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# MATH-TEUR, A MATHEMATICAL CARD GAME FOR (NOT ONLY) FUN

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**Abstract**
*We present the game Math-teur, a mathematical card game used successfully at the John von Neumann University for teaching remedial mathematics as part of a complex university catch-up program. In the paper we describe the game itself and summarize the considerations used for the game design.*

## 1 Introduction

In recent years, the use of games in education has become widespread also in Hungary, ranging from simple logical puzzles to sophisticated computer games. There have been several articles published on international experiences [1]–[10]. In higher education, there are relatively few examples of this practice, especially on board and card games, e.g. [1], [9]. We tried to integrate board games into classes designed to repeat prerequisite knowledge in high school mathematics. Thus, we introduced the Games and Mathematics course at the John von Neumann University in Spring 2015, which has been very popular among students since then. The course was implemented as part of a complex university catch-up program. The aim of the program, and the corresponding course, is to improve students' mathematical competences through cooperative methods. We mainly used games to achieve this goal. During the sessions, we primarily adapted commercially available board and card games and filled them with mathematical content.

One of the most suitable games for the course was the card game Sabotuer [11]. In this paper, we present a mathematical version of this card game. During the game, players build a maze between a start and a goal card using appropriate path cards. We have changed the original path cards to mathematical path cards, i.e. cards containing a number, or a mathematical expression. The goal of the mathematical version is to build a monotonously increasing sequence of path cards from the start to the goal card. The mathematical version of the card game Saboteur is called Math-teur.

We describe the game itself in Section 2. In Section 3, we explain the main issues of the game design. We present the actual card sets in Section 4. Finally, a short discussion section closes the paper.

## 2 How to play Math-teur

### 2.1 Math-teur

Game components: 44 road cards, 27 action cards, 28 gold cards and role cards: 7 gold digger and 4 saboteur cards.

The number of players can be between 3-10, the recommended age depends on the mathematical content.

The playing time is about 30 minutes.

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## 2.2 Object of the Game

The game consists of three rounds, during which players collect gold cards. At the beginning of each round, players are randomly assigned the role of a gold digger or a saboteur. The role of the player must be kept secret, players can only guess other players' role. At the same time, players in the same team have to cooperate, helping each other to achieve a common goal. In the case of gold miners, this goal is to build the path from the start to the gold goal card. In the case of saboteurs, the goal is to prevent building this path. Members of the team that achieved their goal will receive gold cards as a reward. The winner of the game is the one who collects the most gold in the three rounds.


*Figure 1. Game start setup*

## 2.3 Setup of a round

First, roles are assigned. Players randomly draw their role card from the role card deck containing one more card than the number of players. According to the number of players, 1 saboteur remains in the deck for 3 and 4 players, 2 saboteurs for 5 and 6 players, 3 saboteur role cards for 7, 8, and 9 players. For 10 players, all role cards are drawn.

Players look at their role, flip the role card in front of them, but do not tell each other what role they play, as roles may only be revealed at the end of the round. The one undrawn role card is set aside by players without viewing.

Then, the start and goal cards are removed from the path cards. The start card (showing a ladder) is placed the table face up. The three goal cards (one showing the gold and the other two showing stones) are shuffled and placed on the table face down. A distance of seven cards are kept between the start and goal cards, and a distance of one card between the goal cards, as shown in Figure 1. During the game a maze is built from the start card to the goal cards using the path cards.

A volunteer mixes the remaining 40 road cards with the 27 action cards and, depending on the number of players, deals 6 cards to each player if 3, 4 or 5 players play; 5 cards if 6 or 7 players play; and 4 cards if 8, 9 or 10 players are playing. The gold cards are also mixed and set aside until the end of the round.

## 2.4 How to play a Round

The youngest player starts the first round. In the remaining rounds, the player to the left of the one who finished the previous round will start.

