

CI-PROPERTY OF $C_p^2 \times C_n$ AND $C_p^2 \times C_q^2$ FOR DIGRAPHS

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ABSTRACT. We prove that the direct product of two coprime order elementary abelian groups of rank two, as well as the direct product of a cyclic group of prime order and a cyclic group of square free order are DCI-groups. The latter is a generalization of Muzychuk's result on cyclic groups (J. Combin. Theory Ser. A, 1995).

1. INTRODUCTION

Investigation of the isomorphism problem of Cayley graphs started in 1967 with the following conjecture of Ádám [1]. He asked whether two circulant graphs on n vertices are isomorphic if and only if they are isomorphic via a multiplication with an integer coprime to n .

A generalisation of the question using a different terminology was introduced in [3]. Let G be a finite group and let S be a subset of $G \setminus \{e\} = G^\#$. The vertices of the *Cayley graph* $\text{Cay}(G, S)$ are the elements of G and $g \in G$ is connected to $h \in G$ if and only if $hg^{-1} \in S$. A right multiplication by a group element $g \in G$ is an automorphism of an arbitrary Cayley graph and hence $\text{Aut}(\text{Cay}(G, S))$ contains a regular subgroup isomorphic to G .

Any automorphism α of G induces an isomorphism between the two Cayley graphs $\text{Cay}(G, S)$ and $\text{Cay}(G, S^\alpha)$. In this case these graphs are called *Cayley isomorphic*. A subset S of the group G is *CI* if $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ implies that these graphs are Cayley isomorphic. A group G is called a *DCI-group* if S is CI for every $S \subset G^\#$ and it is called a *CI-group* if the same holds for symmetric ($S^{-1} = S$) subsets of $G^\#$.

The first counterexample for Ádám's conjecture was given by Elspas and Turner [10], and independently by Djoković [6]. The complete description of finite cyclic DCI-groups was given by Muzychuk [24] in 1997, who proved that a cyclic group C_n is DCI if and only if $n = ab$, where $a \mid 4$ and b is a square free odd number.

The class of CI-groups is closed under taking subgroups. It was proved by Babai and Frankl [3] that a finite p -group is a DCI-group only if it is either an elementary abelian p -group or a quaternion group of order 8 or a cyclic group of small order. This poses a strong restriction on the structure of DCI-groups. A collection of the candidates of (D)CI-groups is found in [21]. Recently, further significant restriction was obtained by Dobson et al. [9]. Furthermore, it has been proved by

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Muzychuk [22] that for every prime p , the elementary abelian p -groups of sufficiently large rank are not CI-groups. The current lower bound for the rank of a non-CI elementary abelian p -group is $2p + 3$ [29]. On the other hand, it was proved by Feng and Kovács [13] that C_p^5 is CI-group for every prime p .

It was conjectured by Kovács and Muzychuk [17] that the direct product of DCI-groups of coprime orders is always a DCI-group (see also [7, Conjecture 43]). They proved that $C_p^2 \times C_q$ is a DCI-group for every pair of distinct primes p, q . As a strengthening of this result it was proved that $C_p^3 \times C_q$ and $C_p^4 \times C_q$ are also DCI-groups, see [30, 19]. Furthermore, Dobson [7] settled the conjecture for abelian groups under strong restrictions on the order of the factors.

It was shown by Babai [2] that $S \subseteq G^\#$ is CI if and only if all regular subgroups of $\text{Aut}(\text{Cay}(G, S))$ isomorphic to G are conjugate in the automorphism group. This observation gives us one of the main tools in the study of (D)CI-groups and allows us to use results from group theory. Another basic method in these investigations is the use of Schur rings. It started in the paper of Klin and Pöschel [16], where it was proved that a cyclic group whose order is a product of two different primes is a DCI-group. The method was further developed in a paper of Hirasaka and Muzychuk [15], where the notion of star product was introduced. We refer to the survey paper [25] for more information on Schur rings and their link with combinatorics.

In our paper the techniques developed in [30] will be combined with a criterion given in [18] to lead to our results.

Theorem 1.1. *For any prime p and any square free number n the group $C_p \times C_n$ is a DCI-group.*

If n is not divisible by p , then $C_p \times C_n \cong C_{pn}$ is a cyclic group of square free order, so this result includes Muzychuk's theorem [23] and our methods provide an independent proof for that. If p divides n , then $C_p \times C_n \cong C_p^2 \times C_{n/p}$ belongs to a new class of groups for which we establish the CI property.

Theorem 1.2. *If p and q are different primes, then $C_p^2 \times C_q^2$ is a DCI-group.*

This theorem provides the first example besides elementary abelian p -groups of an infinite family of DCI-groups, which are not Burnside groups. The proof of Theorem 1.2 uses some techniques from finite geometry.

As a consequence of Theorems 1.1 and 1.2, and results in [15, 24, 27, 30], we have the complete list of abelian DCI-groups whose order is a product of four not necessarily distinct primes.

Theorem 1.3. *The abelian DCI-groups whose order is a product of four not necessarily distinct primes are the following groups:*

$$C_p^4, C_p^3 \times C_q, C_p^2 \times C_q^2, C_p^2 \times C_{qr}, C_r^2 \times C_4, C_{4rs}, C_{pqrs},$$

where p, q, r, s are pairwise distinct primes and $r, s > 2$.

The paper is organised as follows. The concept of Schur rings is presented in Section 2. The next two sections are devoted to preparation for the proof of our two main theorems. The main result in Section 3 is Lemma 3.5 that can certainly be applied to other infinite families of abelian groups. Section 4 collects results on different types of products of CI-S-rings. The proof of Theorem 1.1 is contained in Section 5. Section 6 is devoted to the investigation of uniprimitive groups containing a regular subgroup isomorphic to $C_p^2 \times C_q^2$ using translation nets. The proof of Theorem 1.2 is contained in Section 7.

Notation. The set of non-identity elements of a group G is denoted by $G^\#$.

For a subset $X \subseteq G$, the set $\{x^{-1} : x \in X\}$ is denoted by X^{-1} and the subgroup generated by X is denoted by $\langle X \rangle$. The element $\sum_{x \in X} x$ of the group ring $\mathbb{Z}G$ is denoted by \underline{X} .

For $L \trianglelefteq G$, the canonical epimorphism from G to G/L is denoted by $\pi_{G/L}$.

The group of all permutations of a set Ω is denoted by $\text{Sym}(\Omega)$ and the identity element of $\text{Sym}(\Omega)$ by id_Ω .

For $A \leq \text{Sym}(\Omega)$ and $\alpha \in \Omega$, the stabiliser of α in A is denoted by A_α , the orbit of α under A by α^A , and the set of all orbits under A by $\text{Orb}(A, \Omega)$.

The right regular representation of G is denoted by ρ_G , i.e., for $x, y \in G$, $x^{\rho_G(y)} = xy$. The image $\rho_G(G)$ is also denoted by G_R .

The set of all permutation groups of G containing G_R is denoted by $\text{Sup}(G_R)$.

For a set $\Delta \subseteq \text{Sym}(G)$ and a section $S = U/L$ of G , we set

$$\Delta^S = \{\varphi^S : \varphi \in \Delta, S^\varphi = S\},$$

where $S^\varphi = S$ means that φ maps U to itself and permutes the L -cosets in U among themselves and φ^S denotes the bijection of S induced by φ .

2. S-RINGS

Let G be a finite group with identity element e and $\mathbb{Z}G$ be the integer group ring. A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an *S-ring* (or *Schur ring*) over G if there exists a partition $\mathcal{S}(\mathcal{A})$ of G such that

- (1) $\{e\} \in \mathcal{S}(\mathcal{A})$,
- (2) if $X \in \mathcal{S}(\mathcal{A})$ then $X^{-1} \in \mathcal{S}(\mathcal{A})$,
- (3) $\mathcal{A} = \text{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}(\mathcal{A})\}$.

The elements in $\mathcal{S}(\mathcal{A})$ are called the *basic sets* of \mathcal{A} and the number $\text{rk}(\mathcal{A}) := |\mathcal{S}(\mathcal{A})|$ is called the *rank* of \mathcal{A} . The definition of an S-ring is due to Wielandt (see [32, Chapter IV]). The motivation comes from the following result of Schur (see [32, Theorem 24.1]).

Theorem 2.1. ([28]) *If $A \in \text{Sup}(G_R)$, then the \mathbb{Z} -submodule $\text{Span}_{\mathbb{Z}}\{\underline{X} : X \in \text{Orb}(A_e, G)\}$ is a subring of $\mathbb{Z}G$.*

Clearly, the ring in the theorem is an example of an S-ring, also called the *transitivity module* over G induced by A and denoted by $V(G, A_e)$. An S-ring \mathcal{A} is called *schurian* if $\mathcal{A} = V(G, A_e)$ for some permutation group $A \in \text{Sup}(G_R)$. We remark that not all S-rings are schurian (see [32]). In the particular case when $A = G_R K$ for some subgroup $K \leq \text{Aut}(G)$, the S-ring $V(G, A_e)$ is called *cyclotomic* and also denoted by $\text{Cyc}(K, G)$. In this case the basic sets are the orbits under K .

Let \mathcal{A} be an S-ring over a group G . A set $X \subseteq G$ is called an *\mathcal{A} -set* if $\underline{X} \in \mathcal{A}$, and a subgroup $H \leq G$ is called an *\mathcal{A} -subgroup* if $\underline{H} \in \mathcal{A}$. The S-ring \mathcal{A} is *primitive* if G contains no non-trivial proper \mathcal{A} -subgroup. Suppose that $\mathcal{A} = V(G, A_e)$ for some permutation group $A \in \text{Sup}(G_R)$. Then $H \leq G$ is an \mathcal{A} -subgroup if and only if the partition of G into its right H -cosets is A -invariant. Hence A is primitive if and only if so is $V(G, A_e)$.

Proposition 2.2. ([31]) *Suppose that G is an abelian group of composite order having a cyclic Sylow subgroup. Then every primitive S-ring over G is of rank 2.*

For a subset $X \subseteq G$ and integer m , define $X^{(m)} = \{x^m : x \in X\}$; and for a group ring element $\eta = \sum_{g \in G} c_g g$, define $\eta^{(m)} = \sum_{g \in G} c_g g^m$. Two useful properties of S-rings over abelian groups are invoked next. The statement in part (i) is [32, Theorem 23.9(a)] and the statement in part (ii) follows from the proof of [32, Theorem 23.9(b)]. For an abelian group G and a prime divisor p of the order of G we will use the notation $G[p] = \{g \in G : g^p = e\}$.

Proposition 2.3. ([32]) *Let \mathcal{A} be an S-ring over an abelian group G .*

- (i) If m is an integer coprime to $|G|$ and $\eta \in \mathcal{A}$, then $\eta^{(m)} \in \mathcal{A}$. In particular, $X^{(m)} \in \mathcal{S}(\mathcal{A})$ whenever $X \in \mathcal{S}(\mathcal{A})$.
- (ii) If p is a prime divisor of $|G|$, $1 \leq k \leq p-1$ and $X \subseteq G$ is an \mathcal{A} -set, then the set

$$X^{[p,k]} := \{x^p : x \in X \text{ and } |X \cap xG[p]| \equiv k \pmod{p}\}$$

is an \mathcal{A} -set (possibly empty). Hence the set

$$X^{[p]} := \{x^p : x \in X \text{ and } |X \cap xG[p]| \not\equiv 0 \pmod{p}\}$$

is also an \mathcal{A} -set.

Let G be an arbitrary group and \mathcal{A} be an S-ring over G . With each \mathcal{A} -set X one can naturally associate two \mathcal{A} -subgroups, namely, $\langle X \rangle$ and

$$\text{rad}(X) := \{g \in G : gX = Xg = X\}.$$

Let $L \trianglelefteq U \leq G$. The section U/L is called an \mathcal{A} -section if U and L are \mathcal{A} -subgroups. If $S = U/L$ is an \mathcal{A} -section, then the module

$$\mathcal{A}_S := \text{Span}_{\mathbb{Z}}\{\underline{X^{\pi_{U/L}}} : X \in \mathcal{S}(\mathcal{A}), X \subseteq U\}$$

is an S-ring over S . Note that, if $\mathcal{A} = V(G, A_e)$ and S is an \mathcal{A} -section, then $\mathcal{A}_S = V(S, (A^S)_{e_S})$ and so \mathcal{A}_S is schurian (see [15, Proposition 2.8]). Here e_S denotes the identity element of S .

