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ABSTRACT

It is shown that two vectors with coordinates in the finite q -element field of characteristic p belong to the same orbit under the natural action of the symmetric group if each of the elementary symmetric polynomials of degree $p^k, 2p^k, \dots, (q-1)p^k$, $k = 0, 1, 2, \dots$ has the same value on them. This separating set of polynomial invariants for the natural permutation representation of the symmetric group is not far from being minimal when $q = p$ and the dimension is large compared to p . A relatively small separating set of multisymmetric polynomials over the field of q elements is derived.

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1. Introduction

Throughout this paper F stands for an arbitrary field, q stands for a power of a prime p , and \mathbb{F}_q stands for the field of q elements. The symmetric group S_n acts on the vector space F^n by permuting coordinates: for $\pi \in S_n$ and $v = (v_1, \dots, v_n) \in F^n$ we

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have $\pi \cdot v = (v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(n)})$. Denote by x_1, \dots, x_n the basis of the dual space of F^n dual to the standard basis in F^n . We have an induced action of S_n via F -algebra automorphisms on the polynomial algebra $F[x_1, \dots, x_n]$. In particular, $\pi \cdot x_i = x_{\pi(i)}$ for $\pi \in S_n$ and $i \in \{1, \dots, n\}$. The algebra of S_n -invariant polynomials is $F[x_1, \dots, x_n]^{S_n} = \{f \in F[x_1, \dots, x_n] \mid \forall \pi \in S_n : \pi \cdot f = f\}$. A subset T of $F[x_1, \dots, x_n]^{S_n}$ is *separating* if for any $v, w \in F^n$ with different S_n -orbits there exists an element $f \in T$ such that $f(v) \neq f(w)$. This is a special case of the notion of *separating set of polynomial invariants* of (not necessarily finite) groups; for the general notion and basic facts about it we refer to [2, Section 2.4]. By a *minimal separating set* we shall mean a separating set none of whose proper subsets are separating (i.e. a separating set minimal with respect to inclusion). It is well known that every separating set has a finite separating subset (by a straightforward modification of the proof of [2, Theorem 2.4.8]), therefore any separating set contains a minimal separating subset, and a minimal separating set is necessarily finite. On the other hand, different minimal separating sets may have different cardinality, so a minimal separating set does not necessarily have minimal possible cardinality.

It is well known that the algebra $F[x_1, \dots, x_n]^{S_n}$ is minimally generated by the *elementary symmetric polynomials* $s_k^{(n)} = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}$. Moreover, when F is algebraically closed (or $F = \mathbb{R}$), then $\{s_k^{(n)} \mid k = 1, \dots, n\}$ is a separating set in $F[x_1, \dots, x_n]^{S_n}$ having the least possible number of elements (in particular, it is a minimal separating set). However, when F is finite, in general the above separating set is not even minimal with respect to inclusion. For a real number v we shall write $\lfloor v \rfloor$ for the largest integer not strictly bigger than v , and write $\lceil v \rceil$ for the smallest integer which is not strictly smaller than v . Kemper, Lopatin and Reimers [7, Lemma 4.3] proved that $\{s_{2^k}^{(n)} \mid k = 0, 1, \dots, \lfloor \log_2 n \rfloor\}$ is a minimal separating set in $\mathbb{F}_2[x_1, \dots, x_n]^{S_n}$ (and this separating set has the least possible number of elements). Our aim in this note is to extend the above result from the 2-element field \mathbb{F}_2 to any finite field \mathbb{F}_q . For a positive integer n set

$$[n]_q := \{jp^k \mid j \in \{1, \dots, q-1\}, \quad k \in \mathbb{Z}_{\geq 0}, \quad jp^k \leq n\}.$$

We shall prove the following:

Theorem 1.1. *The elementary symmetric polynomials $s_m^{(n)}$ with $m \in [n]_q$ form a separating subset in $\mathbb{F}_q[x_1, \dots, x_n]^{S_n}$.*

While the study of separating sets of polynomial invariants of groups became rather popular in the past two decades, as far as we know, the recent paper [7] is the first studying separating sets of polynomial invariants over finite fields. In particular, [7, Theorem 1.1] gives the exact value of a separating set of minimal cardinality for any group acting by linear substitutions on a polynomial algebra over a finite field; see also that paper for motivation (for example, for the connection to the graph isomorphism problem).