**On his turn a player chooses from the following actions:**

1. Add a path card to the existing maze,
2. Place an action card, i.e. put the action card face up in front of any players,

3. or pass, i.e. discard a card into the deck.

At the end of their turn, each player draws a new card from the deck and takes it in their hands. When the draw deck runs out of cards, players can only play the cards in their hands.

1. Playing a path card

A path card containing a mathematical expression shall be added to the maze such that it shall fit with at least one path card in the existing maze. The path card cannot be rotated.
The goal of gold diggers is to create a passage that leads uninterruptedly from the start card to one of the target cards, in such a way that the value of mathematical expressions must be monotonously increasing. By monotone increase we do not mean strict monotonicity, that is, we allow cards of equal value in the sequence. Saboteurs try to prevent this path by interrupting the monotone sequence of numbers or expressions.
While adding a path card to the maze, players must mark the ordering with the help of a matchstick so that the head of the match points towards the greater (or equal) number or expression on the cards.

*Figure 2. Action cards*

2. Playing an action card.

When playing a "Map" card, the player can view one of the three goal cards. The player can't show or tell others what he saw on it. The played "map" card is added to the discard pile.
When playing the "Rockfall" card, the player can remove any path card from the table, except for the start and goal cards. Both the path card and the played "Rockfall" card are added to the discard pile. The gap created by the "Rockfall" action card can be filled later with any path card.
If any of a "Broken tools" card is in front of a player, he may not build the maze, i.e. is not allowed to play any path cards. "Broken tools" cards appear in three different forms by showing a broken pickaxe, a broken mine cart or a broken lantern. More than one different „Broken tools” card can be in front of a player. A player who has at least one broken object in front of him may not use path cards, but he can use other action cards.
A "Fix tools" card can be used to remove a matching "Broken tools" card from the game. The upcoming player can use a "Fix tools" card to take a "Broken tools" card from himself or from a fellow player. Both cards are then placed in the discard pile. "Fix tools" cards show one or two types of a fixed tool. Cards showing two fixed tools can be used to fix one matching tool only, but players are free to choose which one. The action cards are shown in Figure 2 and 3.

*Figure 3. Action cards*

3. Pass

A player who cannot or does not want to play a card, passes. The player places a card face down from the cards in his hand in the deck.

### 2.5 End of a Round

If a player reaches a goal card with his path card and the completed passage leads uninterruptedly from the starting to the goal card, the player can flip that goal card.
Then two cases are possible:

1. If the target card shows a gold nugget, the round is over, and the gold diggers won the round.
2. If the target card shows a rock, the round continues. The face up goal card shall be positioned in such a way that it fit to the last path card.

The round ends even if no player has any cards left to play. In this case, players flip their role cards and reveal their roles.

### 2.6 Handing out the gold

Gold diggers won, if a monotonously increasing series led from the start to the goal card showing the gold. In this case, the player who played the last path card takes as many gold cards from the gold card deck, as the number of gold diggers.

The gold-digging player who placed the last path card chooses one of the gold cards in his hand. He passes the remaining cards to the next gold digger to his left. This player also chooses a gold card and passes the remaining cards to the next gold digger sitting to the left. And so on until each gold miner gets one gold card.

The saboteurs won, if the gold diggers did not reach the gold nugget goal card in a monotonously increasing manner. If there is only one saboteur in the round, he will receive a total of four gold nuggets from the gold card deck. For two or three saboteurs, each of them receives 3 gold nuggets, and for four saboteurs, each receives 2 gold nuggets.

Players keep the number of their gold nuggets secret until the end of the game. After the gold nuggets are handed out, a next round begins.

Figure 4 shows the path cards of a round won by gold diggers.

*Figure 4. Path cards of a round won by gold diggers*

## 3 Planning the cards

We have changed the 40 path cards in the original commercial card game to cards including mathematical content. The numbers and expressions on the cards were chosen according to the following criteria:

1) Mathematical areas to be developed.
2) Didactical considerations
3) Game design

In the following we discuss the above three criteria in detail.