Let \mathcal{A} be an S-ring over a group G and \mathcal{B} be an S-ring over a group H . A bijection $\varphi : G \rightarrow H$ is called an *isomorphism* from \mathcal{A} to \mathcal{B} if $\text{rk}(\mathcal{A}) = \text{rk}(\mathcal{B}) = r$, and there is an ordering X_1, \dots, X_r of the basic sets in $\mathcal{S}(\mathcal{A})$ and an ordering Y_1, \dots, Y_r of the basic sets in $\mathcal{S}(\mathcal{B})$ such that φ is an isomorphism from $\text{Cay}(G, X_i)$ to $\text{Cay}(H, Y_i)$ for every $1 \leq i \leq r$. If there is an isomorphism from \mathcal{A} to \mathcal{B} , then we say that \mathcal{A} and \mathcal{B} are *isomorphic* and write $\mathcal{A} \cong \mathcal{B}$. Let $\text{Iso}(\mathcal{A}, \mathcal{B})$ denote the set of all isomorphisms from \mathcal{A} to \mathcal{B} . An isomorphism $\varphi \in \text{Iso}(\mathcal{A}, \mathcal{B})$ is called *normalised* if it maps the identity element e_G to the identity element e_H . If φ is normalised, then $X_i^\varphi \in \mathcal{S}(\mathcal{B})$ for every basic set $X_i \in \mathcal{S}(\mathcal{A})$ and φ also satisfies the condition:

$$\forall 1 \leq i, j \leq r : (X_i X_j)^\varphi = X_i^\varphi X_j^\varphi. \quad (1)$$

Some further properties are collected below.

Proposition 2.4. ([15, Proposition 2.7]) *Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a normalised isomorphism from an S-ring \mathcal{A} over a group G to an S-ring \mathcal{B} over a group H , and let $E \leq G$ be an \mathcal{A} -subgroup.*

- (i) *The image E^φ is a \mathcal{B} -subgroup of H . Moreover, the restriction $\varphi_E : E \rightarrow E^\varphi$ is an isomorphism between \mathcal{A}_E and \mathcal{B}_{E^φ} .*
- (ii) *For each $x \in G$, $(Ex)^\varphi = E^\varphi x^\varphi$.*
- (iii) *If $E \trianglelefteq G$ and $E^\varphi \trianglelefteq H$, then the mapping $\varphi^{G/E} : G/E \rightarrow H/E^\varphi$, defined by $(Ex)^{\varphi^{G/E}} = E^\varphi x^\varphi$ is a normalised isomorphism between $\mathcal{A}_{G/E}$ and $\mathcal{B}_{H/E^\varphi}$.*

We are interested in isomorphisms between S-rings over the same group and set

$$\text{Iso}(\mathcal{A}) = \bigcup_{\substack{\mathcal{B} \text{ is an S-ring} \\ \text{over } G}} \text{Iso}(\mathcal{A}, \mathcal{B}) \text{ and } \text{Iso}_e(\mathcal{A}) = \{\varphi \in \text{Iso}(\mathcal{A}) : e^\varphi = e\}.$$

Clearly, $\text{Iso}(\mathcal{A}, \mathcal{A})$ is a subgroup of $\text{Sym}(G)$, which contains the normal subgroup defined as

$$\text{Aut}(\mathcal{A}) = \bigcap_{X \in \mathcal{S}(\mathcal{A})} \text{Aut}(\text{Cay}(G, X)).$$

This is called the *automorphism group* of \mathcal{A} (see [16]). Clearly, $G_R \leq \text{Aut}(\mathcal{A})$.

3. DCI-GROUPS AND CI-S-RINGS

Babai [2] gave the following group theoretical criterion for a subset $X \subseteq G$ to be a CI-subset.

Proposition 3.1. ([2, Lemma 3.1]) *A subset $X \subseteq G$ is a CI-subset if and only if any two regular subgroups of $\text{Aut}(\text{Cay}(G, X))$ isomorphic to G are conjugate in $\text{Aut}(\text{Cay}(G, X))$.*

Let $A, B \in \text{Sup}(G_R)$ such that $A \leq B$. Then A is said to be a G_R -complete subgroup of B , denoted by $A \preceq_G B$, if for every $\varphi \in \text{Sym}(G)$, the inclusion $(G_R)^\varphi \leq B$ implies $(G_R)^{\varphi\psi} \leq A$ for some $\psi \in B$ (see [15, Definition 2]). Notice that, the relation \preceq_G is a partial order on $\text{Sup}(G_R)$. In this context Proposition 3.1 reads as

$$X \subseteq G \text{ is a CI-subset} \iff G_R \preceq_G \text{Aut}(\text{Cay}(G, X)). \quad (2)$$

Let $A \leq \text{Sym}(G)$. The 2-closure $A^{(2)}$ is the largest permutation group of G satisfying

$$\text{Orb}(A^{(2)}, G \times G) = \text{Orb}(A, G \times G),$$

where the groups $A^{(2)}$ and A act on $G \times G$ coordinate-wise. The group A is called 2-closed if $A^{(2)} = A$. If $\mathcal{A} = V(G, A_e)$, then $\text{Aut}(\mathcal{A}) = A^{(2)}$. It is well-known that $\text{Aut}(\text{Cay}(G, X))$ is 2-closed for any subset $X \subseteq G$. It follows from this and (2) that G is a DCI-group if $G_R \preceq_G A$ for every 2-closed permutation group $A \in \text{Sup}(G_R)$.

Proposition 3.2. ([15, Theorem 2.6]) *Let $A \in \text{Sup}(G_R)$ be a 2-closed permutation group and $\mathcal{A} = V(G, A_e)$. Then the following statements are equivalent.*

- (i) $G_R \preceq_G A$.
- (ii) $\text{Iso}(\mathcal{A}) = \text{Aut}(\mathcal{A}) \text{Aut}(G)$.
- (iii) $\text{Iso}_e(\mathcal{A}) = \text{Aut}(\mathcal{A})_e \text{Aut}(G)$.

An S-ring \mathcal{A} over G is called a *CI-S-ring* (or *CI* for short) if $\text{Aut}(\mathcal{A}) \text{Aut}(G) = \text{Iso}(\mathcal{A})$ (see [15, Definition 3]).

Proposition 3.3. ([30, Proposition 2.4]) *Let $A, B \in \text{Sup}(G_R)$ such that $B \leq A$, $\mathcal{A} = V(G, A_e)$ and $\mathcal{B} = V(G, B_e)$. If $B \preceq_G \text{Aut}(\mathcal{A})$ and \mathcal{B} is CI, then \mathcal{A} is also CI.*

This allows us to consider only minimal elements of the poset $(\text{Sup}(G_R), \preceq_G)$. The set of such elements will be denoted by $\text{Sup}^{\min}(G_R)$.

Corollary 3.4. *If $V(G, A_e)$ is CI for every $A \in \text{Sup}^{\min}(G_R)$, then G is a DCI-group.*

In fact, we are going to derive Theorems 1.1 and 1.2 by showing that the condition in Corollary 3.4 holds whenever G is one of the groups in the cited theorems.

We conclude the section with a useful lemma.

Lemma 3.5. *Let G be an abelian group, $A \in \text{Sup}^{\min}(G_R)$ and $\mathcal{A} = V(G, A_e)$. Suppose that there exist \mathcal{A} -subgroups $L < U \leq G$ such that $|U/L| = np^t$ for a prime p and $1 < n < p$. Then LU_p is an \mathcal{A} -subgroup, where U_p is the Sylow p -subgroup of U .*

Proof. Let $F_1 := G_R, F_2, \dots, F_k$ be a complete set of representatives of the conjugacy classes of regular subgroups of A isomorphic to G . Then $A = \langle F_1, \dots, F_k \rangle$ because of $A \in \text{Sup}^{\min}(G_R)$. For an easier notation, write \overline{B} for $B^{G/L}$, where $B \leq A$ is any subgroup, and \bar{e} for the identity element of G/L . Furthermore, denote by $\overline{B}_{\{U/L\}}$ the setwise stabiliser of U/L in \overline{B} .

For $1 \leq i \leq k$, let P_i be the Sylow p -subgroup of $(\overline{F_i})_{\{U/L\}}$ and P be a Sylow p -subgroup of $\overline{A}_{\{U/L\}}$ such that $P_1 \leq P$. It follows from $|U/L| = np^t$ and $n < p$ that a Sylow p -subgroup of $\text{Sym}(U/L)$ has n orbits, each containing p^t elements (see [5, Example 2.6.1]). On the other

hand, acting on U/L , the orbits under P_1 are equal to the cosets of LU_p/L in U/L , and therefore, $\text{Orb}(P_1, U/L) = \text{Orb}(P, U/L)$.

Fix i , $2 \leq i \leq k$. By Sylow's theorem $P_i^{\delta_i} \leq P$ for some $\delta_i \in \overline{A}_{\{U/L\}}$. Using also that \overline{F}_i is abelian, we find that the partition of U/L into its LU_p/L -cosets is $\overline{F}_i^{\delta_i}$ -invariant. Thus it is also D -invariant for $D := \langle \overline{F}_1, \overline{F}_2^{\delta_2}, \dots, \overline{F}_k^{\delta_k} \rangle$. In other words, $LU_p/L \in V(G/L, D_{\bar{e}})$.

Let γ_i be a preimage of δ_i under the epimorphism $A \rightarrow \overline{A}$. Then

$$A = V(G, A_e) = V(G, \langle F_1, \dots, F_k \rangle_e) = V(G, \langle F_1, F_2^{\gamma_2}, \dots, F_k^{\gamma_k} \rangle_e),$$

and so

$$\mathcal{A}_{G/L} = V(G/L, \overline{\langle F_1, F_2^{\gamma_2}, \dots, F_k^{\gamma_k} \rangle_{\bar{e}}}) = V(G/L, \langle \overline{F}_1, \overline{F}_2^{\delta_2}, \dots, \overline{F}_k^{\delta_k} \rangle_{\bar{e}}) = V(G/L, D_{\bar{e}}).$$

This shows that $LU_p/L \in \mathcal{A}_{G/L}$, implying that $LU_p \in \mathcal{A}$. \square

4. PRODUCTS OF CI-S-RINGS

In this section we review the star and the generalised wreath product of S-rings. The former was introduced by Hirasaka and Muzychuk [15] and the latter by Evdokimov and Ponomarenko [12] and independently by Leung and Man [20] under the name wedge product.

4.1. Star product. Let \mathcal{A} be an S-ring over a group G and $V, W \leq G$ be two \mathcal{A} -subgroups. The S-ring \mathcal{A} is the *star product* of \mathcal{A}_V with \mathcal{A}_W , written as $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$, if

- (1) $V \cap W \triangleleft W$,
- (2) every $X \in \mathcal{S}(\mathcal{A})$, $X \subseteq W \setminus V$ is a union of some $(V \cap W)$ -cosets,
- (3) for every $X \in \mathcal{S}(\mathcal{A})$ with $X \subseteq G \setminus (V \cup W)$, there exist basic sets $Y, Z \in \mathcal{S}(\mathcal{A})$ such that $X = YZ$, $Y \subseteq V$ and $Z \subseteq W$.

The star product is *non-trivial* if $V \neq \{e\}$, G . In the special case when $V \cap W = \{e\}$ it is also called the *tensor product* and written as $\mathcal{A}_V \otimes \mathcal{A}_W$.

Proposition 4.1. (cf. [17, Proposition 3.2 and Theorem 4.1]) *Let G be a direct product of elementary abelian groups, $A \in \text{Sup}(G_R)$ and $\mathcal{A} = V(G, A_e)$. If $\mathcal{A} = \mathcal{A}_V \star \mathcal{A}_W$ and both S-rings \mathcal{A}_V and $\mathcal{A}_{W/(V \cap W)}$ are CI, then \mathcal{A} is also CI.*

Corollary 4.2. *In particular, if $\mathcal{A} = \mathcal{A}_V \otimes \mathcal{A}_W$ and both S-rings \mathcal{A}_V and \mathcal{A}_W are CI, then \mathcal{A} is also CI.*

Proposition 4.3. ([11, Lemma 2.3.(2)]) *Let G be an abelian group and \mathcal{A} be an S-ring over G . Suppose that $G = H_1 \times H_2$ with \mathcal{A} -subgroups H_1, H_2 . Then $\mathcal{A} \supseteq \mathcal{A}_{H_1} \otimes \mathcal{A}_{H_2}$, and the equality is attained whenever $\mathcal{A}_{H_1} = \mathbb{Z}H_1$ or $\mathcal{A}_{H_2} = \mathbb{Z}H_2$.*

Lemma 4.4. *Let G be an abelian group, $A \in \text{Sup}^{\text{min}}(G_R)$ and $\mathcal{A} = V(G, A_e)$. Suppose that $G = H_1 \times H_2$ with \mathcal{A} -subgroups H_1, H_2 . Then $\mathcal{A} = \mathcal{A}_{H_1} \otimes \mathcal{A}_{H_2}$. If \mathcal{A}_{H_1} and \mathcal{A}_{H_2} are CI, then \mathcal{A} is also CI.*

Proof. Let K_i be the kernel of the action of A on the set of H_i -cosets where $i = 1, 2$. The groups K_1, K_2 are normal in A and intersect trivially because $H_1 \cap H_2 = \{e\}$. Pick a regular abelian subgroup $F \leq A$. Then $F = (F \cap K_1) \times (F \cap K_2) \leq K_1 K_2$. Therefore, any regular abelian subgroup of A is contained in $K_1 K_2$, implying that $K_1 K_2 \leq_G A$. By \leq_G -minimality of A we conclude that $A = K_1 K_2$.

