

ON THE FUNCTION $g(t) = \limsup_{x \rightarrow +\infty} (f(x+t) - f(x))$

by

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The following problem has occurred in the theory of regularly increasing functions:¹

If for a real-valued measurable function $f(x)$ the quantity

$$(1) \quad g(t) = \limsup_{x \rightarrow +\infty} (f(x+t) - f(x))$$

is finite for every possible t , does this imply that the function $g(t)$ is bounded in every finite interval?

The aim of the present note is to show that

a) if (1) is finite for every real number t , then $g(t)$ is bounded in every finite interval;

b) if (1) is finite for every positive number t , then $g(t)$ is bounded in every closed subinterval of the open half-line $(0, +\infty)$ but not necessarily bounded in the neighbourhood of 0.

Case a) Suppose that $g(t)$ is not bounded in some finite interval I ; then there exists a sequence of real numbers $t_n \in I$ such that $g(t_n) > n$ ($n = 1, 2, \dots$). Then, by (1), one could find a sequence $x_n \rightarrow +\infty$ such that

$$(2) \quad f(x_n + t_n) - f(x_n) > n \quad (n = 1, 2, \dots).$$

Take now an arbitrary finite interval J and consider the sets

$$(3) \quad Y_{1,n} = \{y : f(x_n + y) - f(x_n) > \frac{n}{2}; \quad y \in J\},$$

$$Y_{2,n} = \{y : f(x_n + t_n) - f(x_n + y) > \frac{n}{2}; \quad y \in J\}.$$

These sets are measurable for each n , and since by (2) $Y_{1,n} \cup Y_{2,n} = J$, we have either $\mu(Y_{1,n}) \geq \frac{1}{2} \mu(J)$ or $\mu(Y_{2,n}) \geq \frac{1}{2} \mu(J)$ or both (μ denotes the Lebesgue measure).

¹The problem was told to P. ERDŐS by R. BOJANIC and J. KARAMATA. In a paper of W. MATUSZEWSKA (Regularly increasing functions in connection with the theory of L^{*p} -spaces, *Studia Math.* **21** (1962) 317–344) a proof of statement a) is given; in it, however, there is a gap: $g(t)$ is implicitly assumed to be measurable, although the measurability of $f(t)$ does not imply the measurability of $g(t)$.

Put now

$$(4) \quad Z_n = \{z : f(x_n + t_n) - f(x_n + t_n - z) > \frac{n}{2}; t_n - z \in J\}.$$

Then obviously $\mu(Z_n) = \mu(Y_{2,n})$ and thus we have either

$$(5) \quad \mu(Y_{1,n}) \geq \frac{1}{2} \mu(J) \quad \text{infinitely often}$$

or

$$(6) \quad \mu(Z_n) \geq \frac{1}{2} \mu(J) \quad \text{infinitely often}$$

(or both), where all the $Y_{1,n}$'s and Z_n 's are subsets of a fixed finite interval. This clearly implies the existence of a real number y_0 or z_0 contained in infinitely many $Y_{1,n}$ or Z_n , respectively.² But then — by the definitions (3), (4) and (1) — we would have $g(y_0) = +\infty$ or $g(z_0) = +\infty$, respectively, contradicting to the assumed finiteness of $g(t)$.

Case b) The first statement follows in the same way as in case a). We only have to place the interval J between the point 0 and the interval I ; then, if the statement were false, one would obtain a *positive* number y_0 or z_0 , with $g(y_0) = +\infty$ or $g(z_0) = +\infty$. Now we show by a counterexample, that $g(t)$ need not be bounded in the neighbourhood of 0. Let us define the function $f(x)$ in the following way:

$$\begin{aligned} f(0) &= -2 \\ f(2n-1) &= f(2n) = -2^{n+1} \quad (n = 1, 2, \dots). \end{aligned}$$

Further put for $n = 1, 2, \dots$

$$f\left(2n-1 + \frac{1}{2^k}\right) = f(2n-1) + 2^k = -2^{n+1} + 2^k \quad (k = n, n-1, \dots, 1)$$

and

$$f\left(2n-1 - \frac{1}{2^n}\right) = f\left(2n-1 + \frac{1}{2^n}\right) = -2^n.$$