### 3.1 Mathematical areas to be developed.

We have designed these cards according to the mathematical areas to be developed. These areas had been identified in advance in collaboration with secondary school teachers, and with instructors of other faculties of the university. The areas were the following:

1) Logic, elementary combinatorics, probability
2) Basic algebra (operations with integers and rational numbers, order of operations, use of parentheses)

3) Elementary algebra (factoring, completing the squares, rational expressions)
4) Linear equations and inequalities, Quadratic equation and inequalities, Systems of linear equations
5) Exponents, logarithm
6) Functions

Finally, Math-teur was used for topics regarding numbers and algebra (thus topics 2 and 3) and exponents and logarithms (topic 5), as these topics were closely connected to the ordering principles crucial for the game design.

### 3.2 Didactical considerations

Some of the cards contain expressions that caused difficulty for students based on our previous experience. We have tried to include expressions addressing typical difficulties in a particular topic. Thus, these cards are useful to trigger mathematical discourse, but are not so numerous to discourage students from playing. The ratio of the cards addressing complex ideas depends also on the purpose of the game. We need more simple expressions for an introductory face, and more sophisticated problems when the card set serves mainly as a review, as it did in our case.

Ordering the numbers and expressions served not only organizing principles for the game. Recent studies showed that ordering skills play an important role in mathematical ability both in children and in adults [9], [12]. Ordering was also used to introduce and reinforce related mathematical techniques, such as using common denominators or the use of monotonicity in the case of exponential and logarithmic expressions. Thus, in some cases, we deliberately tried to avoid having to evaluate the expressions numerically, i.e. to give the exact rational value of the term. Since the goal of the game is to create a monotone increasing sequence of "numbers", this goal can be achieved for monotone functions by ordering the arguments.

### 3.3 Game design

The sets contain cards where the corresponding operations are easy to perform to make the game smooth. While cards requiring intensive work can be useful for building the corresponding mathematical competencies, a too high number of these problems might take the fun out of the game. We usually planned the sets that about half of the cards are very simple and serve mainly this smoothness. The remaining cards focus generally on one aspect of the corresponding mathematical techniques. The purpose of these cards is to introduce and to reinforce the technique. The sets usually contain a few complex cards, where several of these techniques should be applied. The aim of these cards was to facilitate group work and mathematical discourse between students. If the number of these cards is too low, then players can avoid it by keeping it in hand, or simply passing it. Dealing with the expression might be too difficult for one student, but the group does not have a chance even to see it. On the other hand, too many cards of this type will make the game stuck. The difficulty of the game can be adjusted to the particular group by changing the ratio of the simple and complex cards in the card set.

Another design consideration was the distribution of "numbers", the results of the evaluated mathematical expressions. Since ordering is determined by the distribution of these values, their distribution plays an important role in the game design. We chose values close to each, in order to make the corresponding calculation inevitable. Originally, the 40 cards were spread evenly over an appropriate, relatively short interval. Later, as a result of test games, the distribution of the cards was modified to be distributed more normally i.e. as you moved towards the middle of the interval, the value would appear more and more often.

## 4 The set of cards

In this section we will describe the main ideas behind the design of sets of cards on specific mathematical topics, primarily intended for remedial courses at universities. First, we present the choice of the topic and the main goal of the card set. Then we describe the main didactic considerations we focused on. We also present the main design elements we have considered to create a smooth but mathematically challenging game. We also show the distribution of the cards.

Finally, we give some ideas for improvements based on the experience of games already played. Some of these improvements have already been implemented, and the card sets have been changed accordingly, some are just mentioned as ideas for further improvement. The expressions used for each card set can be found in an appendix.

