$ ,  $\sum_k c_k^2 \le 1$  is universal if and only if there exists a finite measure m on  $A^{\infty}$  such that

(3.15) 
$$\sup_{t \in A^{\infty}} \int_0^1 \frac{\mathrm{d}\varepsilon}{\sqrt{m((t - \varepsilon^2, t + \varepsilon^2))}} < \infty.$$

A measure m such that (3.15) holds is called a *majorizing measure*. Paszkiewicz showed with this deep result the great success of the majorizing measure approach, a technic which has been considerably developed over the years by Talagrand, more recently by Bednorz, and also applied by the second named author to some convergence problems in analysis. Paszkiewicz further obtained other characterizations involving convolution powers of some natural operator.

# 4. Proof of Theorem 2.1

Let  $\mathcal{N} = \{n_k, k \geq 1\}$  be an increasing sequence of positive integers. Let  $\mu > 1$  and consider the "trace" of  $\mathcal{N}$  over the geometric partition of  $\mathbb{N}$  associated to the sequence  $\{\mu^j, j \geq 0\}$ , namely the sets

$$N_j = \mathcal{N} \cap [\mu^j, \mu^{j+1}], \qquad j = 0, 1, \dots$$

Some of them may be empty, so let  $\{N_j^*, j \ge 0\}$  denote the subsequence obtained after having removed all empty sets. By assumption,

$$\sup_{j\geq 0}\vartheta_{\mathcal{N}\cap[\mu^j,\mu^{j+1}[}<\infty.$$

Let  $K \subset N_j^*$  for some j. Let  $M > \mu.$  Applying Lemma 1.10 to

$$T_K(v) = \sum_{k \in K} c_k \sum_{M^v \le j \le M^{v+1}} a_j e_{kj}$$

gives

(4.1) 
$$\left| \left\| T_K(v) \right\|_2^2 - \sum_{M^v \le j \le M^{v+1}} a_j^2 \sum_{k \in K} c_k^2 \right| \le C \vartheta_K M^{v+1} \left( \sup_{M^v \le j \le M^{v+1}} a_j^2 \right) \left( \sum_{k \in K} c_k^2 \right).$$

Using Corollary 3.9, we can bound as follows

$$(4.2) \qquad \begin{aligned} \left\|\sum_{k\in K} c_k f_k\right\|_2^2 &\leq 3\sum_{v=0}^{\infty} \left\|T_K(v)\right\|_2^2 \\ &\leq C(1+\vartheta_K) \Big(\sum_{k\in K} c_k^2\Big) \sum_{v=0}^{\infty} M^{v+1} \Big(\sup_{M^v \leq j \leq M^{v+1}} a_j^2\Big) \\ &\leq C_{\mathcal{N},\mu,f} \Big(\sum_{k\in K} c_k^2\Big). \end{aligned}$$

By taking  $K = N_j^*$  and summing over j, we get

$$\sum_{j\geq 0} \left\| \sum_{k\in N_j^*} c_k f_k \right\|_2^2 \le C_{\mathcal{N},\mu} \|f\|_2^2 \sum_{j\geq 0} \sum_{k\in N_j^*} c_k^2 \le C_{\mathcal{N},\mu} \|f\|_2^2 \sum_{k\geq 0} c_k^2,$$

as claimed. Now, in the case the coefficient sequences have each constant signs, we appeal to the second part of Corollary 3.9 and use the fact that  $\vartheta_{\mathcal{N}\cap[\mu^j,\mu^{j+1}]} = o(1)$ , by assumption (2.3) to conclude.

# 5. Proof of Theorem 2.4

We decompose f into a regular part and an irregular part,  $f = f^{\flat} + f^{\sharp}$ . Here  $f^{\flat} = \sum_{\ell} a_{\ell}^{\flat} e_{\ell}, a_{\ell}^{\flat} = a_{\ell} \chi\{|a_{\ell}| > \varepsilon_{\ell}\}$  is the regular component of f and will be directly controlled by means of Carleson-Hunt's theorem [14]. For the control of the irregular component  $f^{\sharp}$ , arithmetical considerations are needed.

