REMARKS ON BETA DISTRIBUTED RANDOM NUMBERS

by

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1. Introduction. According to M. D. Jöhnk [1], algorithms for generating both beta and gamma distributed random numbers can be based on random numbers n_a from the probability distribution

$$F_a(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ x^a & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqq 1\,, \end{cases} \tag{1}$$

where a is real and positive.

Jöhnk's method for generating beta distributed random numbers in case of non-integral parameters has two serious difficulties, (i) the method is inefficient for larger values of the parameters, and (ii) the usual method for producing random numbers from the distribution (1), i.e. the transformation of uniform random numbers by root extracting is rather slow. In order to make the process more advantageous, G. Bánkövi [2] suggested an approximative method which seems to be satisfactory in many cases. In the first part of this paper I intend to show that Bánkövi's method can be improved so as to become exact not only in practical but also in strict theoretical sense, moreover, the second method suggested here may serve to speed up Bánkövi's procedure. Both methods affect random numbers from (1). In the second part some improvements are introduced to Jöhnk's original method, which increase efficiency and speed up the whole process.

2. A corrected variant of Bánkövi's method. Let us assume that the method described in [2] is sufficiently fast for producing random numbers n_β from $F_\beta(x) = x^\beta$, whereas our problem is to generate such ones but from (1). Suppose further that a differs from β only by a small amount, the relative difference being

$$0 < \varepsilon = \frac{\beta - a}{\beta} \ll 1\,. \tag{2}$$

Consider now the identity

$$x^a = (1 - \varepsilon)\, x^\beta + \varepsilon\, \frac{\beta x^a - a x^\beta}{\beta - a}\,, \tag{3}$$

which shows that x^a can be represented as a mixture of two probability distribution functions, one of which is $F_\beta(x)$, the other being

$$E(x) = \frac{\beta x^a - a x^\beta}{\beta - a}. \tag{4}$$

Thus, the usual random selection technique can be applied: choose either $F_\beta(x)$ or $E(x)$ with probabilities given by the weights in (3), and, if the result happens to be $F_\beta(x)$, take a random number n_β from the distribution $F_\beta(x)$, but take one from $E(x)$ in the opposite case. Having supposed $\varepsilon \ll 1$, the result will be $F_\beta(x)$ for almost all trials. Sometimes, however, the result will be $E(x)$. Now, it is easy to see that $E(x)$ is the probability distribution function of the *product* of two independent random variables having distributions $F_a(x)$ and $F_\beta(x)$ respectively, so that whenever the result of the trial is $E(x)$, we have to generate two independent random numbers n_a and n_β from the mentioned distributions and take their product. As to the number n_a, at first sight it seems that there is no other way for producing this than that to perform a root extraction procedure we wanted to avoid, but it is not a serious time-loss in the present case, since only the ε-th part of the total set of numbers must be generated by this tedious way.

3. A second improvement to BÁNKÖVI's **method.** As it was just mentioned, in the course of generating random numbers n_a by the presented method, with probability ε one has to produce a random number n_a from the distribution $F_a(x)$. For doing this the identity (3) and the same random selection principle can be applied again with the result that root extracting becomes necessary only with probability ε^2 in total, and iterating this process, we can get rid of it altogether. Moreover, the random selections, which were to be performed step by step, can be unified. Summarizing the ideas sketched here, the method may be presented as follows.

Since the identity

$$F_a(x) = x^a = \sum_{k=0}^{\infty} \frac{a}{\beta} \varepsilon^k \left[x^\beta \sum_{\nu=0}^{k} \frac{\beta^\nu}{\nu!} \left(\log \frac{1}{x}\right)^\nu \right] \qquad (\varepsilon = (\beta - a)/\beta) \tag{5}$$

with the probability distributions

$$F_{\beta k}(x) = x^\beta \sum_{\nu=0}^{k} \frac{\beta^\nu}{\nu!} \left(\log \frac{1}{x}\right)^\nu \qquad (0 < x \leqq 1) \tag{6}$$

expresses x^a as a mixture, and since $F_{\beta k}(x)$ $(k = 0, 1, \ldots)$ is the distribution function of the product of k independent random variables from the distribution $F_\beta(x) = x^\beta$, let us choose a function $F_{\beta\varkappa}(a)$ at random with probability $\frac{a}{\beta}\varepsilon^\varkappa$, and, if the result happens to be $\varkappa = k$, then produce $\varkappa$ random numbers n_β independently of each other and accept their product.