### 4.1 Operations with natural numbers.

The card deck helps to practice the addition and subtraction of natural numbers from 1 to 20. The set is mainly used to introduce the framework of the mathematical game. It is advisable to start with this introductory version of Math-teur instead of the original commercial game Saboteur, as the Math-teur game encodes a different strategy. In the case of the number cards, a road must be built according to the principle of ensuring/breaking monotonicity, whereas in Saboteur the different types of path cards determine the possible paths to be built. It can be played from the age of 6-7, as it helps to practice bridging 10 in first grade. However, we have no experience of using card sets in primary schools, thus an adjustment of the cards might be advisable for this age group.

Approximately half of the cards contained addition and the other half subtraction of numbers between 1 and 20. All the expressions consisted of two term expressions.

The cards originally contained additions and subtractions, resulting in numbers from 1 to 20, two of each value. The distribution of the cards was originally uniform, but it was later modified based on test games. Thus, we added 8 extra cards to the deck to make the card set more 'normally distributed'.

Finally, out of the 48 path cards, there are 22 cards containing additions and 26 containing subtractions, 36 of these operations required bridging 10. Didactical considerations were not of high importance, as for this age group the game was only used to introduce the framework itself. However, we suggest using a different set of cards for the group of first graders. About half of the cards should require bridging 10 with 3-4 cards containing 3-4-term operations as well.

### 4.2 Operations with integers

In addition to introducing the framework of the mathematical game, the aim was to get into the routine of counting negative numbers with the deck of cards. The cards involve adding, subtracting, and multiplying whole numbers. Besides the operations with integers, the deck of cards also practices the use and evaluation of brackets.

More than half of the cards (26 cards) include multiplication of integers. Difficulties may be caused by the order of operations, the use of parentheses (e.g. $(-5) * (-1) - (-2) * 2$, $(-2) - (-2 - 2)$). Simpler cards have only one operation, e.g. $2 + (-2)$, while more complex cards require the evaluation of expressions with several terms (e.g. $(-2) - (-5) + (-2)$; $2 - 3 - (-2) - 2$). Of the simpler cards, 11 cards have an expression with 2 terms each with one integer per term, 5 cards have 2 factors each containing one integer, and 1 card has an expression with 3 integer factors. On the more complex cards, 14 cards have a 2-term expression where one term has two factors; 2 cards have a 2-term expression where one term has 2 factors; 5 cards have 2 factors where one factor has 2 terms; and finally, 2 cards have 2 factors with 2 terms in each.

The results of the cards range from -9 to 10, where each integer result occurs twice. The deck of cards can be used from fifth grade onwards. The ratio of two-term or two-factor expressions should be increased for a younger age group to make the game smoother.

### 4.3 Operations with rational numbers

Some of the algebraic difficulties encountered in university courses can be traced back to the difficulty corresponding to operations with fractions. This deck of cards aims to improve the ability to perform fast, accurate routine calculations.

The deck contains 17 cards (approximately half of the deck) that do not require any operation for the evaluation, as the expressions on the cards are ordinary fractions. The remaining 23 cards involve addition, subtraction, multiplication, and division of ordinary fractions. Of the operations with rational numbers, 9 cards have two-term operations, 12 cards have two-factor operations, and two cards have complex two-term operations where one of the terms consists of two factors. The main difficulty in the game was the comparison of fractions on the cards (e.g. $2/5 : (-3/2)$ and $3/2 + 2/5$), thus the difficulty comes from the ordering as well. After finding the common denominator, it is sufficient to compare the numerators only.

The distribution of the cards is determined by the denominators that are used. In this starter pack, fractions between $-13/6$ and $13/6$ occur symmetrically about the origin. However, the corresponding operations resulting the additive inverse are different.

As a starting point, we have designed the game so that the denominators of the fractions can include the primes 2, 3 and 5 as multiplication factors. The denominators were chosen to help the use of different strategies for individual comparisons [See e.g. in [13]] and provide a manageable common denominator for ordering a series of numbers. Thus, the common denominator used for comparison will be 6, 10, 15 or 30. Here, the difficulty can be increased by adding additional primes to the denominator as the main difficulty comes from ordering fractions.