Therefore, the permutation group $A = K_1 K_2$ acting on $G = H_1 H_2$ is permutation isomorphic to the permutation direct product $K_1^{H_1} \times K_2^{H_2}$ acting on $H_1 \times H_2$ (see [5, p. 17]). This implies that $\mathcal{A} = \mathcal{A}_{H_1} \otimes \mathcal{A}_{H_2}$, as required. If \mathcal{A}_{H_i} and \mathcal{A}_{H_2} are CI, then so is \mathcal{A} by Corollary 4.2. \square

4.2. Generalised wreath product. Let \mathcal{A} be an S-ring over a group G and $S = U/L$ be an \mathcal{A} -section of G . The S-ring \mathcal{A} is the S -wreath product (also called the *generalised wreath product* of \mathcal{A}_U with $\mathcal{A}_{G/L}$), written as $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$, if

- (1) $L \trianglelefteq G$,
- (2) every $X \in \mathcal{S}(\mathcal{A})$, $X \subseteq G \setminus U$ is union of some L -cosets.

The S -wreath product is *non-trivial* if $L \neq \{e\}$ and $U \neq G$. Notice the following relation with the star product. If $\mathcal{A}_V \star \mathcal{A}_W$ is defined over the group G such that $V \cap W \trianglelefteq G$, then the latter star product becomes the $V/(V \cap W)$ -wreath product.

An S-ring \mathcal{A} is called *decomposable* if it can be expressed as a non-trivial S -wreath product and *indecomposable* otherwise. In the special case when $U = L$, i.e., S is trivial, the S -wreath product is also called *wreath product* and written as $\mathcal{A}_U \wr \mathcal{A}_{G/U}$.

The following result is a sufficient condition for the CI-property of a generalised wreath product. To state the condition, we set the notation: $\text{Aut}_G(\mathcal{A}) := \text{Aut}(\mathcal{A}) \cap \text{Aut}(G)$. Clearly, if S is an \mathcal{A} -section of G , then $\text{Aut}_G(\mathcal{A})^S \leq \text{Aut}_S(\mathcal{A}_S)$.

Proposition 4.5. ([18, Theorem 1.1]) *Let G be a direct product of elementary abelian groups, and $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ be a non-trivial $S = U/L$ -wreath product such that both \mathcal{A}_U and $\mathcal{A}_{G/L}$ are CI. Then \mathcal{A} is CI whenever*

$$\text{Aut}_S(\mathcal{A}_S) = \text{Aut}_U(\mathcal{A}_U)^S \text{Aut}_{G/L}(\mathcal{A}_{G/L})^S.$$

Note that, if $\mathcal{A}_S = \mathbb{Z}S$ in Proposition 4.5, then $\text{Aut}_S(\mathcal{A}_S)$ is trivial, so we obtain the following.

Corollary 4.6. *If $\mathcal{A}_S = \mathbb{Z}S$ in Proposition 4.5, then \mathcal{A} is CI.*

Two subgroups $K_1, K_2 \leq \text{Aut}(G)$ are *Cayley equivalent*, written as $K_1 \approx_{\text{Cay}} K_2$, if $\text{Orb}(K_1, G) = \text{Orb}(K_2, G)$ (see [18]). A cyclotomic S-ring \mathcal{A} over G is said to be *Cayley minimal* if

$$\{K \leq \text{Aut}(G) : K \approx_{\text{Cay}} \text{Aut}_G(\mathcal{A})\} = \{\text{Aut}_G(\mathcal{A})\}.$$

Proposition 4.7. ([19, Lemma 4.2]) *With the assumptions in Proposition 4.5, suppose that at least one of the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ is cyclotomic and \mathcal{A}_S is Cayley minimal. Then \mathcal{A} is CI.*

This proposition will be especially useful in conjunction with the following lemma.

Lemma 4.8. *Let \mathcal{A} be an S-ring over a cyclic group G of order n .*

- (i) *If n is a prime, then \mathcal{A} is cyclotomic.*
- (ii) *If $n = pq$ for distinct primes p, q and $\text{rk}(\mathcal{A}) \neq 2$, then \mathcal{A} is cyclotomic or a non-trivial wreath product of two S-rings.*
- (iii) *If \mathcal{A} is cyclotomic, then it is Cayley minimal.*

Proof. The statement in (i) follows from Proposition 2.3(i). The statement in (ii) follows from [16, Theorem 2.10].

For (iii) let $\mathcal{A} = \text{Cyc}(K, G)$ for a subgroup $K \leq \text{Aut}(G)$. Let x be a generator of G and $X \in \mathcal{S}(\mathcal{A})$ be the basic set containing x . It is easy to see that K is regular on X , hence $|K| = |X|$. This implies that $\text{Aut}_G(\mathcal{A}) = K$ and $K' \not\approx_{\text{Cay}} K$ for any proper subgroup $K' < K$, i.e., \mathcal{A} is Cayley minimal. \square

Dobson and Witte [8] described the groups in $\text{Sup}(G_R)$ where $G \cong C_p^2$ for a prime p (the description of the imprimitive groups were obtained earlier by Jones [14]). The proposition below follows from their result and for our convenience it is formulated here in the language of S-rings.

Proposition 4.9. (cf. [8, Theorem 14]) *Let $G \cong C_p^2$ for a prime p , $A \in \text{Sup}(G_R)$ and $\mathcal{A} = V(G, A_e)$. If G contains exactly one \mathcal{A} -subgroup of order p , say L , then $\mathcal{A} = \mathcal{A}_L \wr \mathcal{A}_{G/L}$.*

An S-ring \mathcal{A} over a group G is a p -S-ring if G is a p -group and for every $X \in \mathcal{S}(\mathcal{A})$, $|X|$ is equal to a power of p .

Proposition 4.10. ([17, Lemma 5.2]) *Let G be an abelian group, $A \in \text{Sup}^{\min}(G_R)$ and $\mathcal{A} = V(G, A_e)$. Suppose that U is an \mathcal{A} -subgroup such that G/U is a p -group for a prime p . Then $\mathcal{A}_{G/U}$ is a p -S-ring.*

It is obvious that $\mathbb{Z}C_p$ is the only p -S-ring over C_p . Furthermore, it is well-known that up to isomorphism, there are two p -S-rings over C_p^2 , namely

$$\mathbb{Z}C_p^2 \text{ and } \mathbb{Z}C_p \wr \mathbb{Z}C_p, \quad (3)$$

(see, e.g., [15, Section 3.1]).

For the next two propositions let G be an abelian group such that $q \mid |G|$ and $q^2 \nmid |G|$ for a prime q and let \mathcal{A} be an S-ring over G . Let Q be the least \mathcal{A} -subgroup of order divisible by q and H be the unique maximal \mathcal{A} -subgroup of order coprime to q .

Proposition 4.11. ([30, Corollary 3.2]) *With notation as above, \mathcal{A} is the HQ/Q -wreath product.*

Proposition 4.12. ([30, Propositions 3.4 and 3.5]) *With notation as above, if $|HQ/H| \neq q$ or $\mathcal{A}_{HQ/H} \cong \mathbb{Z}C_q$, then $\mathcal{A}_{HQ} = \mathcal{A}_H \star \mathcal{A}_Q$.*

Lemma 4.13. *With the assumptions in Proposition 4.5, \mathcal{A} is CI whenever $\mathcal{A}_{G/L} = \mathcal{A}_S \otimes \mathcal{A}_H$ for some $\mathcal{A}_{G/L}$ -subgroup $H < G/L$.*

Proof. The following containment is clear:

$$\text{Aut}_{G/L}(\mathcal{A}_{G/L})^S \geq (\text{Aut}_S(\mathcal{A}_S) \times \text{Aut}_H(\mathcal{A}_H))^S = \text{Aut}_S(\mathcal{A}_S).$$

On the other hand, $\text{Aut}_{G/L}(\mathcal{A}_{G/L})^S \leq \text{Aut}_S(\mathcal{A}_S)$ and therefore, $\text{Aut}_{G/L}(\mathcal{A}_{G/L})^S = \text{Aut}_S(\mathcal{A}_S)$. Then \mathcal{A} is CI by Proposition 4.5. \square

Lemma 4.14. *Let G be an abelian group, $A \in \text{Sup}^{\min}(G_R)$ and $\mathcal{A} = V(G, A_e)$. Suppose that \mathcal{A} is indecomposable and L is an \mathcal{A} -subgroup of prime order. Then $\rho_G(L) \leq Z(A)$. Moreover, for each $u \in L$, $\{u\} \in \mathcal{S}(\mathcal{A})$.*

Proof. Let $p = |L|$ and write $\hat{L} = \rho_G(L)$. Let K be the kernel of the action of A on the set of L -cosets in G . For $x \in G$, let K_{Lx} denote the pointwise stabiliser of Lx in K . Define the binary relation \sim on the set of L -cosets in G by $Lx \sim Ly$ if and only if $K_{Lx} = K_{Ly}$. It is obvious that \sim is an equivalence relation. Also, for arbitrary $\gamma \in A$, $K_{(Lx)\gamma} = (K_{Lx})^\gamma$, implying that \sim is also A -invariant. This shows that the set $\{Lx : Lx \sim L\}$ is a block for A acting on the set of L -cosets in G . Consequently, the set $U := \bigcup_{Lx \sim L} Lx$ is a block for A acting on G , and so U is an \mathcal{A} -subgroup. Clearly, $L \leq U$.

Let $\gamma \in K_{Lx}$ for some $x \in U$. By the definition of U , $\gamma \in K_U$, the pointwise stabiliser of U in K . In other words, K^U is faithful on Lx for every $x \in U$.

Assume for the moment that $U < G$. Let $x \notin U$. The group K acts primitively on Lx because $|Lx| = p$, and $K_L \triangleleft K$. Since $x \notin U$, $L \not\sim Lx$, and hence $(K_L)^{Lx} \neq 1$. We obtain that the orbit $x^{K_L} = Lx$, so $L \leq \text{rad}(x^{A_e})$. This shows that \mathcal{A} is the non-trivial U/L -wreath product, a contradiction. Thus $U = G$. As $K = K^U$ is faithful on L , \hat{L} is the unique Sylow p -subgroup of K . On the other hand, if $F \leq A$ is any abelian regular subgroup, then $K \cap F$ has order p , and thus we find $K \cap F = \hat{L}$, in particular, $\hat{L} \leq Z(F)$. This yields $\hat{L} \leq Z(A)$ because A is generated by its regular subgroups isomorphic to G due to the condition $A \in \text{Sup}^{\min}(G_R)$.

For the second assertion, choose $u \in L$ and let $X \in \mathcal{S}(\mathcal{A})$ be the basic set containing u . Then $X = u^{A_e}$ and as $\hat{L} \leq Z(A)$, we obtain $X = e^{\rho_G(u)A_e} = (e^{A_e})^{\rho_G(u)} = \{u\}$. \square

5. PROOF OF THEOREM 1.1

Throughout this section we keep the following notation:

$G \cong C_p \times C_n$ for a prime p and a square-free number n (that may or may not be divisible by p), $A \in \text{Sup}^{\min}(G_R)$ and $\mathcal{A} = V(G, A_e)$.

In view of Corollary 3.4, it is sufficient to show that \mathcal{A} is CI. We proceed by induction on the total number of prime divisors (counted with multiplicities) of $|G|$, that we will denote by $\Omega(|G|)$.

If $\Omega(|G|) = 1$, then $G \cong C_p$. It follows from Proposition 3.1 via Sylow's theorem that $A = G_R$, hence $\mathcal{A} = \mathbb{Z}G$, which is clearly CI.