Suppose now that the algorithm for producing a single number n_β requires N uniform random numbers using BÁNKÖVI's method. The efficiency E_{ff}, measured by the reciprocal of this number is then $1/N$, — if the numbers

n_β are accepted as satisfactory approximations of the n_α numbers. The efficiency of the presented method, measured similarly, can be calculated from (5) with the result

$$E_{ff} = \left[\left(\frac{\alpha}{\beta} N + \frac{\alpha}{\beta} \varepsilon \cdot 2N + \frac{\alpha}{\beta} \varepsilon^2 \cdot 3N + \ldots\right) + 1\right]^{-1} = \left(\frac{\beta}{\alpha} N + 1\right)^{-1}. \tag{7}$$

Supposing ε to be small, the loss of efficiency does not seem to be of any importance.

If T denotes operating time needed for producing one number by method of BÁNKÖVI, a similar calculation shows that the operating time required by our algorithm will approximately be

$$T_1 = \frac{\beta}{\alpha} (T + \varepsilon M), \tag{8}$$

where M is the time for a multiplication, although the fact that the general machine program will be longer, and that the selection procedure needs an extra and not at all negligible amount of time, is disregarded here.

The main advantage of the presented method, however, lies not in its theoretical correctness, but rather in the speeding up of the generating procedure. Let us begin with an example, which represents a somewhat extreme case. Let us take $\alpha = 0{,}9 + \mu$, where μ is practically negligible small, to be concrete, put $|\mu| < 0{,}01$. When applying BÁNKÖVI's method, the first task we have is to find integer numbers a_k for which

$$0{,}9 = \sum_{k=1}^{N} \frac{1}{a_k}$$

with an error less than 0,01. If the number 0,9 is given in binary representation, then we obtain

$$0{,}9 \approx \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{64}.$$

The efficiency will be 0,25 and the number S_M of the required multiplications for each number n_β amounts to 12. One may try another representations, e.g.

$$0{,}9 = \frac{1}{2} + \frac{1}{5} + \frac{1}{5} \qquad (E_{ff} = 1/3\,,\ S_M = 7)\,,$$

or

$$0{,}9 \approx \frac{1}{2} + \frac{1}{3} + \frac{1}{16} \qquad (E_{ff} = 1/3\,,\ S_M = 7)\,,$$

but these, though better, are too tricky for a machine to find them out. Despite of this, let us accept $E_{ff} = 1/3$, $S_M = 7$ as best characteristics.

When applying our method, put $\beta = 1$, then ε becomes 0,1 approximately and for E_{ff} and S_M we have 0,47 and 0,11 respectively.

The results of further examples, some of which is taken from BÁNKÖVI's paper, are summarized in Table 1 below. Representations, which are better approximations of the numbers α in Table 1 are not treated there, since both E_{ff} and S_M would be even much worser for BÁNKÖVI's method.

Table 1

Bánkövi's method			Improved method			
a	E_{ff}	s_M	β	ε	E_{ff}	s_M
$\pi/4 \sim 2^{-1} + 2^{-2} + 2^{-5}$	0,33	8	1	0,215	0,44	0,27
$1-\pi/4 \sim 5^{-1} + 68^{-1}$	0,50	10	2^{-2}	0,142	0,46	2,5
$e^{-1} \sim 2^{-2} + 2^{-3}$	0,50	5	2^{-1}	0,264	0,42	1,7
$1-e^{-1} \sim 2^{-1} + 2^{-3}$	0,50	4	1	0,368	0,39	0,58
$10{,}2 \sim 10 + 2^{-3} + 2^{-4}$	0,083	7	11	0,073	0,078	0,079
$10{,}2 \sim 10 + 5^{-1}$	0,091	4	$10 + 2^{-2}$	0,005	0,083	2,0
			3	0,300	0,19	0,43
$2{,}1 \sim 2 + 2^{-4} + 2^{-5}$	0,25	9	$2 + 2^{-3}$	0,012	0,25	3,05
			2	0,450	0,22	0,82
			$1 + 2^{-1}$	0,267	0,27	1,7
$1{,}1 \sim 1 + 5^{-1} + 5^{-1}$	0,33	6	$1 + 2^{-2}$	0,120	0,31	2,4
			$1 + 2^{-3}$	0,022	0,33	3,1
$0{,}33 \sim 2^{-2} + 2^{-4}$	0,5	6	2^{-1}	0,340	0,40	2,03
$0{,}33 \sim 3^{-1}$	1	2	3^{-1}	0,010	0,50	2,03
$0{,}1 \sim 2^{-4} + 2^{-5}$	0,5	9	2^{-3}	0,200	0,44	4,0
$0{,}03 \sim 2^{-5}$	1	5	2^{-5}	0,040	0,49	5,3