### 4.4 Concept of Exponentiation

Working with exponential expressions often causes difficulty for university students [14] The aim of the set was to review the identities of exponentiation, so we chose the same base for all the expressions, namely the base of three.

Students can use the product, quotient, and power rules to evaluate the exponential expressions with powers of three, and to order the corresponding expressions. However, since all expressions have the same base, the monotonicity property of exponential functions means, that it is sufficient to compare only the exponents. It is therefore not necessary to calculate the exact value of an expression for the comparison.

The cards contain expressions of different levels of difficulty, depending on the type of exponent rules to be used for evaluation. In the pack of 40 cards, the evaluation of 17 expressions (approximately half of the deck) requires the use of the definition only where negative and rational exponents are also included (for example $3^{-8}$), 17 cards require the use of the power rule (e.g. $(3^{-1})^{-1}$ ); 6 expressions require the use of the product rule and 6 the use of the quotient rule.

The distribution of the cards is also specified by arguments: integers between -9 and 9 appear as arguments, each argument appearing twice. This package has also been extended by 6 cards to assure the approximately normal distribution.

During the games students started to evaluate the expressions on the cards while gradually reviewing the definitions of negative and rational exponents. They slowly started to use the corresponding exponentiation rules sometimes with the need for further clarification. With the help of the exponential rules, students gradually changed the complex expressions to a power of three and used it for the comparison. Finally, they realized that it was enough to compare the exponents only, as the base was the same for all cards. (e.g. $3^2 \cdot 3^5 = 3^7$) Thus, the deck was appropriate to realize the monotone property of the exponential function, e.g. if $x < y$, then $3^x < 3^y$, (e.g. $(-2)(-3) < 2 + 5$, thus $(3^{-3})^{-2} < 3^2 \cdot 3^5$), thus students compared the exponents only. This final step was encouraged by the fact that the expressions on the cards (although all rational) were sometimes a bit tedious to calculate and compare. Ordering the exponents of the same base offered a considerably easier method for the comparison.

Although the card set was appropriate to make students compare exponentials using the monotonicity property, it was proven to be too easy in the case of several groups. Adding more complex cards (e.g. $\frac{(3^{-3})^{-2} \cdot 3^5}{3^2}$) to the deck would improve the coordination between the various exponential rules.

### 4.5 Concept of Logarithm, Logarithm Rules.

The concept of logarithms provides a challenge for many students [15]–[17]. The card set was designed to be more challenging than the previous ones, as this was the last in the series of decks.

The main goal of the game was to use the rules of logarithms and finally to order the logarithmic expressions. According to our goal, we chose the same base of the logarithm, i.e. the base of the logarithm is two for all the expressions. Of the 40 cards, 9 required knowledge of the definition of logarithm (approximately half of it resulted a positive, half a negative number, and 0 was also included among the expressions), the remaining 31 cards required other operations as well, 17 of these contained a one-term, 14 a two-term expression. Although expressions could be evaluated using various methods, 20 of these focused primarily on logarithmic rules (2 on the product rule, 8 on the quotient rule and 10 on the power rule) and only 15 were complex cards, i.e. containing expressions that required the use of at least two identities. The common base made it possible to use the product and quotient rule. (The rules are $\log_a x + \log_a y = \log_a x \cdot y$, $\log_a x - \log_a y = \log_a \frac{x}{y}$, $n \cdot \log_a x = \log_a x^n$, where $a \in \mathbb{R}, a > 0, a \neq 1; x, y \in \mathbb{R}, x, y > 0; n \in \mathbb{R}$ .)

According to the monotone property of logarithm for a base greater than 1, the logarithm function is increasing, thus $x < y$ implies $\log_2 x < \log_2 y$. To encourage the use of this property for the comparisons, we included 6 non-rational valued two-term expressions (expressions contain $\log_2 3$). Students first approximated the values using a calculator or estimated them based on the definition. Then gradually started to use the monotone property of the logarithm function.