Assume that $\Omega(|G|) > 1$ and the assertion holds for any group $\tilde{G} \cong C_{\tilde{p}} \times C_{\tilde{n}}$, where \tilde{p} is a prime, \tilde{n} is square-free and $\Omega(|\tilde{G}|) < \Omega(|G|)$. Note that, this implies that every schurian S-ring over \tilde{G} is CI.

If $G \cong C_p^2$, then $\mathcal{A} \cong \mathbb{Z}C_p^2$ or $\mathbb{Z}C_p \wr \mathbb{Z}C_p$ by Proposition 4.10 and (3). In either case, \mathcal{A} is CI (for the latter S-ring, see Corollary 4.6).

Now, let $n_{p'} > 1$, where $n_{p'}$ denotes the p' -part of n . Let $n_{p'} = q_1 \cdots q_k$ be the prime decomposition of $n_{p'}$, P be the Sylow p -subgroup and C be the Hall p' -subgroup of G . Then $P \cong C_p$ or C_p^2 and $C \cong C_{n_{p'}}$. For $1 \leq i \leq k$, let Q_i be the least \mathcal{A} -subgroup of G of order divisible by q_i , and H_i be the unique maximal \mathcal{A} -subgroup of order coprime to q_i .

Claim 1. \mathcal{A} is CI, unless $H_i Q_i \neq G$ for every $1 \leq i \leq k$.

Suppose that $H_i Q_i = G$ for some $1 \leq i \leq k$. Then $\mathcal{A} = \mathcal{A}_{H_i} \star \mathcal{A}_{Q_i}$. This follows from Proposition 4.12 if $|G/H_i| \neq q_i$, and from Propositions 4.10 and 4.12 if $|G/H_i| = q_i$.

If $Q_i/(Q_i \cap H_i) \not\cong G$, then the induction hypothesis guarantees that both \mathcal{A}_{H_i} and $\mathcal{A}_{Q_i/(Q_i \cap H_i)}$ are CI, and hence so is \mathcal{A} by Proposition 4.1.

Let $Q_i/(Q_i \cap H_i) \cong G$. Then $Q_i = G$, $H_i = \{e\}$, and these imply that \mathcal{A} is primitive. By Proposition 2.2, $\text{rk}(\mathcal{A}) = 2$, and so \mathcal{A} is CI in this case as well. This completes the proof of Claim 1.

Claim 2. \mathcal{A} is CI, unless C is an \mathcal{A} -subgroup and $\mathcal{A}_{G/C} \cong \mathbb{Z}C_p^2$.

In view of Claim 1, we may assume that $H_i Q_i \neq G$ for every $1 \leq i \leq k$. Then \mathcal{A} is the non-trivial $H_i Q_i / Q_i$ -wreath product by Proposition 4.11.

Assume first that $P \cong C_p$, i.e., G is a cyclic group. Let $X \in \mathcal{S}(\mathcal{A})$ be a basic set containing a generator of G , say x , and let $V = \text{rad}(X)$. Then for every $1 \leq i \leq k$, $x \notin H_i Q_i$, and so $V \geq Q_i$. We obtain $V = C$, in particular, $\underline{C} \in \mathcal{A}$. By Proposition 4.10, $\mathcal{A}_{G/V} \cong \mathbb{Z}C_p$, and it follows that $X = Vx$. This and Proposition 2.3(i) imply that $\mathcal{A} = \mathcal{A}_V \wr \mathcal{A}_{G/V}$, hence \mathcal{A} is CI by Corollary 4.6.

Now, suppose that $P \cong C_p^2$ and let c be a generator of C .

Assume for the moment that some cyclic subgroup of index p is not an \mathcal{A} -subgroup, i.e., $\langle xc \rangle \notin \mathcal{A}$ for some $x \in P^\#$. Let $X \in \mathcal{S}(\mathcal{A})$ be the basic set containing xc and let $V = \text{rad}(X)$. If $xc \in H_i Q_i$ for some $1 \leq i \leq k$, then $\langle xc \rangle = H_i Q_i$ because $|\langle xc \rangle| = |G|/p$ and $H_i Q_i < G$, and this contradicts that $\langle xc \rangle \notin \mathcal{A}$. Thus $xc \notin H_i Q_i$ for every $1 \leq i \leq k$, and one finds as above that $V \geq C$. If $V = C$, then $\underline{C} \in \mathcal{A}$. The basic set $X/V \in \mathcal{S}(\mathcal{A}_{G/V})$ satisfies $|\text{rad}(X/V)| = 1$. On the other hand, $\mathcal{A}_{G/V} = \mathcal{A}_{G/C} \cong \mathbb{Z}C_p^2$ or $\mathbb{Z}C_p \wr \mathbb{Z}C_p$ by Proposition 4.10 and (3). We conclude that $X = Vx$, and so $\langle xc \rangle = \langle X \rangle \in \mathcal{A}$, which is impossible. Thus $V > C$, and it can be deduced from this in the same way as before that $\mathcal{A} = \mathcal{A}_V \wr \mathcal{A}_{G/V}$, so \mathcal{A} is CI.

To sum up, \mathcal{A} is CI, unless $\langle xc \rangle \in \mathcal{A}$ for every $x \in P^\#$. It is easy to see that this implies $\underline{C} \in \mathcal{A}$ and $\mathcal{A}_{G/C} \cong \mathbb{Z}C_p^2$.

Claim 3. \mathcal{A} is CI.

In view of Claim 2, we may assume that $\underline{C} \in \mathcal{A}$ and $\mathcal{A}_{G/C} \cong \mathbb{Z}C_p^2$. By Proposition 3.2, \mathcal{A} is CI exactly when $\text{Iso}_e(\mathcal{A}) = \text{Aut}(\mathcal{A})_e \text{Aut}(G)$. Let $\varphi \in \text{Iso}_e(\mathcal{A})$. We finish the proof of Claim 3 by finding an automorphism $\alpha \in \text{Aut}(G)$ such that

$$\forall X \in \mathcal{S}(\mathcal{A}) : X^\varphi = X^\alpha. \quad (4)$$

By Proposition 2.4(i), C^φ is a subgroup of G of order $n_{p'}$. Thus $C^\varphi = C$, and the restriction φ_C induces a normalised isomorphism of \mathcal{A}_C , see Proposition 2.4(i). Furthermore, $\varphi^{G/C}$ is a normalised isomorphism of $\mathcal{A}_{G/C}$ defined in Proposition 2.4(iii). Since both \mathcal{A}_C and $\mathcal{A}_{G/C}$ are schurian, these are also CI by the induction hypothesis. Thus there exists $\alpha_1 \in \text{Aut}(C)$ such that

$$\forall X \in \mathcal{S}(\mathcal{A}), X \subseteq C : X^{\varphi_C} = X^{\alpha_1}. \quad (5)$$

Also, there exists $\alpha_2 \in \text{Aut}(G/C)$ such that

$$\forall X \in \mathcal{S}(\mathcal{A}) : (X^{\pi_{G/C}})^{\varphi^{G/C}} = (X^{\pi_{G/C}})^{\alpha_2}. \quad (6)$$

Since $G \cong C \times G/C$, there exists a unique automorphism $\alpha \in \text{Aut}(G)$ such that $\alpha^C = \alpha_1$ and $\alpha^{G/C} = \alpha_2$. We claim that α satisfies the condition in Eq. (4).

If $X \in \mathcal{S}(\mathcal{A})$, $X \subseteq C$, then by Eq. (5), $X^\varphi = X^{\varphi_C} = X^{\alpha_1} = X^\alpha$.

Let $X \in \mathcal{S}(\mathcal{A})$, $X \not\subseteq C$. Since $\mathcal{A}_{G/C} \cong \mathbb{Z}C_p^2$, $X \subseteq Cx$ for some element $x \in P^\#$. Let $U = \langle C, x \rangle$. Then $U = \langle X \rangle C$, showing that $\underline{U} \in \mathcal{A}$. Let P_1 be the minimal \mathcal{A} -subgroup contained in U whose order is divisible by p . By Proposition 4.12, $\mathcal{A}_U = \mathcal{A}_C \star \mathcal{A}_{P_1}$. Moreover, letting $D = P_1 \cap C$, the basic set X can be written in the form

$$X = YDx, \quad Y \in \mathcal{S}(\mathcal{A}), \quad Y \subseteq C. \quad (7)$$

Assume first that $Y = \{e\}$ in Eq. (7), i.e., $X = Dx$. Let $\psi = \varphi\alpha^{-1} \in \text{Sym}(G)$. Then $\psi \in \text{Iso}_e(\mathcal{A})$. Using Proposition 2.4(ii)–(iii) and Eq. (6), we can write

$$(Cx)^\varphi = Cx^\varphi = (Cx)^{\varphi^{G/C}} = (Cx)^{\alpha_2} = (Cx)^\alpha.$$

This shows that $(Cx)^\psi = Cx$ and so $U^\psi = U$. Thus $P_1^\psi \leq U$, and as $|P_1^\psi| = |P_1|$ also holds, $P_1^\psi = P_1$. We conclude

$$(Dx)^\psi = (Cx \cap P_1)^\psi = Cx \cap P_1 = Dx.$$

Equivalently, $(Dx)^\varphi = (Dx)^\alpha$.

Finally, let $Y \neq \{e\}$ in Eq. (7). Then by Eq. (1), $X^\varphi = (YDx)^\varphi = Y^\varphi(Dx)^\varphi = Y^\alpha(Dx)^\alpha = X^\alpha$. This completes the proof of Claim 3 as well as the proof of Theorem 1.1.

6. PRIMITIVE RATIONAL S-RINGS OVER $C_p^2 \times C_q^2$ AND TRANSLATION NETS

Let G be an abelian group and $\exp(G)$ be its exponent. Let $\mathbf{P}(G)$ be the subgroup of $\text{Aut}(G)$ consisting of the power automorphisms

$$\pi_m : x \mapsto x^m, \quad x \in G,$$

where $1 \leq m \leq \exp(G)$ and $\gcd(m, \exp(G)) = 1$.

The *trace* X° of a subset $X \subseteq G$ is defined by $X^\circ = \bigcup_{\pi_m \in \mathbf{P}(G)} X^{\pi_m}$. The cyclotomic S-ring $\text{Cyc}(\mathbf{P}(G), G)$ is also known as the *complete S-ring of traces* over G and denoted by $W(G)$. If \mathcal{A} is an S-ring over G , then its *rational closure* \mathcal{A}° is defined by $\mathcal{A}^\circ = \mathcal{A} \cap W(G)$. The S-ring \mathcal{A} is called *rational* if $\mathcal{A}^\circ = \mathcal{A}$, i.e., $\mathcal{A} \subseteq W(G)$. In this case $X^\circ = X$ holds for every basic set $X \in \mathcal{S}(\mathcal{A})$. We also say that X is rational if $X^\circ = X$.

Lemma 6.1. (i) *Let \mathcal{A} be an S-ring over the abelian group G and $X \in \mathcal{S}(\mathcal{A})$ be a basic set. If X contains elements of coprime orders, then X is rational.*

(ii) Let G be an abelian group whose order is divisible by at least two distinct primes and let \mathcal{A} be an S-ring over G . If \mathcal{A}° is of rank 2, then so is \mathcal{A} .

Proof. (i): Assume that $x_1, x_2 \in X$ have coprime orders. Then we can write G as a direct product $G = G_1 \times G_2$, where $x_1 \in G_1$, $x_2 \in G_2$ and $\gcd(|G_1|, |G_2|) = 1$. Let m be an integer such that $\gcd(m, \exp(G)) = 1$. We have to show that $X^{\pi_m} = X$. By the Chinese remainder theorem we can find m_1 and m_2 satisfying

$$\begin{aligned} m_1 &\equiv 1 \pmod{\exp(G_1)}, & m_1 &\equiv m \pmod{\exp(G_2)}, \\ m_2 &\equiv m \pmod{\exp(G_1)}, & m_2 &\equiv 1 \pmod{\exp(G_2)}. \end{aligned}$$

Then $m_1 m_2 \equiv m \pmod{\exp(G)}$. By Proposition 2.3(i) we have that $X^{\pi_{m_1}}, X^{\pi_{m_2}} \in \mathcal{S}(\mathcal{A})$. Since $x_1 \in X \cap X^{\pi_{m_1}}$ and $x_2 \in X \cap X^{\pi_{m_2}}$ we obtain $X^{\pi_{m_1}} = X^{\pi_{m_2}} = X$, hence $X^{\pi_m} = X$ as well.