Table 1 shows (in so far as any general conclusions may be drawn from such a small collection of examples) that for $a < 1$ the simplest and in many cases the best strategy is to put β equal to the nearest integer power of 2^{-1} exceeding a. Accepting this as a general principle, we obtain a simplified form of the method having nothing common with that of Bánkövi, because its basic idea of using ordered samples is left out. As an advantage, there is no need for a comparison algorithm. For $a > 1$, however, putting β equal to the nearest integer exceeding a, we obtain another reduced algorithm, since now the ordered sample will consist of simple uniform random numbers.

Returning to the random selection procedure, it was tacitly assumed that this can be performed by using a single uniform random number, involving that the probabilities in question are previously computed and stored for each a. Instead of storing probabilities, one may apply the following simple algorithm: Take uniform random numbers one after another until one happens to be smaller than a/β. If this is the k-th one, then take the product of k independent random numbers η_β from the distribution x^β.

4. Two variants of Jöhnk's method for generating beta distributed random numbers. The efficiency of Jöhnk's method strongly decreases with increasing parameter values. Let the density function be

$$f_{pq}(x) = C_{pq} x^{p-1}(1-x)^{q-1}\,,$$

where

$$C_{pq} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)},$$

then the efficiency of the quoted method, being equal to the probability of acceptance, turns out to be[1]

$$\frac{1}{2}\int_0^1 qx^{p-1}(1-x)^{q-1}\,dx = \frac{q}{2C_{pq}} \tag{9}$$

(for non-integral values of the parameters p and q). Not only the inefficiency in itself makes the method tedious for larger values of p and q, but also that the random numbers needed are not simple uniform random ones; these must be taken from a distribution of type (1). Though root extraction can be avoided as we have seen above, the procedure seems to be too lengthy.

Let us write the density function as

$$f_{p+\alpha,\,q+\beta}(x) = C_{p+\alpha,\,q+\beta} \cdot x^{p+\alpha-1}(1-x)^{q+\beta-1},$$

where now both α and β are non-negative real numbers smaller than 1, and both p and q are positive integers. Let b_{pq} be a random number from a beta distribution with parameters p, q, and let s be a uniform random number generated independently of b_{pq}.

Accept b_{pq}, if

$$s < \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \cdot \beta^\beta}\, b_{pq}^\alpha (1-b_{pq})^\beta, \tag{10}$$

and reject it in the opposite case. It is easy to show that the b_{pq}'s will be beta distributed but with parameters $p+\alpha$, $q+\beta$ if selected by this rejection condition (10). The total efficiency is given by

$$E_{ff} = \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \cdot \beta^\beta} \cdot \frac{C_{pq}}{C_{p+\alpha,\,q+\beta}} \cdot \frac{1}{p+q},$$

since the numbers b_{pq} can be generated by ordered sets of $p+q-1$ uniform random numbers, as described in [1].

Condition (10) is inconvenient in general, because it involves root extraction. However, for special values of the parameters the algorithm may work fairly well. Let us put e.g. $\alpha = \beta = {}^1/_2$, then the condition (10) takes the form

$$s^2 < 4\, b_{pq}(1-b_{pq})$$

and efficiency will approximately be

$$E_{ff} \approx 2\frac{\sqrt{pq}}{(p+q)^2} \qquad (p \gg 1,\ q \gg 1)$$

calculated by Stirling's formula.

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[1] The factor $1/2$ takes account of the fact that one needs always a *pair* of random numbers for each trial.

Root extraction can be avoided altogether when using a rejection technique with double acceptance condition. Let us take two uniform random numbers s_1 and s_2, and let b_{pq} generated from the beta distribution as above. For $\alpha > \beta$ accept b_{pq} if

$$s_1^{1/\beta} < 4\, b_{pq}(1 - b_{pq}) \cap s_2^{1/(\alpha-\beta)} < b_{pq} \,,$$

and in the opposite case $\alpha < \beta$ accept b_{pq} if

$$s_1^{1/\alpha} < 4\, b_{pq}(1 - b_{pq}) \cap s_2^{1/(\beta-\alpha)} < 1 - b_{pq} \,.$$

As for $\alpha = \beta$, the second condition involving s_2 should simply be omitted. Since both $s_1^{1/\beta}$ and $s_2^{1/(\alpha-\beta)}$ are random numbers from distributions of type (1), root extraction will be not necessary when the former described method is applied.