Plainly,

$$\sup_{V \le u \le v \le W} \left| \sum_{u \le n \le v} c_n f_{k_n}^{\flat} \right| = \sup_{V \le u \le v \le W} \left| \sum_{\ell} a_{\ell}^{\flat} \sum_{u \le n \le v} c_n e_{\ell k_n} \right|$$

(5.1) 
$$\leq \sup_{\substack{V \leq u \leq v \leq W}} \sum_{\ell} |a_{\ell}^{\flat}| \Big| \sum_{\substack{u \leq n \leq v \\ u \leq v \leq W}} c_n e_{\ell k_n} \Big| \\ \leq \sum_{\ell} |a_{\ell}^{\flat}| \sup_{\substack{V \leq u \leq v \leq W}} \Big| \sum_{\substack{u \leq n \leq v \\ u \leq v \leq W}} c_n e_{\ell k_n} \Big|.$$

By using Carleson-Hunt's theorem [14],

$$\left\| \sup_{V \le u \le v \le W} \left| \sum_{n=u}^{v} c_n f_{k_n}^{\flat} \right| \right\|_2 \le \sum_{\ell} |a_{\ell}^{\flat}| \left\| \sup_{V \le u \le v \le W} \left| \sum_{u \le n \le v} c_n e_{\ell k_n} \right| \right\|_2$$

$$(5.2) \le A \sup_{\ell} \left\| \sup_{V \le u \le v \le W} \left| \sum_{u \le n \le v} c_n e_{\ell k_n} \right| \right\|_2 \le CA \left( \sum_{k=V}^{W} c_k^2 \right)^{1/2}.$$

Therefore, the sequence  $\left\{\sum_{n=1}^{N} c_n f_{k_n}^{\flat}, N \ge 1\right\}$  has oscillation near infinity tending to zero a.e. In other words, the series  $\sum_n c_n f_{k_n}^{\flat}$  converges a.e.

To control the sums related to the irregular component we need an extra lemma.

Lemma 5.1. For any finite set K,

$$\Big\| \sum_{k \in K} c_k f_k^{\sharp} \Big\|_2 \le CB \Big( \sum_{k \in K} c_k^2 \vartheta_K(k) \Big)^{1/2},$$

where B is defined in assumption (2.4). If  $j_s = M^s$  for some M > 1, let  $B_1 = \sum_s M^{s+1} \varepsilon_{Ms}^2$ . Then,

$$\left\|\sum_{k\in K} c_k f_k^{\sharp}\right\|_2 \le CB_1^{1/2} \left(\sum_{k\in K} c_k^2 \vartheta_K(k)\right)^{1/2},$$

*Proof.* Let  $J_s = [j_s, j_{s+1}]$ . By Lemma 1.10,

$$\begin{aligned} \|\sum_{k\in K} c_k \sum_{j\in J_s} a_j^{\sharp} e_{kj} \|_2^2 &\leq \sum_{j\in J_s} \sum_{k\in K} a_j^{\sharp^2} c_k^2 + C(j_{s+1} - j_s) \varepsilon_{j_s}^2 \sum_{k\in K} c_k^2 \vartheta_K(k) \\ &\leq C(j_{s+1} - j_s) \varepsilon_{j_s}^2 \Big( \sum_{k\in K} c_k^2 \max(1, \vartheta_K(k)) \Big). \end{aligned}$$

Thus

$$\begin{aligned} \left\| \sum_{k \in K} c_k f_k^{\sharp} \right\|_2 &\leq \sum_s \left\| \sum_{k \in K} c_k \sum_{j \in J_s} a_j^{\sharp} e_{kj} \right\|_2 \leq C \Big( \sum_s j_{s+1}^{1/2} \varepsilon_{j_s} \Big) \Big( \sum_{k \in K} c_k^2 \vartheta_K(k) \Big)^{1/2} \\ &= CB \Big( \sum_{k \in K} c_k^2 \vartheta_K(k) \Big)^{1/2} \end{aligned}$$