(ii): Let $\{e\} \neq X \in \mathcal{S}(\mathcal{A})$. Then $X^\circ = G^\#$, hence X contains elements of every prime order occurring in G . By (i), it follows that $X = G^\#$. \square

For the next lemma we define a particular subgroup of $\mathbf{P}(G)$. If p is a prime divisor of $|G|$, then let

$$\mathbf{P}_p(G) = \{\pi_m \in \mathbf{P}(G) : m \equiv 1 \pmod{\exp(G)_{p'}}\}.$$

Lemma 6.2. Let G be an abelian group with Sylow p -subgroup $G_p \cong C_p^2$ and assume that $G_p \neq G$. Let \mathcal{A} be a primitive S-ring over G , $X \in \mathcal{S}(\mathcal{A})$ a $\mathbf{P}_p(G)$ -invariant basic set and $x \in G^\#$ a p' -element. Then $X \cap G_p x$ is one of the following sets: \emptyset , Rx or $(G_p \setminus R)x$ for a subgroup $R \leq G_p$ of order p , G_p .

Proof. Consider the set $X^{[p]}$. It is contained in $G_{p'}$, the Hall p' -subgroup of G , and by Proposition 2.3(ii), it is an \mathcal{A} -set. If $p \nmid |X \cap G_p x|$, then $\langle X^{[p]} \rangle$ is a non-trivial proper \mathcal{A} -subgroup. But this is impossible as \mathcal{A} is primitive, hence $p \mid |X \cap G_p x|$. Now $X \cap G_p x$ is mapped to itself by every automorphism in $\mathbf{P}_p(G)$, hence $X \cap G_p x = (X \cap \{x\}) \cup \bigcup_{i=1}^m R_i^\# x$ with some p -element subgroups $R_1, \dots, R_m \leq G_p$ ($0 \leq m \leq p+1$). Thus $|X \cap G_p x| = f + m(p-1)$, where $f \in \{0, 1\}$. As $p \mid |X \cap G_p x|$, we obtain that $(f, m) = (0, 0), (1, 1), (0, p)$ or $(1, p+1)$, and the result follows. \square

We analyse rational S-rings over $G \cong C_{p_1}^2 \times \dots \times C_{p_k}^2$, where p_1, \dots, p_k are pairwise distinct primes. Clearly,

$$W(G) = W(G_{p_1}) \otimes \dots \otimes W(G_{p_k}),$$

where G_{p_i} is the Sylow p_i -subgroup of G . The basic sets of $W(G_{p_i})$ distinct from $\{e\}$ are in one-to-one correspondence with the p_i+1 proper non-trivial subgroups of G_{p_i} , denoted by $L_{i,1}, \dots, L_{i,p_i+1}$. The basic set corresponding to $L_{i,j}$ is $X_{i,j} := L_{i,j}^\#$. Furthermore, we set the notation $X_{i,0}$ for the basic set $\{e\}$. Writing $[0, n]$ for $\{0, 1, \dots, n\}$, we obtain

$$\mathcal{S}(W(G)) = \left\{ \prod_{i=1}^k X_{i,t_i} : (t_1, \dots, t_k) \in [0, p_1+1] \times \dots \times [0, p_k+1] \right\}.$$

This shows that $W(G)$ is of rank $\prod_{i=1}^k (p_i+2)$.

Let now \mathcal{A} be an arbitrary rational S-ring over G , i.e., $\mathcal{A} \subseteq W(G)$. Every basic set of \mathcal{A} is a union of basic sets of $W(G)$, and for this reason, it is encoded by a non-empty subset of $[0, p_1+1] \times \dots \times [0, p_k+1]$. More precisely, if $T \subseteq [0, p_1+1] \times \dots \times [0, p_k+1]$, then the corresponding basic set X is given as

$$X = \bigcup_{(t_1, \dots, t_k) \in T} \prod_{i=1}^k X_{i,t_i}. \quad (8)$$

Given an arbitrary subset $T \subseteq [0, p_1 + 1] \times \cdots \times [0, p_k + 1]$, a $(k - 1)$ -tuple $a \in \mathbb{Z}^{k-1}$ and a number $1 \leq i \leq k$, define the subset $T_i(a) \subseteq [0, p_i + 1]$ as follows:

$$T_i(a) = \{t_i : \exists t = (t_1, \dots, t_k) \in T \text{ such that } (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k) = a\}.$$

Lemma 6.3. *With the notation as above, suppose that \mathcal{A} is primitive, $k \geq 2$ and $T \subseteq [0, p_1 + 1] \times \cdots \times [0, p_k + 1]$ corresponds to a basic set of \mathcal{A} . Then for each non-zero $(k - 1)$ -tuple $a \in \mathbb{Z}^{k-1}$ and $1 \leq i \leq k$,*

$$T_i(a) = \emptyset \text{ or } \{0, \ell\} \text{ or } [0, p_i + 1] \setminus \{0, \ell\} \text{ or } [0, p_i + 1]$$

for some $1 \leq \ell \leq p_i + 1$.

Proof. Suppose that $T_i(a) \neq \emptyset$. It follows from Eq. (8) that $a_j \in [0, p_j + 1]$ if $1 \leq j < i$ and $a_j \in [0, p_{j+1} + 1]$ if $i \leq j \leq k - 1$. Let X be the basic set corresponding to T . Then

$$X \cap G_{p_i} \prod_{j=1}^{i-1} X_{j,a_j} \prod_{j=i}^{k-1} X_{j+1,a_j} = \bigcup_{\ell \in T_i(a)} \left(X_{i,\ell} \prod_{j=1}^{i-1} X_{j,a_j} \prod_{j=i}^{k-1} X_{j+1,a_j} \right).$$

Thus, for a fixed $x \in \prod_{j=1}^{i-1} X_{j,a_j} \prod_{j=i}^{k-1} X_{j+1,a_j}$, $X \cap G_{p_i} x = \left(\bigcup_{j \in T_i(a)} X_{i,j} \right) x$. Then $x \neq e$ because at least one of the entries a_j is non-zero, and Lemma 6.2 can be applied to $X \cap G_{p_i} x$. We conclude that, either there exists a subgroup $R \leq G_{p_i}$ of order p_i such that $\bigcup_{j \in T_i(a)} X_{i,j} = R$ or $G_{p_i} \setminus R$, or else $X_{i,j} = G_{p_i}$ (recall that, $T_i(a) \neq \emptyset$ was assumed at the beginning of the proof). \square

The next theorem is the main result of this section, which is of independent interest.

Theorem 6.4. *Suppose that $G \cong C_p^2 \times C_q^2$ for distinct odd primes p, q and \mathcal{A} is a primitive rational S -ring over G of rank at least 3. Then \mathcal{A} has a basic set of the form*

$$H_1^\# \cup \cdots \cup H_m^\#,$$

where each $H_i \leq G$ is of order pq and $H_i \cap H_j = \{e\}$ if $i \neq j$.

Proof. We consider the partition of $G^\#$ into the basic sets of \mathcal{A} distinct from $\{e\}$. Denote the latter basic sets by X, Y, Z , etc. We imagine such partition as a $(p + 1) \times (q + 1)$ matrix M filled up with the letters X, Y, Z , etc. More precisely, let $M_{i,j} = X$ if and only if $(i, j) \in [1, p + 1] \times [1, q + 1]$ belongs to the subset of $[0, p + 1] \times [0, q + 1]$ corresponding to X . Here and in what follows, we use the description of the basic sets of \mathcal{A} established in Eq. (8) with abbreviation $p_1 = p$ and $p_2 = q$.

Notice that, Lemma 6.3 implies that the subsets of $[0, p + 1] \times [0, q + 1]$ corresponding to the basic sets X, Y, Z , etc. are determined uniquely by M (hence as well as \mathcal{A}). We will freely use symmetries arising by permuting the rows, the columns, the letters, and transposing the matrix.

Suppose that $M_{i,j} = X$ and $M_{i,j'} \neq X$ if $j' \neq j$. Let $T \subseteq [0, p + 1] \times [0, q + 1]$ be the subset corresponding to X . Then $T_2(i) = \{0, j\}$ by Lemma 6.3. In particular, $(i, 0) \in T$, and so $X_{1,i} X_{2,0} = L_{1,i}^\# \subseteq X$. This shows that X is the only letter in the i^{th} row occurring exactly once. Applying Lemma 6.3 again, one concludes that each row and each column is filled up with either a single letter or with the same letter with one exception. We will call the letter that occurs there with at most one exception the *dominant letter* of the row or column.

Assume to the contrary to the claim in the theorem that no basic set of \mathcal{A} is the union of pairwise disjoint subgroups of order pq without the identity element. This means that every letter is dominant in at least one row or column. As $\text{rk}(\mathcal{A}) \geq 3$, there are at least two letters. Moreover, primitivity implies that every letter occurs in at least two rows and in at least two columns.

There cannot be three different dominant letters of rows, since in that case there would be a column containing three different letters. As $p, q > 2$, we may assume w.l.o.g. that the number of rows is at least 6. Thus, there are 3 rows with the same dominant letter, say, X , hence we see

that with at most one exception the dominant letter of the columns is X . Now, since the number of columns is at least 4, we also conclude that with at most one exception the dominant letter of the rows is X .

We show next that there cannot be three different letters. W.l.o.g. the first p rows are dominated by X , the last row by Y , the first q columns by X , and the last column by Z . The letter Y should appear somewhere in the first p rows (as it was pointed out, this follows from primitivity). Since no column is dominated by Y , this is the only Y in its column. Hence up to permuting the rows and columns, the matrix M must look like this:

$$M = \begin{pmatrix} X & X & \dots & X & X & Y \\ X & X & \dots & X & X & Z \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X & X & \dots & X & X & Z \\ Y & Y & \dots & Y & Y & Z \end{pmatrix}.$$

Then Z occurs only in the last column, but primitivity does not allow this. So there are exactly two letters, X and Y .

Case 1. Both the last row and column are dominated by Y and all others by X .

Subcase 1A. If the entry in the lower right corner is X , then the matrix is uniquely determined:

$$M = M_1 := \begin{pmatrix} X & X & \dots & X & X & Y \\ X & X & \dots & X & X & Y \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X & X & \dots & X & X & Y \\ Y & Y & \dots & Y & Y & X \end{pmatrix}.$$

Subcase 1B. If the entry in the lower right corner is Y , then up to permuting rows and columns the matrix looks like this:

$$M = \begin{pmatrix} ? & X & \dots & X & X & ? \\ X & X & \dots & X & X & Y \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X & X & \dots & X & X & Y \\ ? & Y & \dots & Y & Y & Y \end{pmatrix}$$

where each question mark should be replaced by X or Y in such a way that in the first row, as well as in the first column, there can be at most one Y . That allows the following possibilities for the corner elements of the matrix (up to transposition):

$$M_2 := \begin{pmatrix} X & \dots & X \\ \vdots & & \vdots \\ X & \dots & Y \end{pmatrix}, \quad M_3 := \begin{pmatrix} X & \dots & Y \\ \vdots & & \vdots \\ X & \dots & Y \end{pmatrix}, \quad M_4 := \begin{pmatrix} X & \dots & Y \\ \vdots & & \vdots \\ Y & \dots & Y \end{pmatrix}, \quad M_5 := \begin{pmatrix} Y & \dots & X \\ \vdots & & \vdots \\ X & \dots & Y \end{pmatrix}.$$

Case 2. The last row is dominated by Y and all other rows as well as every column is dominated by X .

By primitivity, Y must occur somewhere in the first p rows. No two Y 's can be in the same column, because each one is dominated by X . Hence up to permutations, we have a unique possibility:

$$M = M_6 := \begin{pmatrix} X & X & \dots & X & X & Y \\ X & X & \dots & X & X & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X & X & \dots & X & X & X \\ Y & Y & \dots & Y & Y & X \end{pmatrix}.$$

What is left to show is that none of the matrices M_1, \dots, M_6 defines a primitive S-ring (of rank 3). Let P and Q be the Sylow p - and q -subgroups of G , respectively.

We start with the matrix M_1 . Let $T \subset [0, p+1] \times [0, q+1]$ be the subset corresponding to Y . Using Lemma 6.3, we find

$$T = \{(i, q+1), (i, 0) : 1 \leq i \leq p\} \cup \{(p+1, j), (0, j) : 1 \leq j \leq q\}.$$

Then due to Eq. (8), $Y = L_{1,p+1}(Q \setminus L_{2,q+1}) \cup (P \setminus L_{1,p+1})L_{2,q+1}$. It can be seen easily that $Yx = Y$ for every $x \in L_{1,p+1}L_{2,q+1}$. Thus $\text{rad}(Y)$ is a non-trivial \mathcal{A} -subgroup, contradicting that \mathcal{A} is primitive.