The efficiency of this second variant is smaller than that of the first one, and bot variants become practically useless for pairs of values p,q with $p/q \ll 1$ or $p/q \gg 1$.

5. Another form of improvement. It is well known ([3], p. 153, Theorem 5) that the random variable

$$\xi_{pq} = \frac{\eta_p}{\eta_p + \eta_q} \tag{11}$$

is beta distributed if η_p, η_q are independent gamma variates. In possession of a fast algorithm for generating gamma distributed random numbers the relation (11) offers a possibility to produce beta distributed ones. As to the random numbers from a gamma distribution with non-integral parameter value, the method of M. SIBUYA [4] completed by that of I. TAKAHASHI [5] may be used. TAKAHASHI's rejection method however, though very efficient, has the disadvantage that it needs logarithms when the acceptance condition will be tested. Our proposition is therefore to use JÖHNK's second method instead of that given by TAKAHASHI.

Thus, the suggested algorithm consists of four steps as follows.

1. Produce two independent random numbers g_p and g_q from the gamma distributions

$$\frac{1}{\Gamma(p)} \int_0^x t^{p-1} e^{-t}\, dt \quad \text{and} \quad \frac{1}{\Gamma(q)} \int_0^x t^{q-1} e^{-t}\, dt$$

respectively, generated e.g. by SIBUYA's method. Alternatively, BÁNKÖVI's method [6] for generating exponential random numbers may also be applied.

2. Generate two independent exponential random numbers e_1, e_2.

3. Generate two independent beta distributed random numbers $b_{\alpha,1-\alpha}$ and $b_{\beta,(1-\beta)}$ by JÖHNK's second method.

4. Compute

$$\frac{g_p + e_1 b_{\alpha,1-\alpha}}{g_p + e_1 \cdot b_{\alpha,1-\alpha} + g_q + e_2 \cdot b_{\beta,1-\beta}}$$

which gives the required number $b_{p+\alpha,\, q+\beta}$.

All this may appear to be rather complicated, but, on the other side, JÖHNK's original procedure needs over 130 (x^a-distributed) random numbers for a single beta distributed one, when the parameters $p + a$, $q + \beta$ are as small as 4,5. Our method requires in average

$$\frac{1}{E_{ff}} = p + q + 2 + \frac{2}{\Gamma(a+1)\,\Gamma(2-a)} + \frac{2}{\Gamma(\beta+1)\,\Gamma(2-\beta)} \leqq \leqq p + q + 2 + \frac{16}{\pi} \tag{12}$$

uniform random numbers, if root extractions and logarithmic transformations are admitted, and it demands only slightly more when improved techniques are used. For $p = q = 4$, $a = \beta = 0{,}5$ equation (12) gives $1/E_{ff} = 15{,}1$.

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ЗАМЕЧАНИЯ К ПРОБЛЕМЕ ПОЛУЧЕНИЯ СЛУЧАЙНЫХ ЧИСЕЛ С БЕТА — РАСПРЕДЕЛЕНИЕМ

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Резюме

Настоящая работа связана с статьями M. D. JÖHNK [1] и G. BÁNKÖVI [2]. BÁNKÖVI указал приблизительный способ для получения случайных чисел с законом распределения x^a, имеющий преимущество, что в его алгоритме нет извлечения корни. В п. 2 настоящей работы показывается, что дополнив способ BÁNKÖVI некоторым алгоритмом, он становится теоретически точным; а в п. 3 предлагается более общий, опирающийся на метод BÁNKÖVI способ, который оказывается точным и во многих случаях более скорым.

В своей вышеупомянутой работе JÖHNK указывает два способа для получения случайных чисел с бета — распределением, но эти методы оказываются малоэффективными, если параметры распределения являются большими и не целыми числами. В пп. 4 и 5 настоящей работы предлагаются разные, опирающиеся на способ JÖHNK методы, но в упомянутых случаях они являются гораздо более эффективными оригинального алгоритма.