Our Theorem 1.1 has an equivalent reformulation not referring to the notion of separating sets of polynomial invariants, but as a statement about univariate polynomials over finite fields as follows:

Theorem 1.2. *Let $f, g \in \mathbb{F}_q[x]$ be monic polynomials of degree n , such that both f and g split as a product of root factors over \mathbb{F}_q . Assume that for all $j \in [n]_q$, the degree $n - j$ coefficient of f coincides with the degree $n - j$ coefficient of g . Then we have $f = g$.*

We also investigate whether the separating set given in Theorem 1.1 is minimal. It turns out that it is minimal (with respect to inclusion) for $q = 3, 4, 5$ with arbitrary n and for $q = 7$ with “most” n (see Corollary 4.9). However, computer calculations show that it is not always minimal (see the case $q = 7$, $n = 5$ in Corollary 4.9, or the results in Section 4.5 for $p = 11$). On the other hand, we point out in Proposition 4.12 that when $q = p$ and n is large compared to p , then in a certain sense, the separating set given in Theorem 1.1 is not far from being minimal.

In Section 5 we turn to the study of multisymmetric polynomials over the field \mathbb{F}_q . Separating sets of multisymmetric polynomials are studied in [10], [8], and a minimal separating set of multisymmetric polynomials over \mathbb{F}_2 is given in [7, Theorem 4.8]. Here we shall exploit Theorem 1.1 to obtain a relatively small separating set of multisymmetric polynomials over \mathbb{F}_q for an arbitrary prime power q in Theorem 5.3.

2. Preliminaries on polynomials

Lemma 2.1. *Let F be an arbitrary field, $f = \sum_{i=0}^d c_i x^i \in F[x]$ a polynomial whose formal derivative f' is not zero (i.e. $c_i \neq 0$ for some i not divisible by the characteristic of F). Assume that $c_0 \neq 0$ and $c_1 = \dots = c_m = 0$. Then f has at least $m + 1$ distinct roots in the algebraic closure of F .*

Proof. The formal derivative of f is

$$f' = (m+1)c_{m+1}x^m + \sum_{j=m+2}^d j c_j x^{j-1} = x^m h$$

for some non-zero polynomial $h \in F[x]$. We have

$$\deg(h) = \deg(f') - m \leq \deg(f) - 1 - m. \quad (1)$$

Recall that the number of distinct roots of f in the algebraic closure of F is greater than or equal to the difference of the degree of f and the degree of the greatest common divisor $\gcd_{F[x]}(f, f')$ of f and f' . As x does not divide f (recall that $c_0 \neq 0$), we have

$$\gcd_{F[x]}(f, f') = \gcd_{F[x]}(f, h).$$

Consequently,

$$\deg(\gcd_{F[x]}(f, f')) \leq \deg(h) \text{ and } \deg(f) - \deg(\gcd_{F[x]}(f, f')) \geq \deg(f) - \deg(h).$$

It follows by (1) that the number of distinct roots of f in the algebraic closure of f is at least

$$\deg(f) - \deg(h) \geq m + 1. \quad \square$$

We shall denote by \mathbb{F}_q^\times the set of non-zero elements in \mathbb{F}_q .

Corollary 2.2. *Given a map $\mathcal{O} : \mathbb{F}_q^\times \rightarrow \mathbb{Z}_{\geq 0}$ consider the polynomial*

$$G_{\mathcal{O}}(x) := \prod_{a \in \mathbb{F}_q^\times} (1 + ax)^{\mathcal{O}(a)} \in \mathbb{F}_q[x].$$

Assume that all terms of $G_{\mathcal{O}}(x)$ of degree $1, 2, \dots, q-1$ have coefficient zero. Then p divides $\mathcal{O}(a)$ for all $a \in \mathbb{F}_q^\times$.

Proof. Suppose for contradiction that p does not divide $\mathcal{O}(a)$ for some $a \in \mathbb{F}_q^\times$. Then the formal derivative $G'_{\mathcal{O}}$ of $G_{\mathcal{O}}$ is not the zero polynomial in $\mathbb{F}_q[x]$. Thus Lemma 2.1 applies for $G_{\mathcal{O}}$, and we conclude that $G_{\mathcal{O}}$ has at least $(q-1)+1 = q$ distinct roots in the algebraic closure of \mathbb{F}_q . However, $G_{\mathcal{O}}$ splits as a product of root factors already over \mathbb{F}_q , and all its roots are non-zero, so $G_{\mathcal{O}}$ has at most $q-1$ distinct roots in \mathbb{F}_q (and hence in its algebraic closure), a contradiction. \square

Lemma 2.3. *Take two maps $\mathcal{O}, \mathcal{P} : \mathbb{F}_q^\times \rightarrow \{0, 1, \dots, q-1\}$ and consider the polynomials*

$$G_{\mathcal{O}}(x) := \prod_{a \in \mathbb{F}_q^\times} (1 + ax)^{\mathcal{O}(a)} = \sum_j b_j x^j \quad \text{and} \quad G_{\mathcal{P}}(x) := \prod_{a \in \mathbb{F}_q^\times} (1 + ax)^{\mathcal{P}(a)} = \sum_j c_j x^j.$$

Suppose that $b_1 = c_1, b_2 = c_2, \dots, b_{q-1} = c_{q-1}$. Then we have $\mathcal{O}(a) \equiv \mathcal{P}(a)$ modulo p for all $a \in \mathbb{F}_q^\times$.

