DECOMPOSITIONS OF COMPLETE GRAPHS INTO FORESTS

by

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The arboricity $a(G)$ of a graph G is the minimum number of forests whose union is G. In a recent paper [2], NASH—WILLIAMS determined the arboricity of all graphs. In this note we provide explicit constructions for the fewest forests needed for two classes of graphs. These are the *complete graphs* K_p, having p points with every pair adjacent, and the *complete bipartite graphs* $K_{m,n}$, having m light points and n dark points with every light point adjacent to every dark one.

Theorem 1. *The arboricity of the complete graph* K_p *is*

$$a(K_p) = \left[\frac{p+1}{2}\right].$$

Proof. Since K_p has p points and $\frac{1}{2}\,p(p-1)$ lines, $a(K_p) \geqq \frac{p}{2}$; that is, $a(K_p) \geqq \left[\frac{p+1}{2}\right]$.

In providing constructions for the reverse inequality, we first let p be even. Take p points as the vertices of a regular polygon, and label them clockwise $1, 2, \ldots, p-1, 0$. Form T_1 as the path whose consecutive points are $1, 0, 2, p-1, 3, \ldots, \frac{1}{2}\,p, \frac{1}{2}\,p+1$. This is illustrated in Figure 1 for the case $p = 8$. For $i = 2, 3, \ldots, \frac{1}{2}\,p$, form the path T_i from the path T_{i-1} by leaving the points fixed and rotating the lines one position clockwise. Again, see Figure 1. The path T_i can be defined more explicitly as follows: If t_j denotes the j'th point of T_1, then the j'th point of T_i is $t_j + i \pmod{p}$. It is quite clear that every line of K_p appears in exactly one of the $\frac{p}{2}$ paths formed in this way. Hence, $a(K_p) = \frac{p}{2}$.

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To form $\frac{p+1}{2}$ forests whose union is K_p when p is odd, first form the paths as described above for K_{p-1}. From these, construct $\frac{p-1}{2}$ forests by adding an isolated p'th point; then construct another graph by placing each of the original $p-1$ points adjacent to the p'th point, forming a tree. See Figure 2 for $p = 9$. This completes the constructions.

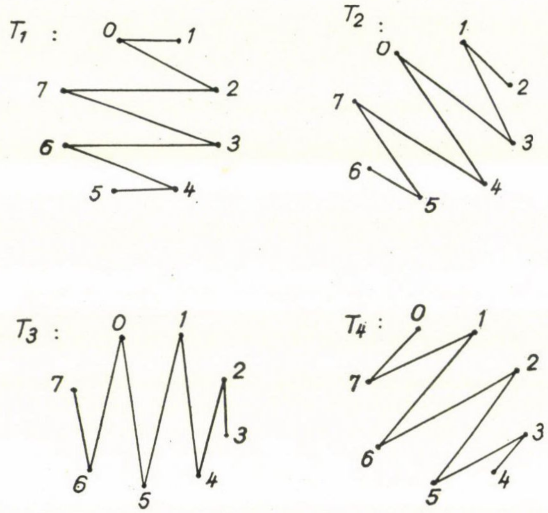

Figure 1.

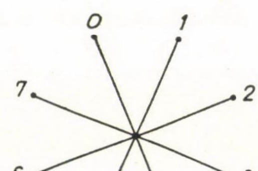

Figure 2.

As a corollary to this proof, we note that K_p is the union of $\frac{p}{2}$ line-disjoint paths when p is even, and that K_p is the union of $\frac{p-1}{2}$ line-disjoint cycles when p is odd. This last assertion follows from the proof by adding a p'th point adjacent to the end points of the paths formed for K_{p-1}.

Our second theorem gives the arboricity of complete bipartite graphs, and the devices used in the proof are similar to those in [1]. We find it helpful to begin by developing some preliminary results.

Lemma 1. *Let m and k be fixed positive integers with* $1 \leqq \frac{m}{2} < k < m$. *Let* $r = \left[\frac{k(m-1)}{m-k}\right]$ *and* $f(x) = \frac{mx}{m+x-1}$. *Then r is the greatest integral value of x for which* $\{f(x)\} = k$.

Proof. Since $m \geqq 2$ by hypothesis, $f(x)$ is a strictly increasing function of the positive real variable x. If $f(x) = k$, then $x = \frac{k(m-1)}{m-k}$. Hence, $\{f(r)\} \leqq k$ and $\{f(r+1)\} \geqq k+1$. Since $f(r+1) - f(r) = \frac{m(m-1)}{(m+r)(m+r-1)} <$ < 1, it follows that $\{f(r+1)\} - \{f(r)\} \leqq 1$, so $\{f(r)\} = k$ and $\{f(r+1)\} =$ $= k+1$. The lemma now follows immediately from the fact that $\{f(x)\}$ is a nondecreasing function of x.