Similarly, if $M = M_3, M_4$, or M_6 we can find a non-trivial \mathcal{A} -subgroup contradicting the primitivity of \mathcal{A} . Namely, if $M = M_3$, then $Y = L_{1,p+1}(Q \setminus L_{2,1}) \cup (P \setminus L_{1,p+1})L_{2,q+1}$, hence $\text{rad}(Y) \geq L_{1,p+1}$. If $M = M_4$, then $X = (P \setminus L_{1,p+1})(Q \setminus L_{2,q+1})$, so $\text{rad}(X) \geq L_{1,p+1}L_{2,q+1}$. If $M = M_6$, then $Y = Q^\# \cup L_{1,1}^\#L_{2,q+1} \cup L_{1,p+1}^\#(Q \setminus L_{2,q+1})$, hence $\text{rad}(Y \cup \{e\}) \geq L_{2,q+1}$.

Let $M = M_2$. Then one obtains $Y = A \cup B \cup C$, where

$$A = (P \setminus (L_{1,1} \cup L_{1,p+1}))L_{2,q+1}, \quad B = L_{1,p+1}(Q \setminus (L_{2,1} \cup L_{2,q+1})) \quad \text{and} \quad C = L_{1,p+1}^\#L_{2,q+1}^\#. \quad (9)$$

Write $(\underline{Y})^2 = \sum_{g \in G} c_g g$. A straightforward computation shows that $c_h = p(q-2)^2 + 2(p-1)(q-2)$ for $h \in Q \setminus (L_{2,1} \cup L_{2,q+1})$, and $c_{h'} = (p-2)^2q + 2(p-2)(q-1)$ for $h \in P \setminus (L_{1,1} \cup L_{1,p+1})$. Since Y is a basic set of \mathcal{A} and $h, h' \in Y$, $c_h = c_{h'}$ must hold. This gives that $(q-p)(pq-2) = 0$, a contradiction.

Finally, let $M = M_5$. Then $Y = A \cup B \cup C \cup (L_{1,1}L_{2,1})^\#$, where A, B and C are defined in (9). In this case write $(\underline{Y})^2 = \sum_{g \in G} \tilde{c}_g g$. Comparing the coefficients with the previous case ($M = M_2$) we see that $\tilde{c}_h = c_h + 2(q-2)$ for $h \in Q \setminus (L_{2,1} \cup L_{2,q+1})$, and $\tilde{c}_{h'} = c_{h'} + 2(p-2)$ for $h \in P \setminus (L_{1,1} \cup L_{1,p+1})$. As $\tilde{c}_h = \tilde{c}_{h'}$ must hold, we obtain $(q-p)pq = 0$, a contradiction again. \square

The particular case when $|G| = 36$ is considered separately and the proposition below follows from the database of S-rings over groups of small order due to Reichard [26].

Proposition 6.5. *Suppose that $G \cong C_2^2 \times C_3^2$ and \mathcal{A} is a primitive rational schurian S-ring over G of rank at least 3. Then \mathcal{A} has a basic set in the form $H_1^\# \cup H_2^\#$, where H_1 and H_2 are subgroups of G of order 6 and $H_1 \cap H_2 = \{e\}$.*

The subgroups H_i 's above and as well as in Theorem 6.4 can be used to define a translation net with translation group G_R (see [4, Definition 1]) and this connection will be explored in the proof of Theorem 1.2. The rest of the section is devoted to translation nets.

An (n, k) -net $\mathcal{N} = (\Omega, \mathcal{L})$ consists of a set Ω of n^2 points and a set \mathcal{L} of kn lines such that

- (1) each line $L \in \mathcal{L}$ contains n points,
- (2) \mathcal{L} is partitioned into k parallel classes: $\mathcal{L}_1, \dots, \mathcal{L}_k$,
- (3) any two lines from distinct parallel classes intersect at exactly one point.

The *collinearity graph* $\Gamma_{\mathcal{N}}$ has vertex set Ω , and two points α and β are adjacent if and only if there is a line $L \in \mathcal{L}$ passing through these points. $\Gamma_{\mathcal{N}}$ is a strongly regular graph with parameters $(n^2, k(n-1), n-2 + (k-1)(k-2), k(k-1))$ and non-principal eigenvalues $n-k$ and $-k$.

Following [4], a *weak automorphism* of \mathcal{N} is a permutation of Ω , which preserves the line set \mathcal{L} . By a *strong automorphism* we mean a weak automorphism when, in addition, it also preserves each parallel class \mathcal{L}_i . If \mathcal{N} admits a group H of strong automorphisms acting regularly on Ω , then it is called a *translation net* with *translation group* H .

One way to construct an (n, k) -net is the following. Let H be a group of order n^2 . A *partial congruence partition* of H with degree k (an (n, k) -PCP for short) is a family of k subgroups H_1, \dots, H_k of order n such that $H_i \cap H_j = \{e\}$ whenever $i \neq j$. Letting $\Omega = H$ and \mathcal{L} to

be the set of all right cosets $H_i x$, $1 \leq i \leq k$ and $x \in H$, the pair (Ω, \mathcal{L}) becomes an (n, k) -net whose i -th parallel class \mathcal{L}_i consists of the cosets $H_i x$, $x \in H$. Note that, the collinearity graph is $\Gamma_{(\Omega, \mathcal{L})} = \text{Cay}(H, D)$, where $D = \bigcup_{i=1}^k H_i^\#$. Furthermore, (Ω, \mathcal{L}) is a translation net with translation group H_R .

Proposition 6.6. *Let \mathcal{N} be an (n, k) -net such that $n > (k - 1)^2$. Then the size of a clique of the collinearity graph $\Gamma_{\mathcal{N}}$ is bounded by n , and the lines of \mathcal{L} are the only n -cliques of $\Gamma_{\mathcal{N}}$.*

Proof. Let Δ be a clique not contained in any line. We will show that $|\Delta| \leq (k - 1)^2$. Choose a line L such that $m = |\Delta \cap L|$ is as large as possible. As $\Delta \not\subseteq L$, we can take a point $\delta \in \Delta \setminus L$. Let L_0 be the line through δ parallel to L , and let L_1, \dots, L_m be the lines connecting δ to the points in $\Delta \cap L$. Then L_0, L_1, \dots, L_m are pairwise distinct, so $m + 1 \leq k$. Each point in Δ is connected to δ , hence we obtain $|\Delta| \leq 1 + k(m - 1) \leq (k - 1)^2$. \square

Proposition 6.7. *Let $\mathcal{N} = (\Omega, \mathcal{L})$ be an (n, k) -net such that $k < n$ and H be an abelian group of weak automorphisms of \mathcal{N} , which is regular on Ω . Then every element in H is a strong automorphism.*

Proof. Note, first, that $|H| = |\Omega| = n^2$. Let $L \in \mathcal{L}$ and $\mathcal{O} = \{L^h : h \in H\}$ be the orbit of L under H in its action on \mathcal{L} . It follows from $|\mathcal{O}| \leq |\mathcal{L}| = nk < n^2 = |H|$ that the setwise stabilizer $H_{\{L\}} = \{h \in H : L^h = L\}$ is non-trivial. Since H is abelian, $H_{\{L\}} = H_{\{L^h\}}$ for every $h \in H$. In particular, the intersection $L \cap L^h$ is mapped to itself by $H_{\{L\}}$. Using also that H is semiregular on Ω , this implies that $|L \cap L^h|$ is divisible by $|H_{\{L\}}|$. If $L \neq L^h$, then $|L \cap L^h| \in \{0, 1\}$, and we conclude that the lines in \mathcal{O} are pairwise disjoint. Therefore, $|H/H_{\{L\}}| = |\mathcal{O}| \leq n$, implying that $|H_{\{L\}}| \geq n$. On the other hand, $|H_{\{L\}}| \leq |L| = n$. Therefore, $|H_{\{L\}}| = n = |\mathcal{O}|$ and, consequently, \mathcal{O} is a parallel class of \mathcal{L} . \square

We conclude the section with a sufficient condition for the CI-property of an S-ring over $C_p^2 \times C_q^2$.

Lemma 6.8. *Let G be an abelian group of order $p^2 q^2$ for primes $p < q$, $A \in \text{Sup}^{\min}(G_R)$ and $\mathcal{A} = V(G, A_e)$. Suppose that there exists an \mathcal{A} -set of the form*

$$H_1^\# \cup \dots \cup H_k^\#,$$

where H_1, \dots, H_k are subgroups of G and form a (pq, k) -PCP of G . Then $\underline{H}_i \in \mathcal{A}$ for each $1 \leq i \leq k$. If $k > 1$, then \mathcal{A} is CI.

Proof. Denote by \mathcal{N} the induced translation net, i.e., the point set is G and the lines are the cosets $H_i x$, $1 \leq i \leq k$ and $x \in G$. The collinearity graph is $\Gamma_{\mathcal{N}} = \text{Cay}(G, X)$, where $X = \bigcup_{i=1}^k H_i^\#$. Let $\gamma \in \text{Aut}(\mathcal{A})$. Then $\gamma \in \text{Aut}(\Gamma_{\mathcal{N}})$. Now $k \leq p + 1$, and we have $(k - 1)^2 \leq p^2 < pq$. By Lemma 6.6, the lines of \mathcal{N} are the only n -cliques of $\Gamma_{\mathcal{N}}$. Since $\gamma \in \text{Aut}(\Gamma_{\mathcal{N}})$, it follows that γ maps an n -clique to an n -clique, and we conclude that γ is a weak automorphism of \mathcal{N} .

Let F be a regular and abelian subgroup of A . By Lemma 6.7, F is a group of strong automorphisms of \mathcal{N} , or equivalently, the partition of G into its H_i -cosets is F -invariant. Thus $\underline{H}_i \in \mathcal{A}$ follows from the \leq_G -minimality of A . If $k > 1$, then Lemma 4.4 shows that \mathcal{A} is CI. \square

7. PROOF OF THEOREM 1.2

For this section we fix the following notation:

$$G = P \times Q, \text{ where } P \cong C_p^2, Q \cong C_q^2 \text{ for distinct primes } p \text{ and } q, A \in \text{Sup}^{\min}(G_R) \\ \text{and } \mathcal{A} = V(G, A_e).$$

Again, our goal is to show that \mathcal{A} is CI (see Corollary 3.4). In the proof we shall use the following two lemmas.

Lemma 7.1. *Suppose that $\underline{P} \notin \mathcal{A}$ and let $x \in P^\#$ be such that $\langle x \rangle \notin \mathcal{A}$. Then for every basic set $X \in \mathcal{S}(\mathcal{A})$*

- (i) q divides $|X \cap Qx|$.
- (ii) *If X contains an element of order p , then $X \cap Qx$ can be one of the following: \emptyset , Rx or $(Q \setminus R)x$ for some subgroup $R \leq Q$ of order q , or Qx .*

Proof. For (i) assume on the contrary that $|X \cap Qx|$ is not divisible by q . Consider the \mathcal{A} -set $X^{[q]}$ defined in Proposition 2.3(ii). Then $x^q \in X^{[q]}$, hence $\langle X^{[q]} \rangle = \langle x \rangle$ or P , contradicting that none of these subgroups are \mathcal{A} -subgroups.

If X also contains an element of order p , then it is $\mathbf{P}_q(G)$ -invariant and (ii) follows from this and (i) in the same way as in the proof of Lemma 6.2. \square

Lemma 7.2. *Suppose that $\underline{P} \notin \mathcal{A}$, $\underline{Q} \in \mathcal{A}$, and there exists a subgroup $U \leq G$ of order pq^2 such that $\underline{U} \in \mathcal{A}$. Let $X \in \mathcal{S}(\mathcal{A})$ be a basic set with the following properties: there exist an element $x \in P \setminus U$ and a subgroup $R \leq Q$ of order q such that $X \subseteq Ux$, $X \cap Qx = Rx$, $X \neq Rx$, $X \neq (U \cap P)Rx$. Then \mathcal{A} is CI.*

Proof. Due to Proposition 4.10 and (3), $\mathcal{A}_{G/Q} \cong \mathbb{Z}C_p^2$ or $\mathbb{Z}C_p \wr \mathbb{Z}C_p$. Since $X \neq Rx$, the former case is impossible, hence $|X \cap Qxy^i| = q$ for every $0 \leq i \leq p-1$, where y is a generator of $U_p = U \cap P$. Notice that $\langle x \rangle \notin \mathcal{A}$ by Lemma 4.4, and hence Lemma 7.1(ii) can be applied to X . Thus for each $0 \leq i \leq p-1$,

$$X \cap Qxy^i = R_i xy^i \text{ or } (Q \setminus R_i)xy^i, \quad R_i \leq Q \text{ and } |R_i| = q.$$

The subgroup $R_0 = R$ and if $X \cap Qxy^i = (Q \setminus R_i)xy^i$ for some i , then $q = 2$ must hold.