Further, when  $j_s = M^s$  for some M > 1, by Corollary 3.9, next Lemma 1.10,

$$\begin{aligned} \left\|\sum_{k\in K} c_k f_k^{\sharp}\right\|_2^2 &\leq C\sum_s \left\|\sum_{k\in K} c_k \sum_{j\in J_s} a_j^{\sharp} e_{kj}\right\|_2^2 \leq C \Big(\sum_s j_{s+1} \varepsilon_{j_s}^2\Big) \Big(\sum_{k\in K} c_k^2 \vartheta_K(k)\Big) \\ (5.3) &= CB_1 \Big(\sum_{k\in K} c_k^2 \vartheta_K(k)\Big). \end{aligned}$$

Now we can finish the proof of Theorem 2.4. By Remark 3.11, the series  $\sum_{n\geq 1} c_n f_{k_n}^{\sharp}$ , converges a.e. for any coefficient sequence  $\{c_n, n\geq 1\}$  such that  $\{c_n\sqrt{\vartheta_K(k_n)}, n\geq 1\}$  is universal. And this follows from assumption (2.5). Since we have seen that the series  $\sum_n c_n f_{k_n}^{\sharp}$  converges a.e., we deduce that the series  $\sum_n c_n f_{k_n}$  converges a.e.

### 6. Proof of Corollary 2.6

By Rademacher-Menshov's Theorem, the sequence  $\{c_n\sqrt{\vartheta_K(k_n)}, n \ge 1\}$  is universal if  $\sum_n c_n^2 \vartheta_K(k_n) (\log n)^2 < \infty$ . But by Lemma 1.11,

$$\sum_{v} \sum_{2^{v} < n \le 2^{v+1}} c_{n}^{2} \vartheta_{K}(k_{n}) (\log n)^{2} \le C \sum_{v} \sum_{2^{v} < n \le 2^{v+1}} c_{n}^{2} A_{K}(k_{n}) (\log n)^{2}$$
$$= C \sum_{n} c_{n}^{2} A_{K}(k_{n}) (\log n)^{2} < \infty,$$

by assumption. Hence, by Theorem 2.4 the series  $\sum_n c_n f_{k_n}$  converges a.e. Taking in particular  $K = \mathbb{N}$ , yields that  $\sum_n c_n f_n$  converges a.e., whenever  $\sum_n c_n^2 A(n) (\log n)^2 < \infty$ .

Remark 6.1. Let  $\{k_n, n \ge 1\}$  be an arbitrary increasing sequence of integers. The part of the proof concerning  $f^{\flat}$  also implies that the series  $\sum_n c_n f_{k_n}$  converges a.e. whenever  $f \in \operatorname{Lip}_{\alpha}(\mathbb{T})$  with  $\alpha > 1/2$ , and for any coefficient sequence such that  $\sum_n c_n^2 < \infty$ , which much improves upon Corollaries 2.3, 2.3<sup>\*</sup>, 2.5<sup>\*</sup>, 2.6 in [4].

# 7. Proof of Theorem 2.8

We produce it for  $k_n = n$ , the case of an arbitrary increasing sequence  $k_n$  being treated identically. By specializing (1.26) for  $K = [2^r, 2^{r+1}]$ , we get

(7.1) 
$$\left\|\sum_{2^{r} \leq k < 2^{r+1}} c_k f_k\right\|_2^2 \leq C_s \sum_{2^{r} \leq k < 2^{r+1}} \sigma_{1-2s}(k) c_k^2,$$

when  $1/2 < s \leq 1$ . Thus by assumption (2.12)

$$\sum_{r=1}^{\infty} \int_{0}^{1} r^{2} \Big| \sum_{j=2^{r+1}}^{2^{r+1}} c_{j} f_{j}(x) \Big|^{2} dx \leq C_{s} \sum_{r=1}^{\infty} r^{2} \sum_{k=2^{r+1}}^{2^{r+1}} \sigma_{1-2s}(k) c_{k}^{2}$$
$$\leq C_{s} \sum_{r=1}^{\infty} \sum_{j=2^{r+1}}^{2^{r+1}} \sigma_{1-2s}(k) c_{k}^{2} (\log k)^{2} < \infty.$$