Proof. Note that $b_0 = c_0 = 1$. Therefore by assumption the polynomial $G_{\mathcal{O}} - G_{\mathcal{P}}$ is divisible in $\mathbb{F}_q[x]$ by x^q . Denote by D the greatest common divisor in $\mathbb{F}_q[x]$ of $G_{\mathcal{O}}$ and $G_{\mathcal{P}}$. Set $g := G_{\mathcal{O}}/D$ and $h := G_{\mathcal{P}}/D$, so x^q divides $(g-h)D$. As x does not divide $G_{\mathcal{O}}$, it does not divide D , and therefore x^q divides $g-h$ in $\mathbb{F}_q[x]$. There exist some disjoint subsets A, B of \mathbb{F}_q^\times such that

$$g = \prod_{a \in A} (1 + ax)^{k_a} \quad \text{and} \quad h = \prod_{b \in B} (1 + bx)^{m_b},$$

where k_a, m_b are positive integers less than or equal to $q-1$. Set

$$f := g \prod_{b \in B} (1 + bx)^{q-m_b}.$$

Then

$$f = (g - h) \prod_{b \in B} (1 + bx)^{q-m_b} + \prod_{b \in B} (1 + bx)^q.$$

The first summand on the right hand side above is divisible by x^q , whereas the second summand minus 1 is also divisible by x^q . This implies that $f - 1$ is divisible by x^q . On the other hand, we have

$$f = \prod_{a \in A} (1 + ax)^{k_a} \prod_{b \in B} (1 + bx)^{q-m_b}.$$

By Corollary 2.2 we conclude that p divides k_a for all $a \in A$ and p divides $q - m_b$ (and hence p divides m_b) for all $b \in B$. Define $\mathcal{R} : \mathbb{F}_q^\times \rightarrow \{0, 1, \dots, q-1\}$ by

$$D = \prod_{c \in \mathbb{F}_q^\times} (1 + cx)^{\mathcal{R}(c)}.$$

Then

$$\mathcal{O}(c) = \begin{cases} \mathcal{R}(c) + k_c & \text{for } c \in A \\ \mathcal{R}(c) & \text{for } c \notin A \end{cases} \quad \text{and} \quad \mathcal{P}(c) = \begin{cases} \mathcal{R}(c) + m_c & \text{for } c \in B \\ \mathcal{R}(c) & \text{for } c \notin B. \end{cases}$$

As p divides k_a and m_b for all $a \in A$ and $b \in B$, this clearly implies that both $\mathcal{O}(c)$ and $\mathcal{P}(c)$ are congruent to $\mathcal{R}(c)$ modulo p , hence p divides $\mathcal{O}(c) - \mathcal{P}(c)$ for all $c \in \mathbb{F}_q^\times$. \square

3. A separating set of elementary symmetric polynomials

Denote by $\Pi_{q,n}$ the set of functions $\mathcal{O} : \mathbb{F}_q^\times \rightarrow \mathbb{Z}_{\geq 0}$ satisfying $\sum_{a \in \mathbb{F}_q^\times} \mathcal{O}(a) \leq n$. There is a natural bijection between $\Pi_{q,n}$ and the set of S_n -orbits in \mathbb{F}_q^n ; namely, associate with $\mathcal{O} \in \Pi_{q,n}$ the set of vectors in \mathbb{F}_q^n having $\mathcal{O}(a)$ coordinates equal to a for each $a \in \mathbb{F}_q^\times$, and having $n - \sum_{a \in \mathbb{F}_q^\times} \mathcal{O}(a)$ zero coordinates. We shall write $s_k(\mathcal{O})$ for the value of the elementary symmetric polynomial $s_k^{(n)} \in \mathbb{F}_q[x_1, \dots, x_n]$ on the vectors in \mathbb{F}_q^n that belong to the orbit labelled by \mathcal{O} . For $k > n$ we set $s_k(\mathcal{O}) = 0$, and set $s_0(\mathcal{O}) = 1$.