For any other nonnegative value x define $f(x)$ by linear interpolation (see Figure 1). Then it is easy to see that

$$g\left(\frac{1}{2^n}\right) = \limsup_{x \rightarrow +\infty} \left(f\left(x + \frac{1}{2^n}\right) - f(x) \right) = 2^n$$

and for $t \geq \frac{1}{2^n}$ we have $g(t) \leq 2^n$. Thus $g(t)$ is finite for every positive number t but obviously

$$\lim_{t \rightarrow +0} g(t) = +\infty.$$

² If A_n ($n = 1, 2, \dots$) are arbitrary measurable sets with $\mu(A_n) \geq a$ and $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < +\infty$, then $\mu(\limsup A_n) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{n=k}^{\infty} A_n\right) \geq a$.

Remarks.

1. The assumption that $g(t)$ is finite for every *real* t is obviously equivalent with the assumption that

$$h(t) = \limsup_{x \rightarrow +\infty} |f(x+t) - f(x)|$$

is finite for every *positive* t . Thus also the latter condition implies the boundedness of $g(t)$ — and of $h(t)$ — in every finite interval. In the counterexample given above $h(t) = +\infty$ for every t .

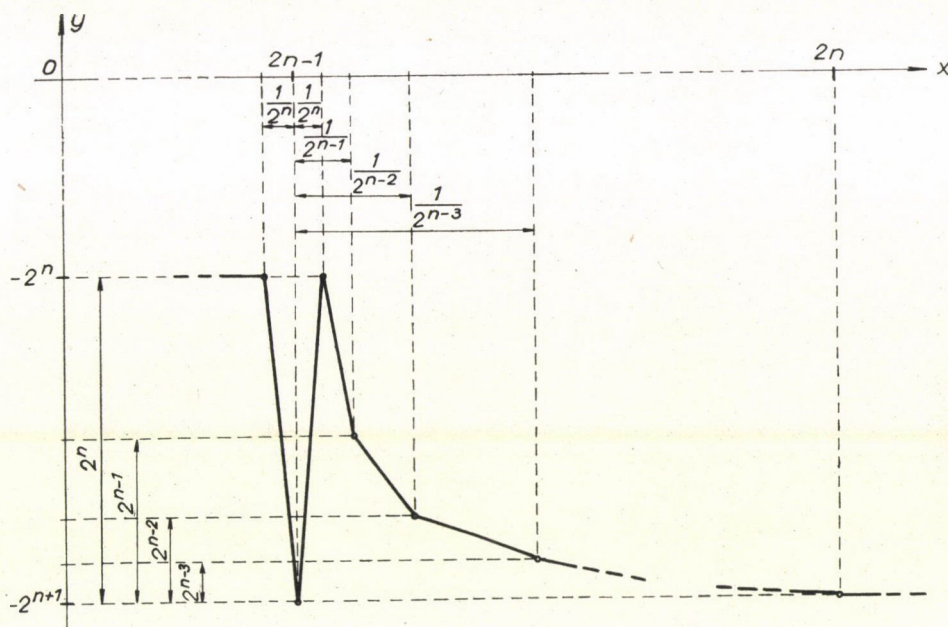


Fig.1

2. If we do not assume the measurability of $f(x)$, then $g(t)$ can be finite for every real t without being bounded in any interval. That is the case e.g. for any non-measurable solution of the Cauchy functional equation $f(x+y) = f(x) + f(y)$.

(Received October 26, 1964.)

О ФУНКЦИИ
$$g(t) = \limsup_{x \rightarrow +\infty} (f(x+t) - f(x))$$

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Резюме

Пусть $f(x)$ — измеримая вещественная функция и положим

$$g(t) = \limsup_{x \rightarrow +\infty} (f(x+t) - f(x)).$$

Доказываются следующие установления:

- а) если функция $g(t)$ конечна при любом вещественном t , то она является ограниченной в любом конечном интервале;
- б) если функция $g(t)$ конечна при любом положительном t , то она является ограниченной в любом закрытом подинтервале открытого интервала $(0, +\infty)$; но не должна быть ограниченной в окрестности 0.