Let m, k, and r be as in the lemma. We define an $m \times r$ array A whose cells contain finite sequences of positive integers in the following way. Let

$$c(i,j) = \left\{(i+j)\left(\frac{r}{k}\right)\right\} - \left\{(i+j-1)\left(\frac{r}{k}\right)\right\}$$

be the length of the sequence in the (i, j) cell of A. Let the entries in the first row be consecutive positive integers; that is, the entries in the (1, 1) cell are $1, 2, \ldots, c(1,1)$, in the (1, 2) cell are $c(1,1)+1, \ldots, c(1,1)+c(1,2)$; and so on. Now define the entries in the j'th column inductively: Assuming the entries in the $(i-1, j)$ cell are given, let the (i, j) cell contain $c(i, j)$ consecutive integers beginning with the last entry in the $(i-1, j)$ cell. Now reduce all entries modulo r. We illustrate with $m = 6$, $k = 4$, $r = 10$:

$$\begin{pmatrix} 123 & 45 & 678 & 90 \\ 34 & 567 & 89 & 012 \\ 456 & 78 & 901 & 23 \\ 67 & 890 & 12 & 345 \\ 789 & 01 & 234 & 56 \\ 90 & 123 & 45 & 678 \end{pmatrix}.$$

Lemma 2. *The array A has the following two properties:*

(i) *The entries in each row are r consecutive integers modulo r.*

(ii) *In each column, if the first entry of all cells except the first is excluded, the remaining entries are consecutive integers modulo r and there are at most r of them.*

Proof. Since the terms being summed telescope, for each i,

$$\sum_{j=1}^{k} c(i,j) = \left\{(i+k)\left(\frac{r}{k}\right)\right\} - \left\{i\left(\frac{r}{k}\right)\right\} = r.$$

Hence each row contains r integers. That these are consecutive integers follows from the obvious fact that

$$c(i, j) = c(i-1, j+1), \text{ for } i = 2, 3, \ldots, m \text{ and } j = 1, 2, \ldots, k-1,$$

and from our choice of the first entry in each cell. This proves that A has property (i).

The total number of entries in the j'th column is, using the telescoping property of the terms,

$$\sum_{i=1}^{m} c(i, j) = \left\{(m+j)\frac{r}{k}\right\} - \left\{j\frac{r}{k}\right\} \leq \left\{\frac{mr}{k}\right\} \leq m + r - 1\,,$$

since $k = \left\{\dfrac{mr}{m+r-1}\right\}$ by Lemma 1. Subtracting the m entries, corresponding to those first integers in each cell appearing in the preceding cell, we have no more than $r - 1$ entries remaining in column j. That these are consecutive residue classes is immediate. Hence, A also has property (ii).

Theorem. *The arboricity of the complete m by n bipartite graph $K_{m,n}$ is* $a(K_{m,n}) = \left\{\dfrac{mn}{m+n-1}\right\}$.

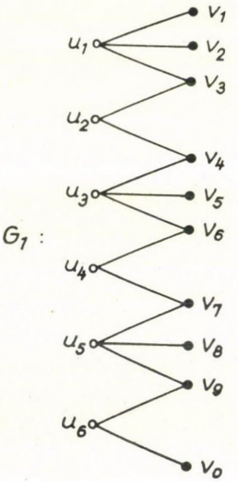

Figure 3.

Let m and n be given. If $m = 1$, then the graph is already a forest. If $n > (m-1)^2$, then $a(K_{m,n}) \geq m$, by Lemma 1. That $a(K_{m,n}) = m$ in this case follows from m copies of the graph $K_{1,m}$. Hence we assume $2 \leq m \leq$ $\leq n \leq (m-1)^2$. Set $k = \left\{\dfrac{mn}{m+n-1}\right\}$. Then $\dfrac{m}{2} < k < m$. Define $r =$ $= \left[\dfrac{k(m-1)}{m-k}\right]$ as in Lemma 1. We will use the array A to show that $a(m, r) \leq k$, from which it will follow that $a(m, n) = k$, since $a(m, n) \geq k$.

Define k graphs $G_1, G_2, \ldots, G_k$ using the k columns of the array. Each graph G_j has m light points $u_1, u_2, \ldots, u_m$ and r dark points $v_1, v_2, \ldots, v_{r-1}, v_0$. In G_j, let u_i be adjacent to v_h if and only if the integer h is in the (i, j) cell of A. That G_j is acyclic follows immediately from property (ii) since no cycle can occur. That the union of the graphs G_j is $K_{m,r}$ follows from (i), because it implies that each u_i, $(i = 1, 2, \ldots, m)$ is adjacent to each v_h, $(h = 0, 1, \ldots, r-1)$ since in the i'th row h appears in some column j. Therefore $a(K_{m,r})$, and hence $a(K_{m,n})$, is at most k. But since a tree contained in $K_{m,n}$ has $m + n - 1$ lines and $K_{m,n}$ has mn lines, $a(K_{m,n}) \geqq \left\{\frac{mn}{m+n-1}\right\} = k$. This proves the theorem.

We illustrate G_1 for the array given above in Figure 3.

In the table below we have listed, for small m and k, the value r. That is, given m and k, r is such that $K_{m,r}$ is the largest complete bipartite graph with arboricity k.

k \ m	3	4	5	6	7	8	9	10	11	12
2	4	3	2	2	2	2	2	2	2	2
3		9	6	5	5	4	4	3	3	3
4			16	10	8	7	6	6	5	5
5				25	15	12	10	9	8	7
6					36	21	15	13	12	11
7						49	28	21	17	15
8							64	36	26	22

The definition and problems involved in this note were proposed by Professor A. Rényi in a seminar conducted by Professor F. Harary, who conjectured the results. I wish to also thank Professor R. Read for this version of the proof of Theorem 1.

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РАЗЛОЖЕНИЕ ПОЛНЫХ ГРАФОВ НА ЛЕСА

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Резюме

Лесом мы называем соединение деревьев без общих точек. Автор дает метод эффективного *конструирования* как для представления полных графов так и для представления полных графов с счетным числом обходов в виде соединения минимального числа лесов. *Существование* разложения на минимальное число лесов было в первые доказано NASH-ом и WILLIAMS-ом.