Case 1. For each $0 \leq i \leq p-1$, $X \cap Qxy^i = R_i xy^i$.

Notice that, $X^{[p,k]} \cup \{e\}$ is just the union of those subgroups R_i that occur exactly k times in the union

$$X = R_0 x \cup R_1 xy \cup \cdots \cup R_{p-1} xy^{p-1}.$$

Since $X \neq (U_p)Rx$, it follows that there exists an integer $1 \leq k \leq p-1$ such that $X^{[p,k]}$ is non-empty \mathcal{A} -set, see Proposition 2.3(ii). Hence $X^{[p,k]} \cup \{e\} = \bigcup_{i=1}^r S_r$ with $\{S_1, \dots, S_r\} \subseteq \{R_0, \dots, R_{p-1}\}$. By Proposition 2.3(ii), this is an \mathcal{A} -set. Write $\underline{X} \cdot X^{[p,k]} = \sum_{g \in G} c_g g$ and fix a generator u_i of R_i for each $0 \leq i \leq p-1$. A direct computation shows that $c_{u_i xy^i} = q-1$ or 0 depending on whether $R_i \in \{S_1, \dots, S_r\}$ or not. On the other hand, all of the coefficients $c_{u_i xy^i}$ must be the same, and therefore, $\{S_1, \dots, S_r\} = \{R_0, \dots, R_{p-1}\}$. This means each R_i occurs k times, in particular, k divides p . It follows from $1 \leq k \leq p-1$ that $k = 1$, hence the subgroups R_0, \dots, R_{p-1} are pairwise distinct. Let $H_i = R_i \langle xy^i \rangle$. It is easy to see that the subgroups H_i , $0 \leq i \leq p-1$ form a (pq, p) -PCP of G . Since $X^\circ \cup X^{[p]} = H_0^\# \cup \cdots \cup H_p^\#$ is an \mathcal{A} -set, Lemma 6.8 gives that \mathcal{A} is CI.

Case 2. $q = 2$ and $X \cap Qxy^i = (Q \setminus R_i)xy^i$ for some $1 \leq i \leq p-1$.

We show that this case cannot occur. Assume that $p > 3$. Applying Lemma 3.5 for $L = \{e\}$ and U yields $\underline{U}_p \in \mathcal{A}$. Let $l = |X \cap U_p x|$. As $x \in X$ but $xy^i \notin X$, we have $1 \leq l < p$. Now the coefficient of x in $\underline{X} \underline{U}_p$ is equal to l , hence for any $u \in Q$ for which $X \cap (U_p)xu$ is not empty, we get $|X \cap U_p xu| = l$. Using also that $X \neq U_p Rx$, it follows from this that $|X| = 3l$ or $4l$ since $|Q| = 4$. This contradicts that $|X| = 2p$ and $p > 3$.

If $p = 3$, i.e., $|G| = 36$, then the database of S-rings over groups of small order given in [26] shows that \mathcal{A} does not exist. \square

We focus first on the case when \mathcal{A} is decomposable.

7.1. \mathcal{A} is decomposable. Let \mathcal{A} be a non-trivial $S = U/L$ -wreath product. Since $\{e\} < L \leq U < G$, we have $1 \leq \Omega(|L|) \leq \Omega(|U|) \leq 3$. If $\Omega(|U|) = \Omega(|L|)$, then $U = L$ and \mathcal{A} is CI by Corollary 4.6. So it remains to consider the following cases:

$$(\Omega(|U|), \Omega(|L|)) \in \{(3, 2), (2, 1), (3, 1)\}.$$

Case 1. $(\Omega(|U|), \Omega(|L|)) = (3, 2)$.

In this case $\Omega(|S|) = 1$, hence \mathcal{A}_S is Cayley minimal by Lemma 4.8(i) and (iii). We may assume w.l.o.g. that

$$|G/L| = pq \text{ or } |G/L| = p^2.$$

Let $|G/L| = pq$. As $\underline{S} \in \mathcal{A}_{G/L}$, $\text{rk}(\mathcal{A}_{G/L}) \neq 2$. According to Lemma 4.8(ii) $\mathcal{A}_{G/L}$ is cyclotomic or a non-trivial wreath product of two S-rings. We claim that \mathcal{A} is CI in both cases. Indeed, this follows by Proposition 4.7 if $\mathcal{A}_{G/L}$ is cyclotomic. If $\mathcal{A}_{G/L}$ is a non-trivial wreath product, then $\mathcal{A}_{G/L} = \mathcal{A}_S \wr \mathcal{A}_{(G/L)/S}$. This implies that $\mathcal{A} = \mathcal{A}_U \wr \mathcal{A}_{G/U}$, and so \mathcal{A} is CI by Corollary 4.6.

Now, suppose that $|G/L| = p^2$. By Proposition 4.10, $\mathcal{A}_{G/L}$ is a p -S-ring, in particular, $\mathcal{A}_S = \mathbb{Z}S$, and so \mathcal{A} is CI by Corollary 4.6.

Case 2. $(\Omega(|U|), \Omega(|L|)) = (2, 1)$.

Again, $\Omega(|S|) = 1$ and \mathcal{A}_S is Cayley minimal. We may assume w.l.o.g. that $|L| = p$. Then

$$|U| = pq \text{ or } |U| = p^2.$$

Let $|U| = pq$. As $\underline{L} \in \mathcal{A}$, $\text{rk}(\mathcal{A}_U) \neq 2$. It follows from Lemma 4.8(ii) that \mathcal{A}_U is cyclotomic or a non-trivial wreath product of two S-rings. In the former case \mathcal{A} is CI by Proposition 4.7. In the latter case $\mathcal{A}_U = \mathcal{A}_L \wr \mathcal{A}_{U/L}$, which implies that $\mathcal{A} = \mathcal{A}_L \wr \mathcal{A}_{G/L}$, and so \mathcal{A} is CI by Corollary 4.6.

Now, suppose that $|U| = p^2$, i.e., $U = P$. Note that, S is an $\mathcal{A}_{G/L}$ -subgroup of order p . Denote the maximal q - $\mathcal{A}_{G/L}$ -subgroup of G/L by H . Clearly, $|H| \in \{1, q, q^2\}$.

If $|H| = q^2$, then $\underline{LQ} \in \mathcal{A}$. By Proposition 4.10, $\mathcal{A}_{G/(LQ)} \cong \mathbb{Z}C_p$. This implies that $\mathcal{A}_S = \mathbb{Z}S$, and so \mathcal{A} is CI by Corollary 4.6.

Let $|H| \in \{1, q\}$. By Proposition 4.11, $\mathcal{A}_{G/L}$ is a non-trivial $(HS)/S$ -wreath product. This implies that \mathcal{A} is the $(HS)^{(\pi_{G/L})^{-1}}/U$ -wreath product. One can see that $(\Omega(|(HS)^{(\pi_{G/L})^{-1}}|), \Omega(|U|)) = (2, 2)$ or $(3, 2)$, and hence we are done by Corollary 4.6 or by Case 1, respectively.

Case 3. $(\Omega(|U|), \Omega(|L|)) = (3, 1)$.

In this case $\Omega(|S|) = 2$. We may assume w.l.o.g. that $|U| = p^2q$. By Proposition 4.10, $\mathcal{A}_{G/U} \cong \mathbb{Z}C_q$, and this implies that

$$(\mathcal{A}_{G/L})_{(G/L)/S} \cong \mathbb{Z}C_q. \quad (10)$$

Clearly, $|L| \in \{p, q\}$. Let $|L| = q$. Then $|S| = p^2$. Using this, Eq. (10) and Proposition 4.12, we find that $\mathcal{A}_{G/L} = \mathcal{A}_S \star \mathcal{A}_{Q_1}$, where Q_1 is the least $\mathcal{A}_{G/L}$ -subgroup of G/L of order divisible by q . If $|S \cap Q_1| = 1$, then $\mathcal{A}_{G/L} = \mathcal{A}_S \otimes \mathcal{A}_{Q_1}$. Then \mathcal{A} is CI by Lemma 4.13. If $|S \cap Q_1| > 1$ then $\mathcal{A}_{G/L}$ is the non-trivial $S/(S \cap Q_1)$ -wreath product. Thus \mathcal{A} is the $U/(S \cap Q_1)^{(\pi_{G/L})^{-1}}$ -wreath product. One can see that $(\Omega(|U|), \Omega(|(S \cap Q_1)^{(\pi_{G/L})^{-1}}|)) = (3, 3)$ or $(3, 2)$, and hence we are done by Corollary 4.6 or by Case 1.

Now, suppose that $|L| = p$. Then $|G/L| = pq^2$ and $|S| = pq$. Denote by H the unique maximal q - $\mathcal{A}_{G/L}$ -subgroup of G/L , and by P_1 the least $\mathcal{A}_{G/L}$ -subgroup of order divisible by p . Obviously, $|H| \in \{1, q, q^2\}$.

Let $|H| \in \{1, q\}$. Assume that $H \not\leq S$. Then $|H| = q$ and $|H \cap S| = 1$. So $G/L = H \times S$ and Eq. (10) implies that $\mathcal{A}_H \cong \mathbb{Z}C_q$. Then $\mathcal{A}_{G/L} = \mathcal{A}_S \otimes \mathcal{A}_H$ by Proposition 4.3, and hence \mathcal{A} is CI by Lemma 4.13. Now, let $H \leq S$. Since $|S| = pq$, $P_1 \leq S$. By Proposition 4.11, $\mathcal{A}_{G/L}$ is the

S/P_1 -wreath product. Thus \mathcal{A} is the $U/P_1^{(\pi_{G/L})^{-1}}$ -wreath product. We are done by Corollary 4.6 or Case 1 because $(\Omega(|U|), \Omega(|P_1^{(\pi_{G/L})^{-1}}|)) = (3, 3)$ or $(3, 2)$.

Let $|H| = q^2$, $V = LQ$ and $K = H \cap S$. Then $\underline{V} \in \mathcal{A}$ and $\underline{K} \in \mathcal{A}_{G/L}$. By Proposition 4.10, $\mathcal{A}_{G/V} \cong \mathbb{Z}C_p$, implying that,

$$(\mathcal{A}_{G/L})_{(G/L)/H} \cong \mathbb{Z}C_p. \quad (11)$$

Assume for the moment that H contains an \mathcal{A}_H -subgroup K' of order q such that $K' \neq K$. It follows from Eq. (10) that $(\mathcal{A}_{G/L})_{H/K} \cong \mathbb{Z}C_q$, this implies that $\mathcal{A}_{K'} = \mathbb{Z}K'$. Then $\mathcal{A}_{G/L} = \mathcal{A}_S \otimes \mathcal{A}_{K'}$ by Proposition 4.3, and so \mathcal{A} is CI by Lemma 4.13.

From now on K is assumed to be the only \mathcal{A} -subgroup of H of order q . The S-ring $\mathcal{A}_{G/L} = \mathcal{A}_H \star \mathcal{A}_{P_1}$ by Proposition 4.12 and Eq. (11).

Let $|H \cap P_1| = 1$ then $\underline{P} \in \mathcal{A}$, and so $\mathcal{A}_{G/P}$ is a q -S-ring by Proposition 4.10. This implies that $\mathcal{A}_K = \mathbb{Z}K$. On the other hand, $\mathcal{A}_{P_1} = \mathbb{Z}P_1$ follows from Eq. (11), and we conclude that $\mathcal{A}_S = \mathcal{A}_{P_1K} = \mathbb{Z}S$, and so that \mathcal{A} is CI by Corollary 4.6.

Now, suppose that $|H \cap P_1| > 1$. Then $P_1 = S$ and it follows that $\mathcal{A}_{G/L}$ is the H/K -wreath product. By Proposition 4.9, $\mathcal{A}_H = \mathcal{A}_K \wr \mathcal{A}_{H/K}$, and we conclude that $\mathcal{A}_{G/L} = \mathcal{A}_K \wr \mathcal{A}_{(G/L)/K}$. Thus \mathcal{A} is the $U/K^{(\pi_{G/L})^{-1}}$ -wreath product, and we are done by Case 1 because $(\Omega(|U|), \Omega(|K^{(\pi_{G/L})^{-1}}|)) = (3, 2)$. By this we have considered all cases and shown that \mathcal{A} is CI.