For $\mathcal{O} \in \Pi_{q,n}$ and $a \in \mathbb{F}_q^\times$ we have

$$\mathcal{O}(a) = \sum_{j=0}^{\lfloor \log_p n \rfloor} \mathcal{O}(a)_j p^j$$

for some uniquely determined integers $\mathcal{O}(a)_j \in \{0, 1, \dots, p-1\}$ (i.e. the numbers $\mathcal{O}(a)_j$ are the digits of the non-negative integer $\mathcal{O}(a)$ in the number system with base p). For $k = 0, 1, \dots, \lfloor \log_p n \rfloor$ denote by $\mathcal{O}_{\{k\}} \in \Pi_{q,n}$ the function given by

$$\mathcal{O}_{\{k\}}(a) = \sum_{j=0}^k \mathcal{O}(a)_j p^j, \quad a \in \mathbb{F}_q^\times$$

and let $\mathcal{O}^{[k]} \in \Pi_{q,n}$ be the function given by

$$\mathcal{O}^{[k]}(a) = \mathcal{O}(a) - \mathcal{O}_{\{k\}}(a), \quad a \in \mathbb{F}_q^\times,$$

whereas $\mathcal{O}^{[k]/p^{k+1}} \in \Pi_{q,n}$ is the function given by

$$\mathcal{O}^{[k]/p^{k+1}}(a) = \frac{\mathcal{O}^{[k]}(a)}{p^{k+1}}, \quad a \in \mathbb{F}_q^\times.$$

For $\mathcal{O} \in \Pi_{q,n}$ set

$$G_{\mathcal{O}}(x) := \sum_{j=0}^n s_j(\mathcal{O}) x^j = \prod_{a \in \mathbb{F}_q^\times} (1 + ax)^{\mathcal{O}(a)} \in \mathbb{F}_q[x] \quad (2)$$

(the second equality above follows from Vieta's Formulae expressing the coefficients of a polynomial in terms of its roots). With this notation we have the obvious equalities

$$G_{\mathcal{O}}(x) = G_{\mathcal{O}_{\{k\}}}(x) \cdot G_{\mathcal{O}^{[k]}}(x) \in \mathbb{F}_q[x] \quad (3)$$

and

$$G_{\mathcal{O}^{[k]}}(x) = G_{\mathcal{O}^{[k]}/p^{k+1}}(x)^{p^{k+1}}. \quad (4)$$

Lemma 3.1. *Suppose that $\mathcal{O}, \mathcal{P} \in \Pi_{q,n}$ satisfy*

$$s_j(\mathcal{O}) = s_j(\mathcal{P}) \text{ for } j = 1, 2, \dots, q-1.$$

Then $\mathcal{O}_{\{0\}} = \mathcal{P}_{\{0\}}$ (i.e. $\mathcal{O}(a)$ is congruent to $\mathcal{P}(a)$ modulo p for all $a \in \mathbb{F}_q^\times$).

Proof. We have $q = p^{e+1}$ for some nonnegative integer e . By (3) and (4) we have

$$\mathcal{G}_{\mathcal{O}}(x) = G_{\mathcal{O}_{\{e\}}}(x) \cdot G_{\mathcal{O}^{[e]}/q}(x)^q.$$

The coefficient of x^j in $G_{\mathcal{O}^{[e]}/q}(x)^q$ is non-zero only if q divides j , and the constant term of $G_{\mathcal{O}^{[e]}/q}(x)^q$ is 1. Comparing the degree j coefficients of the two sides of the above equality for $j = 1, \dots, q-1$ we get

$$s_j(\mathcal{O}) = s_j(\mathcal{O}_{\{e\}}) \text{ for } j = 1, \dots, q-1. \quad (5)$$

Similarly we have

$$s_j(\mathcal{P}) = s_j(\mathcal{P}_{\{e\}}) \text{ for } j = 1, \dots, q-1. \quad (6)$$

Note that $\mathcal{O}_{\{e\}}$ and $\mathcal{P}_{\{e\}}$ are maps from \mathbb{F}_q^\times to $\{0, 1, \dots, q-1\}$. Moreover, for $j = 1, \dots, q-1$ the equality $s_j(\mathcal{O}) = s_j(\mathcal{P})$ implies by (5) and (6) that the degree j coefficient of $G_{\mathcal{O}_{\{e\}}}(x)$ coincides with the degree j coefficient of $G_{\mathcal{P}_{\{e\}}}(x)$. Consequently, Lemma 2.3 applies for $\mathcal{O}_{\{e\}}$ and $\mathcal{P}_{\{e\}}$, and yields the desired equality $\mathcal{O}_{\{0\}} = \mathcal{P}_{\{0\}}$. \square

Lemma 3.2. Suppose that $\mathcal{O}, \mathcal{P} \in \Pi_{q,n}$ and for some $k \in \{0, 1, \dots, \lfloor \log_p n \rfloor - 1\}$ we have $\mathcal{O}_{\{k\}} = \mathcal{P}_{\{k\}}$ and $s_{jp^{k+1}}(\mathcal{O}) = s_{jp^{k+1}}(\mathcal{P})$ for $j = 1, 2, \dots, q-1$. Then $\mathcal{O}_{\{k+1\}} = \mathcal{P}_{\{k+1\}}$.