During the games $3 \cdot \log_2 \frac{3}{2^3} - \log_2 2$ appeared to be the most difficult expression to deal with. Students used various ideas to overcome this difficulty including the use of technology. For details see [18]. Overcoming such difficulties, teams had to work in groups, i.e. in addition to the individual interpretation of the logarithmic expressions, the group members did to work together and discuss the corresponding mathematical concepts. We did see a similar pattern as in the case of exponential card set, namely that students first used calculators for evaluation, then turned to the definition itself. They gradually began to apply the logarithm rules. Using the rules made it possible to “rewrite” the expressions using one logarithm only. While using one logarithm, it was enough to compare the arguments for the ordering. Finally, some groups started to rely on the monotone property almost exclusively, namely that in the case of the same base, it is enough to compare the arguments only.

The values of the expressions on the cards ranged from -9 to 6, while their distribution was close to normal with arguments between $2^{-9}$ and 64, see in Figure 5.

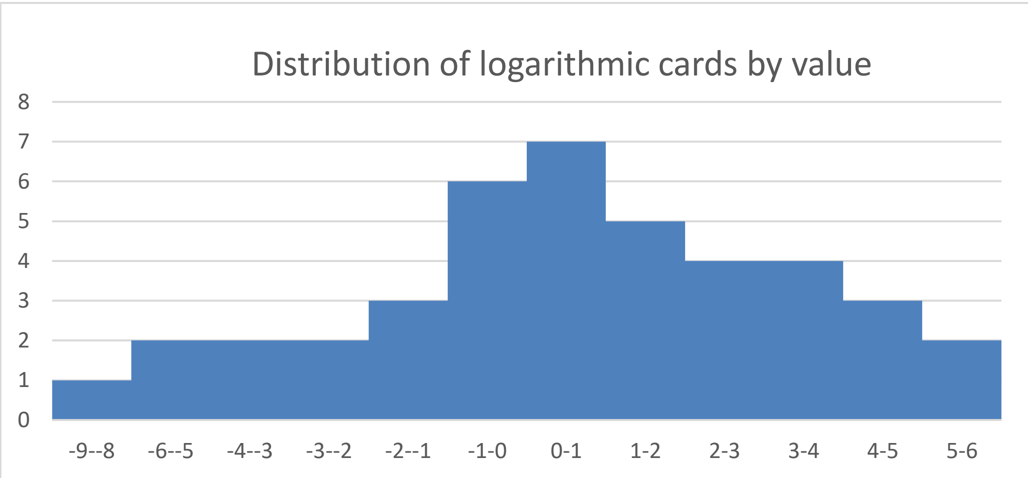


*Figure 5. Distribution of logarithmic cards by value*

We found that this card set was the most successful based on the games played during classes. The set was challenging enough for university students, but it remained at the playable level. Using the same base encouraged the use of logarithm rules and the monotone property, but it did not address the change of base formula.

## 5 Discussion

The mathematical content did not spoil the fun game character of the game Saboteur. The excitement and flow of the original game has been kept and it remained relatively fast-paced. The mathematical part of the game was not easy, but students were very motivated thanks to the game environment.

Students could profit from the cooperative group setting during the game, whereas the competition between the groups provided extra motivation at the same time. The role of chance and the freedom of actions provided by different roles made it easier to face with committing errors in front of other learners. In this relaxed environment we did experience sophisticated mathematical discourse among students [19]. We addressed some of these observations in a previous paper [18]. We plan to investigate in a later publication the similar pattern students went through in the case of the exponential and logarithmic cards set.

The games are easy to adapt for several areas of mathematics. It is advisable to choose topics where ordering plays a special role. However, the original game could be adapted differently, e.g. keeping the original path cards by encoding the directions using trigonometric expressions or complex numbers.