Therefore

$$\sum_{r=1}^{\infty} r^2 \Big| \sum_{j=2^r+1}^{2^{r+1}} c_j f_j(x) \Big|^2 < \infty \qquad \text{a.e.}$$

And the Cauchy-Schwarz inequality yields for any  $1 \leq M < N$ 

$$\begin{aligned} \left|\sum_{j=2^{M}+1}^{2^{N}} c_{j} f_{j}(x)\right|^{2} &\leq \left(\sum_{k=M}^{N-1} \left|\sum_{j=2^{k}+1}^{2^{k+1}} c_{j} f_{j}(x)\right|\right)^{2} \\ &\leq \left(\sum_{k=M}^{N-1} \frac{1}{k^{2}}\right) \left(\sum_{k=M}^{N-1} k^{2} \left|\sum_{j=2^{k}+1}^{2^{k+1}} c_{j} f_{j}(x)\right|^{2}\right) \\ &\leq 2\sum_{k=M}^{\infty} k^{2} \left|\sum_{j=2^{k}+1}^{2^{k+1}} c_{j} f_{j}(x)\right|^{2} \to 0, \end{aligned}$$

as  $M \to \infty$ . This implies that  $\sum_{j=1}^{2^m} c_j f_j(x)$  converges a.e. as  $m \to \infty$ . Now by using again (1.26) and standard maximal inequalities (see e.g. [27], Lemma 8.3.4) we get

$$\begin{split} \sum_{k=1}^{\infty} \left\| \max_{2^{k}+1 \le i \le j \le 2^{k+1}} \left| \sum_{\ell=i}^{j} c_{\ell} f_{\ell} \right| \right\|^{2} & \le \quad C_{s} \sum_{k=1}^{\infty} k^{2} \left( \sum_{\ell=2^{k}+1}^{2^{k+1}} \sigma_{1-2s}(\ell) c_{\ell}^{2} \right) \\ & \le \quad C_{s} \sum_{\ell=1}^{\infty} \sigma_{1-2s}(\ell) c_{\ell}^{2} (\log \ell)^{2} < \infty, \end{split}$$

which implies

(7.2) 
$$\max_{2^{k}+1 \le i \le j \le 2^{k+1}} \left| \sum_{\ell=i}^{j} c_{\ell} f_{\ell}(x) \right| \to 0 \quad \text{a.e}$$

completing the proof of the theorem.

Acknowledgment. We wish to thank Pennti Haukkanen for useful references, notably the recent article of Julien Brémont, which we discovered while this work was much advanced.