Proof. Compare the coefficient of $x^{jp^{k+1}}$ on the two sides of (3). Taking into account that all non-zero terms of $G_{\mathcal{O}_{\{k\}}}(x)$ have degree divisible by p^{k+1} we get that

$$s_{jp^{k+1}}(\mathcal{O}) = \sum_{i=0}^j s_{ip^{k+1}}(\mathcal{O}^{[k]}) s_{(j-i)p^{k+1}}(\mathcal{O}_{\{k\}}) \quad \text{for } j = 1, \dots, q-1. \quad (7)$$

Consider the following system of linear equations for the unknowns y_1, \dots, y_{q-1} :

$$\sum_{i=1}^j s_{(j-i)p^{k+1}}(\mathcal{O}_{\{k\}}) y_i = s_{jp^{k+1}}(\mathcal{O}) - s_{jp^{k+1}}(\mathcal{O}_{\{k\}}), \quad j = 1, \dots, q-1. \quad (8)$$

By (7) we see that $y_j = s_{jp^{k+1}}(\mathcal{O}^{[k]})$, $j = 1, \dots, q-1$ is a solution of the system (8). Moreover, the system (8) has a unique solution: the equation for $j = 1$ gives

$$y_1 = s_{p^{k+1}}(\mathcal{O}) - s_{p^{k+1}}(\mathcal{O}_{\{k\}}),$$

and supposing that we have already fixed the values of y_1, \dots, y_{j-1} for some $j > 1$, we have

$$y_j = s_{jp^{k+1}}(\mathcal{O}) - s_{jp^{k+1}}(\mathcal{O}_{\{k\}}) - \sum_{i=1}^{j-1} s_{(j-i)p^{k+1}}(\mathcal{O}_{\{k\}}) y_i.$$

Similar considerations hold for the values of $s_{jp^{k+1}}(\mathcal{P}^{[k]})$, $j = 1, \dots, q-1$. By assumption we have

$$s_{ip^{k+1}}(\mathcal{P}_{\{k\}}) = s_{ip^{k+1}}(\mathcal{O}_{\{k\}}) \text{ and } s_{ip^{k+1}}(\mathcal{P}) = s_{ip^{k+1}}(\mathcal{O}) \text{ for } i = 1, \dots, q-1.$$

It follows that $y_j = s_{jp^{k+1}}(\mathcal{P}^{[k]})$, $j = 1, \dots, q-1$ is also a solution of the system (8), and by uniqueness of the solution we conclude that

$$s_{jp^{k+1}}(\mathcal{O}^{[k]}) = s_{jp^{k+1}}(\mathcal{P}^{[k]}) \text{ for } j = 1, \dots, q-1. \quad (9)$$

By (4) we have

$$s_{jp^{k+1}}(\mathcal{O}^{[k]}) = s_j(\mathcal{O}^{[k]/p^{k+1}})^{p^{k+1}} \quad (10)$$

and

$$s_{jp^{k+1}}(\mathcal{P}^{[k]}) = s_j(\mathcal{P}^{[k]/p^{k+1}})^{p^{k+1}} \quad (11)$$

By (9), (10), (11) we get that

$$s_j(\mathcal{O}^{[k]/p^{k+1}}) = s_j(\mathcal{P}^{[k]/p^{k+1}}) \text{ for } j = 1, \dots, q-1. \quad (12)$$

We conclude from (12) by Lemma 3.1 that

$$\mathcal{O}_{\{0\}}^{[k]/p^{k+1}} = \mathcal{P}_{\{0\}}^{[k]/p^{k+1}}. \quad (13)$$

Note finally that

$$\mathcal{O}_{\{0\}}^{[k]/p^{k+1}}(a) = \mathcal{O}^{[k]}(a)_{k+1} = \mathcal{O}(a)_{k+1} \text{ for all } a \in \mathbb{F}_q^\times \quad (14)$$

and similarly

$$\mathcal{P}_{\{0\}}^{[k]/p^{k+1}}(a) = \mathcal{P}^{[k]}(a)_{k+1} = \mathcal{P}(a)_{k+1} \text{ for all } a \in \mathbb{F}_q^\times. \quad (15)$$

Now (13), (14), (15) show that $\mathcal{O}(a)_{k+1} = \mathcal{P}(a)_{k+1}$ for all $a \in \mathbb{F}_q^\times$. As we have $\mathcal{O}_{\{k\}} = \mathcal{P}_{\{k\}}$ by assumption, this gives the desired equality $\mathcal{O}_{\{k+1\}} = \mathcal{P}_{\{k+1\}}$. \square

Corollary 3.3. *Let $\mathcal{O}, \mathcal{P} \in \Pi_{q,n}$ and $t \in \{1, \dots, n\}$. Assume that $s_{jp^k}(\mathcal{O}) = s_{jp^k}(\mathcal{P})$ holds for all $j \in \{1, \dots, q-1\}$ and $k \in \{0, 1, \dots, \lfloor \log_p(t) \rfloor\}$. Then we have*

- (i) $\mathcal{O}_{\{\lfloor \log_p(t) \rfloor\}} = \mathcal{P}_{\{\lfloor \log_p(t) \rfloor\}}$
- (ii) $s_t(\mathcal{O}) = s_t(\mathcal{P})$.