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## Appendix: Expressions on the cards

### Operations with Natural numbers

$12-11$, $19-18$, $13-11$, $11-9$, $12-9$, $11-8$, $13-9$, $12-8$, $13-8$, $14-9$, $15-9$, $14-8$, $15-8$, $16-9$, $17-9$, $14-6$, $17-8$, $14-5$, $6+4$, $3+7$, $5+6$, $8+3$, $5+7$, $4+8$, $6+7$, $5+8$, $6+8$, $5+9$, $6+9$, $7+8$, $7+9$, $8+8$, $8+9$, $11+6$, $9+9$, $11+7$, $12+7$, $15+4$ $12+8$, $11+9$,
Extension $16-7$, $15-6$, $15-7$, $13-5$, $14-3$, $13-2$, $17-5$, $18-6$

### Operations with Integers

$(-3)\cdot 3$, $(-5)+(-2-2)$, $(-2)\cdot(-2)\cdot(-2)$, $3\cdot(-2)+(-2)$, $(-5)+(-2)$, $(-3)\cdot 2-1$, $3\cdot(-2)$, $(-2)+(-2)\cdot 2$, $3\cdot(-2)+1$, $(-2)\cdot 2-1$, $(-2)+(-2)$, $(-2)\cdot 2$, $(-5)-(-2)$, $(-7)+4$, $(3-2)\cdot(-2)$, $2-(-2)\cdot(-2)$, $(-2)\cdot(-2)-5$, $5-(-3)\cdot(-2)$, $2+(-2)$, $(-2)-(-2)$, $(-2)\cdot(-2)-3$, $-(1-2)$, $2\cdot 2-2$, $(-2)-(-2-2)$, $5+(-2)$, $-(2-5)$, $(-2)\cdot(-2)$, $2-(-2)$, $2\cdot 3+(-1)$, $9-(-2)\cdot(-2)$, $2-2\cdot(-2)$, $3-(-3)$, $5-(-2)$, $-((-5)+(-2))$, $(-2)\cdot(-5)+(-2)$, $(-2)\cdot(-2-2)$, $(-3)\cdot(-3)$, $(-5)\cdot(-1)-(-2)\cdot 2$, $2\cdot 3-(-2)\cdot 2$, $(2+3)\cdot 2$,

### Operations with Rational numbers

$-\frac{13}{6}$, $\frac{-1}{2}-\frac{5}{3}$, $-\frac{19}{10}$, $-\frac{3}{2}-\frac{2}{5}$, $\frac{-6}{5}$, $\frac{-2}{\frac{5}{3}}$, $-\frac{7}{6}$, $\frac{1}{2}-\frac{5}{3}$, $-\frac{11}{10}$, $-\frac{3}{2}+\frac{2}{5}$, $\frac{-1}{2}\cdot\frac{5}{3}$, $\frac{\frac{5}{3}}{-2}$, $-\frac{3}{5}$, $\frac{3}{2}\cdot\frac{-2}{5}$, $\frac{-3}{10}$, $\frac{\frac{-1}{2}}{\frac{5}{3}}$, $-\frac{4}{15}$, $\frac{\frac{2}{5}}{\frac{-3}{2}}$, $0$, $\frac{6}{6}-\frac{5}{5}$, $\frac{2}{5}\cdot\frac{5}{2}-1$, $\frac{5}{6}-\frac{5}{3}\cdot\frac{1}{2}$, $\frac{4}{15}$, $\frac{\frac{2}{5}}{\frac{3}{2}}$, $\frac{3}{10}$, $\frac{\frac{1}{2}}{\frac{5}{3}}$, $\frac{3}{5}$, $\frac{3}{2}\cdot\frac{2}{5}$, $\frac{1}{2}\cdot\frac{5}{3}$, $\frac{\frac{5}{3}}{2}$, $\frac{11}{10}$, $\frac{3}{2}-\frac{2}{5}$, $\frac{7}{6}$, $\frac{-1}{2}+\frac{5}{3}$, $\frac{6}{5}$, $\frac{2}{\frac{5}{3}}$, $\frac{19}{10}$, $\frac{3}{2}+\frac{2}{5}$, $\frac{13}{6}$, $\frac{1}{2}+\frac{5}{3}$,