7.2. \mathcal{A} is indecomposable. Assume first that \mathcal{A} is primitive. If $\text{rk}(\mathcal{A}) = 2$, then \mathcal{A} is CI, hence let $\text{rk}(\mathcal{A}) > 2$. If p or q is equal to 2 and $|G| > 36$, then by Lemma 3.5 (choose $L = \{e\}$ and $U = G$), $\underline{P} \in \mathcal{A}$ if $q = 2$ and $\underline{Q} \in \mathcal{A}$ if $p = 2$. Hence either $p, q > 2$ or $|G| = 36$. Since $\text{rk}(\mathcal{A}) > 2$, $\text{rk}(\mathcal{A}^\circ) > 2$ as well, see Lemma 6.1. Also, \mathcal{A}° is primitive and as $\mathcal{A}^\circ = \mathcal{A} \cap W(G)$, it is also schurian. Due to Theorem 6.4 and Proposition 6.5, there exists a basic set of \mathcal{A}° of the form $H_1^\# \cup \dots \cup H_k^\#$, where H_1, \dots, H_k are subgroups of G and form an (pq, k) -PCP of G . By Lemma 6.8, each H_i is an \mathcal{A} -subgroup, a contradiction.

Now let \mathcal{A} be imprimitive and U be a proper \mathcal{A} -subgroup of maximal order.

Claim. $|G/U|$ is a prime.

If $|G/U|$ is a prime power, then $\mathcal{A}_{G/U}$ is a primitive p -S-ring, implying that $|G/U|$ is a prime. If $|G/U| = pq$, then by Lemma 3.5, UR is an \mathcal{A} -subgroup, where $R \in \{P, Q\}$ is a Sylow $\max(p, q)$ -subgroup of G . This contradicts the maximality of U .

Thus we have to consider a unique case: U is of prime order. W.l.o.g. $|U| = q$. Suppose there exists a proper \mathcal{A} -subgroup $V \neq U$. Then $U \cap V = 1$ and $UV = G$, and hence $|V| = |G|/|U| > |U|$, contrary to the maximal choice of U . Thus we assume that U is the unique non-trivial proper \mathcal{A} -subgroup. The quotient $\mathcal{A}_{G/U}$ is a primitive S-ring over an abelian group of order p^2q . By Proposition 2.2 it has rank 2. Therefore $TU = G \setminus U$ holds for each basic set T outside $G \setminus U$. It follows from Lemma 4.14 that for each $u \in U$ the singleton $\{u\}$ is a basic set of \mathcal{A} . Therefore Tu is a basic set of \mathcal{A} . Thus either $U \leq \text{rad}(T)$ or $T \cap Tu = \emptyset$ for each $u \in U^\#$. In the first case \mathcal{A} is the wreath product $\mathcal{A}_U \wr \mathcal{A}_{G/U}$, contradicting that \mathcal{A} is indecomposable. In the second case, T, Tu, \dots, Tu^{q-1} are the only basic sets outside U . Therefore $|T| = p^2q - 1$.

If $T \cap Q = \emptyset$, then $Tu \cap Q = \emptyset$ for all $u \in U$. Therefore, every basic set Tu contains q -elements. It follows from $T \cup Tu \cup \dots \cup Tu^{q-1} = G \setminus U$ that at least one of the sets Tu^i contains some p -element. W.l.o.g. $T \cap P \neq \emptyset$. Combining this with $T \cap Q \neq \emptyset$ we conclude that T is rational. Write

$$T = T_0 \cup P_1^\# S_1 \cup \dots \cup P_{p+1}^\# S_{p+1},$$

where P_1, \dots, P_{p+1} are all subgroups of P of order p , $T_0 := T \cap Q$ and $S_i \subseteq Q$, $1 \leq i \leq p+1$ are rational sets. It follows from $TU = G \setminus U$ that each S_i is a transversal of Q/U . Therefore $|S_i| = q$ and S_i is a subgroup of Q of order q such that $S_i \neq U$. Now, it follows from $|T| = |T_0| + (p+1)(p-1)q$ that $|T_0| = q - 1$. Since T_0 is rational, the subset $S_0 := T_0 \cup \{e\}$ is a subgroup of Q distinct from U . Note that the subgroups $S_i < Q$, $0 \leq i \leq p+1$ are not necessarily distinct. Let us choose the indices so that S_0, \dots, S_k is the complete set of distinct subgroups that appear in the list S_0, \dots, S_{p+1} . Denote by n_i the multiplicity with which S_i , $0 \leq i \leq k$ appears in the above list. Then $\sum_{i=0}^k n_i = p+2$.

We finish the proof of the claim by finding a non-trivial proper \mathcal{A} -subgroup distinct from U , which was excluded above. If $0 < i \leq k$ and $n_i \neq p$, then the set $S_i^\#$ is contained in $T^{[p]}$. Therefore, if $k \neq 0$ and $(k, n_0, n_1) \neq (1, 2, p)$, then $T^{[p]}$ is a non-empty subset of Q , which intersects U trivially, and so $\langle T^{[p]} \rangle$ is the required \mathcal{A} -subgroup. If $k = 0$, then $T^{[p]} = \{e\}$ and $T = (PS_0)^\#$, showing that PS_0 is an \mathcal{A} -subgroup of order p^2q . Finally, if $(k, n_0, n_1) = (1, 2, p)$, then $T^{[p]} = \{e\}$ and $T = (P_j S_0)^\# \cup (P \setminus P_j)S_1$ for some $1 < j \leq p+1$. Then $P_j \leq \text{rad}(T \cup \{e\}) < G$, hence $\text{rad}(T \cup \{e\})$ is the required \mathcal{A} -subgroup. The claim is proved.

Assume w.l.o.g. that $|U| = pq^2$. For the rest of the proof fix an element $x \in P \setminus U$. By Proposition 4.10, $\mathcal{A}_{G/U} \cong \mathbb{Z}C_p$, and this shows that Ux is an \mathcal{A} -set. Let I be the intersection of all subgroups $\text{rad}(X)$, $X \in \mathcal{S}(\mathcal{A})$ and $X \subseteq Ux$. Then $\underline{I} \in \mathcal{A}$. Furthermore, it follows from Proposition 2.3(i) that $I \leq \text{rad}(X)$ for any basic set X outside U , and we find that \mathcal{A} is the U/I -wreath product. As \mathcal{A} is indecomposable, $I = \{e\}$. We have shown the following:

$$\bigcap_{\substack{X \in \mathcal{S}(\mathcal{A}) \\ X \subseteq Ux}} \text{rad}(X) = \{e\}. \quad (12)$$

Notice that $\langle x' \rangle \notin \mathcal{A}$ can also be assumed for each $x' \in P \setminus U$, otherwise \mathcal{A} would be CI by Lemma 4.4.

Case 1. $\underline{P} \notin \mathcal{A}$.

Let y be a generator of $U_p = U \cap P$. For $0 \leq i \leq p-1$, let X_i be the basic set containing xy^i . By Lemma 7.1(ii),

$$X_i \cap Qxy^i = R_i xy^i$$

for some non-trivial subgroup $R_i \leq Q$. Note that the sets X_i are not necessarily distinct. In view of Eq. (12), we may assume w.l.o.g. that $|R_0| = q$.

Fix $0 \leq i \leq p-1$. We claim that every basic set $X \in \mathcal{S}(\mathcal{A})$ satisfies

$$X \cap Qxy^i \neq \emptyset, \underline{R}_i \in \mathcal{A} \text{ and } |R_i| = q \implies R_i \leq \text{rad}(X). \quad (13)$$

Indeed, $\{u\} \in \mathcal{S}(\mathcal{A})$ for every $u \in R_i$ because of Lemma 4.14, hence the right multiplications $\rho_G(u)$, $u \in R_i$ map the basic sets of \mathcal{A} having non-empty intersection with Qxy^i to themselves. Lemma 7.1(i) implies that there are at most q such basic sets. Using this and that $X_i R_i = X_i$, we conclude that $X R_i = X$, i.e., (13) holds.

Assume first that $X_0 = U_p R_0 x$. Then $X_i = X_0$ and $R_i = R_0$ for each $1 \leq i \leq p-1$. It follows from Eq. (12) that $U_p \not\leq \text{rad}(X)$ for some basic set $X \subset Ux$. Then $S := \langle X^{[p]} \rangle$ is a non-trivial \mathcal{A} -subgroup contained in Q and distinct from R_0 . If $|S| = q$, then $\{u\} \in \mathcal{S}(\mathcal{A})$ for every $u \in S$ because of Lemma 4.14, and we find that the sets $X_0 u = U_p R_0 x u$, $u \in S$ are the basic sets contained in Ux . This contradicts Eq. (12). Let $S = Q$. Then $R_0 = \text{rad}(X_0) \cap S$, and so $\underline{R}_0 \in \mathcal{A}$. Then (13) implies that $R_0 \leq \text{rad}(X)$ for every basic set $X \subset Ux$, a contradiction to Eq. (12).

Assume second that $X_0 \neq U_p R_0 x$ and $\underline{Q} \in \mathcal{A}$. By Lemma 7.2, we may assume that $X_0 = R_0 x$. Then $\underline{R}_0 \in \mathcal{A}$ and $\mathcal{A}_{G/Q} \cong \mathbb{Z}C_p^2$ by Proposition 4.10. Thus for every $0 \leq i \leq p-1$, $X_i = R_i xy^i$,

and hence $\underline{R}_i \in \mathcal{A}$. It follows from Eq. (12) and the implication in (13) that there exists some $1 \leq i \leq p-1$ such that $|R_i| = q$ and $R_i \neq R_0$. But then $\langle X_0 \rangle = \langle R_0x \rangle$ and $\langle X_i \rangle = \langle R_ixy^i \rangle$ are \mathcal{A} -subgroups intersecting trivially, and hence \mathcal{A} is CI by Lemma 4.4.

We are left with the case when $X_0 \neq U_p R_0x$ and $\underline{Q} \notin \mathcal{A}$. We show that $\underline{R}_0 \in \mathcal{A}$. Assume on the contrary that $\underline{R}_0 \notin \mathcal{A}$. As neither \underline{Q} belongs to \mathcal{A} , $X^{[p]}$ contains no element from $R_0^\#$, and hence $R_0^\#xy^i \subseteq X \cap Qxy^i$ if $1 \leq i \leq p-1$. Using also Lemma 7.1, we find that the latter intersection is equal to R_0xy^i , Qxy^i or $(Q \setminus R)xy^i$ for some subgroup $R < Q$ such that $|R| = q$ and $R \neq R_0$. Furthermore, as $X \neq U_p R_0x$, there exists some $1 \leq i \leq p-1$ such that one of the last two possibilities holds. If $X \cap Qxy^i = Qxy^i$, then $(Q \setminus R_0) \subseteq X^{[p]}$, hence $\langle X^{[p]} \rangle = Q$, a contradiction. Suppose that $X \cap Qxy^i = (Q \setminus R)x$. Then $Q \setminus (R \cup R_0) \subseteq X^{[p]}$, which implies that $q = 2$. If $p > 3$, then Lemma 3.5 applied to \mathcal{A} , where $|G/\{e\}| = 4p^2$, we find that $\underline{P} \in \mathcal{A}$, contradicting our assumption $\underline{P} \notin \mathcal{A}$. If $p = 3$, then $|G| = 36$, and it follows from the database of S-rings of small order in [26] that such an \mathcal{A} cannot exist.

Note that, the above proof also shows that, for any $1 \leq i \leq p-1$, $\underline{R}_i \in \mathcal{A}$ whenever $|R_i| = q$. Now, since $\underline{Q} \notin \mathcal{A}$, R_0 is the only non-trivial \mathcal{A} -subgroup contained in \underline{Q} , and therefore, $R_i = R_0$ or Q for each $1 \leq i \leq p-1$. Then by (13), $R_0 \leq \text{rad}(X)$ for each basic set $X \subseteq Ux$, a contradiction to Eq. (12).

Case 2. $\underline{P} \in \mathcal{A}$.

By Proposition 4.10, there exists an \mathcal{A} -subgroup V of order p^2q . If now $\underline{Q} \notin \mathcal{A}$, then one can copy the argument used in Case 1 with V and Q playing the role of U and P , respectively and deduce that \mathcal{A} is CI. If $\underline{Q} \in \mathcal{A}$, then Lemma 4.4 shows that \mathcal{A} is CI.

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