Proof. Lemma 3.1 and an iterated use of Lemma 3.2 yield (i), so setting $d := \lfloor \log_p(t) \rfloor$ we have $\mathcal{O}_{\{d\}} = \mathcal{P}_{\{d\}}$. By (4) we have $G_{\mathcal{O}^{[d]}}(x) = G_{\mathcal{O}^{[d]/p^{d+1}}}(x)^{p^{d+1}}$, showing that all non-zero terms of $G_{\mathcal{O}^{[d]}}(x)$ of positive degree have degree greater than or equal to $p^{d+1} > t$; moreover, the constant term of $G_{\mathcal{O}^{[d]}}(x)$ is 1. On the other hand, by (3) we have $G_{\mathcal{O}}(x) = G_{\mathcal{O}_{\{d\}}}(x) \cdot G_{\mathcal{O}^{[d]}}(x)$. It follows that $s_t(\mathcal{O}) = s_t(\mathcal{O}_{\{d\}})$. Similarly we have $s_t(\mathcal{P}) = s_t(\mathcal{P}_{\{d\}})$. Taking into account (i) we get

$$s_t(\mathcal{O}) = s_t(\mathcal{O}_{\{d\}}) = s_t(\mathcal{P}_{\{d\}}) = s_t(\mathcal{P}),$$

so (ii) holds as well. \square

Proof of Theorem 1.1. Suppose that for the S_n -orbit corresponding to $\mathcal{O} \in \Pi_{q,n}$ and the S_n -orbit corresponding to $\mathcal{P} \in \Pi_{q,n}$ we have $s_{jp^k}(\mathcal{O}) = s_{jp^k}(\mathcal{P})$ for all $j = 1, \dots, q-1$ and $k \in \mathbb{Z}_{\geq 0}$ (recall that $s_m(\mathcal{O}) = 0 = s_m(\mathcal{P})$ for any $m > n$). Corollary 3.3 (i) in the special case $n = t$ gives

$$\mathcal{O}_{\{\lfloor \log_p n \rfloor\}} = \mathcal{P}_{\{\lfloor \log_p n \rfloor\}}.$$

Taking into account that $\mathcal{O} = \mathcal{O}_{\{\lfloor \log_p n \rfloor\}}$ and $\mathcal{P} = \mathcal{P}_{\{\lfloor \log_p n \rfloor\}}$ we conclude the equality $\mathcal{O} = \mathcal{P}$. This clearly means that the set of elementary symmetric polynomials in the statement is separating. \square

4. Minimality

Lemma 4.1. Suppose that for some subset $A \subseteq \{1, \dots, n\}$ we have that $\{s_i^{(n)} \mid i \in A\}$ is separating in $F[x_1, \dots, x_n]^{S_n}$. Then for all $m \leq n$ we have that $\{s_j^{(m)} \mid j \in A \cap \{1, \dots, m\}\}$ is separating in $F[x_1, \dots, x_m]^{S_m}$.

Proof. For $n \geq m$ we shall treat F^m as the subspace of F^n consisting of the vectors whose last $n - m$ coordinates are zero. With this convention we have that for $v \in F^m$ and $n > m$,

$$s_j^{(n)}(v) = \begin{cases} s_j^{(m)}(v) & \text{for } j \leq m \\ 0 & \text{for } m < j \leq n. \end{cases} \quad (16)$$

Now take $v, w \in F^m$ having different S_m -orbit. Then viewed as elements of F^n , v and w have different S_n -orbit. Hence by assumption there exists an $i \in A$ with $s_i^{(n)}(v) \neq s_i^{(n)}(w)$. In particular, $s_i^{(n)}(v)$ and $s_i^{(n)}(w)$ are not both zero, hence $i \leq m$. Moreover, by (16) we conclude $s_i^{(m)}(v) \neq s_i^{(m)}(w)$. \square

Lemma 4.2. Let K be a field containing F as a subfield. If $\{s_i^{(n)} \mid i \in A\}$ is a separating set in $K[x_1, \dots, x_n]^{S_n}$, then it is also a separating set in $F[x_1, \dots, x_n]^{S_n}$.

Proof. The elementary symmetric polynomials are defined over the prime subfield \mathbb{F}_p of K and F . If a set of elementary symmetric polynomials separates the S_n -orbits in K^n , then it necessarily separates the orbits in the S_n -stable subset F^n of K^n . \square

Definition 4.3. We say that the elementary symmetric polynomial $s_k^{(n)}$ is *irreplaceable* over F if any separating subset of $F[x_1, \dots, x_n]^{S_n}$ that consists of elementary symmetric polynomials necessarily contains $s_k^{(n)}$.