### Concept of Exponentiation

$\left(3^{-3}\right)^3$, $\left(3^3\right)^{-3}$, $3^{-8}$, $\left(3^4\right)^{-2}$, $3^{-7}$, $3^{-2}\cdot 3^{-5}$, $\left(3^{-3}\right)^2$, $\left(3^{-2}\right)^3$, $3^{-5}$, $\left(3^{-1}\right)^5$, $\left(3^{-2}\right)^2$, $\left(3^1\right)^{-4}$, $3^{-3}$, $\left(3^1\right)^{-3}$, $\frac{3^{10}}{3^{12}}$, $\frac{1}{3^2}$, $3^{-1}$, $\frac{3^5}{3^6}$, $\frac{3^5}{3^5}$, $3^5\cdot 3^{-5}$, $3\frac{1}{2}\cdot 3^{-1}$, $3^1$, $\left(3^{-1}\right)^{-1}$, $\frac{3^{-2}}{3^{-4}}$, $\frac{1}{3^{-2}}$, $3^3$, $\left(3^{-3}\right)^{-1}$, $\left(3^2\right)^2$, $\left(3^{-2}\right)^{-2}$, $3^5$, $\left(3^{-1}\right)^{-5}$, $\left(3^3\right)^2$, $\left(3^{-3}\right)^{-2}$, $3^2\cdot 3^5$, $3^3\cdot 3^4$, $3^8$, $3^3\cdot 3^5$, $3^9$, $\left(3^3\right)^3$,
Extension: $3^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $3^{-\frac{1}{3}}$, $3^{-\frac{1}{2}}$, $3^{\frac{1}{5}}$, $3^{-\frac{1}{5}}$

### A Concept of Logarithm, Logarithm Rules

$3\cdot\log_2\frac{1}{8}$, $3\cdot\log_2\frac{1}{4}$, $3\cdot\log_2\frac{3}{2^3}-\log_2 2$, $\log_2\frac{1}{16}$, $\log_2\frac{1}{2^4}$, $\log_2\frac{1}{8}$, $\log_2\frac{1}{2}+2\cdot\log_2\frac{1}{2}$, $\log_2\frac{1}{4}$, $\log_2\frac{1}{2}+\log_2\frac{1}{2}$, $2\cdot\log_2\frac{1}{2}$, $\log_2\frac{1}{2}$, $\frac{1}{3}\cdot\log_2\frac{1}{8}$, $\frac{1}{3}\cdot\log_2\frac{1}{4}$, $\log_2\frac{3}{2}-\log_2 2$, $\frac{1}{3}\cdot\log_2\frac{1}{2}$, $\log_2 1$, $\log_2 2-\log_2 2$, $3\cdot\log_2 1$, $\frac{1}{3}\cdot\log_2 1$, $\log_2 4-\log_2 3$, $\log_2 2^2-\log_2 3$ $\frac{1}{2}\cdot\log_2 2$, $\log_2 2$, $\frac{1}{2}\log_2 2^2$, $\frac{1}{3}\cdot\log_2 8$, $\frac{1}{4}\log_2 4^2$, $\log_2\frac{1}{2}-\log_2\frac{1}{4}$, $3\cdot\log_2\frac{1}{2}-\log_2\frac{1}{2^2}$, $\log_2 4$, $\log_2 2+\log_2 2$, $\log_2 2^2$, $3\cdot\log_2 2-\log_2 2$, $\log_2 8$, $3\cdot\log_2 2$, $\log_2 4+\log_2 3$, $\log_2 2^2+\log_2 3$, $\log_2 16$, $\log_2 4^2$, $3\cdot\log_2 2+\log_2 2$ $2\cdot\log_2 8$,