### References

- [1] Aisleitner C. (2011) Convergence of  $\sum c_k f(kx)$  and the  $Lip_{\alpha}$  class, to appear.
- [2] Bellman R. Introduction to Matrix Analysis, Sd Ed., Classics in Appl. Math. 19 Siam, Philadelphia (1997).
- [3] Berkes I. On the convergence of  $\sum_{n} c_n f(nx)$  and the Lip 1/2 class, Trans. Amer. Math. Soc. **349** no10, 4143-4158.
- [4] Berkes I., Weber M. (2009) On the convergence of  $\sum c_k f(n_k x)$ , Memoirs of the A.M.S. **201** no. **943**, vi+72p.
- [5] Brémont J. (2011) Davenport series and almost sure convergence, Quart. J. Math. 62, 825–843.
- [6] Carleson, L. (1966) On convergence and growth of partial sums of Fourier series, Acta Math. 116, 135–157.
- [7] Gaposhkin V.F. (1968) On convergence and divergence systems, Mat. Zametki 4, 253-260.
- [8] Gaposhkin V.F. (1970). The central limit theorem for certain weakly dependent sequences. (Russian) Teor. Verojatnost. i Primenen. 15, 666–684.
- [9] Gronwall T.H. (1912) Some asymptotic expressions in the theory of numbers, Trans. Am. Math. Soc. 8, 118–122.
- [10] Haukkanen P., Wang J., Sillanpää J. (1997), On Smith's determinant, Linear algebra and its Appl. 258, 251–269.
- [11] Hedenmalm H., Lindqvist P., Seip K. (1997) A Hilbert space of Dirichlet series and systems of dilated functions in L<sup>2</sup>([0, 1]), Duke Math. J. 86, 1-37.
- [12] Hedenmalm H., Lindqvist P., Seip K. (1999) Addendum to "A Hilbert space of Dirichlet series and systems of dilated functions in L<sup>2</sup>([0, 1])", Duke Math. J. 99, 175-178.
- [13] S. Hong, R. Loewy, (2004) Asymptotic behavior of eigenvalues of greatest common divisor matrices, Glasgow Math. J. 46, 551–569.
- [14] Hunt R. (1968) On the convergence of Fourier series, Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, III 1967), 235–255, Southern Illinois Univ. Press, Carbondale.
- [15] Jaffard S. (2004) On Davenport expansions, Proc. of Symp. in Pure Math. 72.1, 273–303.
- [16] Kac, M. (1943) Convergence of certain gap series, Ann. of Math. 44, 411–415.
- [17] van Lint J.H., Richert H.-E. (1965) On primes in arithmetic progressions, Acta Arith. XI, 209– 216.
- [18] Lindqvist P., Seip K. (1998) Note on some greatest common divisor matrices, Acta Arith. LXXXIV 2, 149–154.
- [19] Olevskii A. M. (1975) Fourier series with respect to general orthogonal systems, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 86.
- [20] Paszkiewicz A. (2010) A complete characterization of coefficients of a.e. convergent orthogonal series and majorizing measures, Invent. Math. 180, 55–110.
- [21] Tandori K. (1957) Zur Divergenz der Orthogonal Reihe, Acta Sci. Szeged 18, 57–130.
- [22] Tandori K. (1965) Bemerkung zur Konvergenz der Orthogonal Reihen, Acta Sci. Szeged 26, 249–251.
- [23] Tenenbaum G. (1985) Sur la concentration moyenne des diviseurs, Comment. Math. Helv. 60, 411–428.
- [24] Tenenbaum G. (1990) Introduction à la théorie analytique et probabiliste des nombres, Revue de l'Institut Elie Cartan 13, Département de Mathématiques de l'Université de Nancy I.
- [25] Tóth L. (2010) A survey of the gcd-sum functions, J. Integers Sequences 13, Article 10.8.1.
- [26] Weber M. (2011) On systems of dilated functions, C. R. Acad. Sci. Paris, Sec. 1 **349**, 1261–1263.
- [27] Weber M. (2009) Dynamical Systems and Processes, European Mathematical Society Publishing House, IRMA Lectures in Mathematics and Theoretical Physics 14, xiii+759p.
- [28] Wintner A. (1944): Diophantine approximation and Hilbert space, Amer. Journal of Math. 66, 564-578.
- [29] Zygmund. A [2002]: Trigonometric series, Third Ed. Vol. 1&2 combined, Cambridge Math. Library, Cambridge Univ. Press.

MICHEL WEBER: IRMA, 10 RUE DU GÉNÉRAL ZIMMER, 67084 STRASBOURG CEDEX, FRANCE *E-mail address*: michel.weber@math.unistra.fr ; m.j.g.weber@gmail.com

ISTVAN BERKES, TECHNISCHE UNIVERSITAT GRAZ, INSTITUT FÜR STATISTIK, MÜNZGRABENSTRASSE 11, A-8010 GRAZ, AUSTRIA.

*E-mail address*: berkes@tugraz.at