Remark 4.4. (i) We would like to emphasize that the question whether the k th elementary symmetric polynomial is irreplaceable depends both on the number of variables and on the base field considered.

(ii) It follows from Theorem 1.1 that if $s_m^{(n)}$ is irreplaceable over \mathbb{F}_q , then $m \in [n]_q$.

Lemma 4.1 and Lemma 4.2 have the following immediate consequence:

Corollary 4.5. *If $s_k^{(n)}$ is irreplaceable over F , then $s_k^{(m)}$ is irreplaceable over K for all $m \geq n$ and all overfields K of F .*

Irreplaceable elementary symmetric polynomials have the following obvious characterization:

Lemma 4.6. *The elementary symmetric polynomial $s_k^{(n)}$ is irreplaceable over F if and only if there exist elements $v, w \in F^n$ such that*

$$s_j^{(n)}(v) = s_j^{(n)}(w) \quad \forall j \in \{1, \dots, n\} \setminus \{k\} \text{ and } s_k^{(n)}(v) \neq s_k^{(n)}(w). \quad (17)$$

Lemma 4.7. *If $s_k^{(n)}$ is irreplaceable over \mathbb{F}_q , then $s_{p^j k}^{(m)}$ is irreplaceable over \mathbb{F}_q for all $j \in \mathbb{Z}_{\geq 0}$ and $m \geq p^j n$.*

Proof. Suppose that $s_k^{(n)}$ is irreplaceable over \mathbb{F}_q . Then by Lemma 4.6 there exist $v, w \in \mathbb{F}_q^n$ such that $s_k^{(n)}(v) \neq s_k^{(n)}(w)$ and $s_i^{(n)}(v) = s_i^{(n)}(w)$ for all $i \in \{1, \dots, n\} \setminus \{k\}$. Denote by $\mathcal{O}, \mathcal{P} \in \Pi_{q,n}$ the functions $\mathbb{F}_q^\times \rightarrow \mathbb{Z}_{\geq 0}$ corresponding to the orbits of v, w , so we have $s_i(\mathcal{O}) = s_i(\mathcal{P})$ for $i \in \{1, \dots, n\} \setminus \{k\}$ and $s_k(\mathcal{O}) \neq s_k(\mathcal{P})$. Consider the polynomials $G_{\mathcal{O}}(x)$ and $G_{\mathcal{P}}(x)$. Then these are polynomials of degree at most n , all but their degree k coefficients agree and their degree k coefficient is different. For $m \geq p^j n$ denote by $p^j \mathcal{O} \in \Pi_{q,m}$, $p^j \mathcal{P} \in \Pi_{q,m}$ the functions $a \mapsto p^j \mathcal{O}(a)$ ($a \in \mathbb{F}_q^\times$), $a \mapsto p^j \mathcal{P}(a)$ ($a \in \mathbb{F}_q^\times$). We have the equalities

$$G_{p^j \mathcal{O}}(x) = G_{\mathcal{O}}(x)^{p^j} = \sum_{i=0}^n s_i(\mathcal{O})^{p^j} x^{ip^j} \text{ and } G_{p^j \mathcal{P}}(x) = G_{\mathcal{P}}(x)^{p^j} = \sum_{i=0}^n s_i(\mathcal{P})^{p^j} x^{ip^j}.$$

This shows that $s_{p^j k}(p^j \mathcal{O}) \neq s_{p^j k}(p^j \mathcal{P})$ and $s_i(p^j \mathcal{O}) = s_i(p^j \mathcal{P})$ for all $i \in \{1, \dots, m\} \setminus \{k\}$. Consequently, $s_{p^j k}^{(m)}$ is irreplaceable over \mathbb{F}_q by Lemma 4.6. \square

Corollary 4.8. *Suppose that each elementary symmetric polynomial $s_k^{(k)}$ is irreplaceable over \mathbb{F}_q for $k \in \{1, 2, \dots, q-1\}$. Then for an arbitrary n the separating subset $\{s_m^{(n)} \mid m \in [n]_q\}$ of $\mathbb{F}_q[x_1, \dots, x_n]^{S_n}$ given in Theorem 1.1 is minimal (with respect to inclusion).*

4.1. Some irreplaceable elementary symmetric polynomials

Below for certain prime powers q and certain integers n, k we provide pairs of vectors v, w in \mathbb{F}_q^n satisfying (17), showing by Lemma 4.6 that $s_k^{(n)}$ is irreplaceable over \mathbb{F}_q . The elements of the p -element field \mathbb{F}_p will be denoted by $0, 1, 2, \dots, p-1$ in the obvious way.

$q = 3$		$s_1^{(1)}$	$s_2^{(2)}$
v		[1]	[1, 2]
w		[0]	[0, 0]

$q = 5$		$s_1^{(1)}$	$s_2^{(2)}$	$s_3^{(3)}$	$s_4^{(4)}$
v		[1]	[1, 4]	[1, 4, 4]	[1, 2, 3, 4]
w		[0]	[0, 0]	[2, 2, 0]	[0, 0, 0, 0]

$q = 7$		$s_1^{(1)}$	$s_2^{(2)}$	$s_3^{(3)}$	$s_4^{(4)}$	$s_5^{(6)}$	$s_6^{(6)}$
v		[1]	[1, 6]	[1, 2, 4]	[1, 1, 6, 6]	[1, 2, 3, 4, 5, 6]	[2, 2, 2, 3, 6, 6]
w		[0]	[0, 0]	[0, 0, 0]	[3, 4, 0, 0]	[0, 0, 0, 0, 0, 0]	[1, 1, 4, 5, 5, 5]

$q = 11$		$s_1^{(1)}$	$s_2^{(2)}$	$s_3^{(3)}$	$s_4^{(4)}$	$s_5^{(5)}$	$s_6^{(6)}$
v		[1]	[5, 6]	[5, 7, 9]	[1, 2, 2, 5]	[1, 3, 9, 5, 4]	[1, 1, 3, 6, 8, 8]
w		[0]	[0, 0]	[10, 0, 0]	[10, 0, 0, 0]	[0, 0, 0, 0, 0, 0]	[9, 9, 10, 10, 0, 0]

$q = 11$		$s_7^{(8)}$
v		[1, 2, 2, 2, 2, 4, 4, 5]
w		[6, 7, 7, 9, 9, 9, 9, 10]

4.2. The polynomial $s_k^{(k)}$ for a divisor k of $q - 1$ for an arbitrary prime power q

We may take as v the vector whose coordinates are the k roots in \mathbb{F}_q of the polynomial $x^k - 1$, and for w the zero vector. Then we have $0 = s_1^{(k)}(w) = s_1^{(k)}(v) = s_2^{(k)}(w) = s_2^{(k)}(v) = \dots = s_{k-1}^{(k)}(w) = s_{k-1}^{(k)}(v) = s_k^{(k)}(w)$ whereas $s_k^{(k)}(v) = (-1)^{k+1} \in \mathbb{F}_q$.

4.3. The polynomial $s_k^{(k)}$ for an arbitrary k and $q \geq k! - k + 1$

The assumption on q guarantees that the number $\binom{q-1+k}{k}$ of S_k -orbits in \mathbb{F}_q^k is greater than the number q^{k-1} which is an obvious upper bound for the number of possible values of the $(k-1)$ -tuples $(s_1^{(k)}(z), \dots, s_{k-1}^{(k)}(z))$ (where $z \in \mathbb{F}_q$), hence the desired pair of vectors in \mathbb{F}_q^k exists by the pigeonhole principle. From the special case $k = 3$ we get that $s_3^{(3)}$ is irreplaceable over \mathbb{F}_q for all prime powers $q \geq 4$.

4.4. Result obtained by computer for $q = 7$

$\{s_k^{(5)} \mid k = 1, 2, 3, 4\}$ is a separating set in $\mathbb{F}_7[x_1, x_2, x_3, x_4, x_5]^{S_5}$, hence the elementary symmetric polynomial $s_5^{(5)}$ is not irreplaceable over \mathbb{F}_7 .

4.5. Further results obtained by computer for $q = 11$

The elementary symmetric polynomial $s_7^{(7)}$ is not irreplaceable over \mathbb{F}_{11} . On the other hand, $\{s_i^{(10)} \mid i = 1, 2, 3, 4, 5, 6, 7, 10\}$ is a minimal separating set in $\mathbb{F}_{11}[x_1, \dots, x_{10}]^{S_{10}}$.

Consequently, none of $s_8^{(8)}, s_8^{(9)}, s_8^{(10)}, s_9^{(9)}, s_9^{(10)}$ is irreplaceable over \mathbb{F}_{11} . Further computer calculations showed that none of $s_8^{(11)}, s_8^{(12)}, s_8^{(13)}$ is irreplaceable over \mathbb{F}_{11} .

The results of Section 4.1 (and Section 4.4) imply by Lemma 4.7 and Corollary 4.8 the following:

Corollary 4.9. *The separating subset $\{s_m^{(n)} \mid m \in [n]_q\}$ of $\mathbb{F}_q[x_1, \dots, x_n]^{S_n}$ given in Theorem 1.1 is minimal (with respect to inclusion) for $q = 3, 4, 5$ with arbitrary n , and for $q = 7$ with $\log_7 n - \lfloor \log_7 n \rfloor < \log_7 5$ or $\log_7 n - \lfloor \log_7 n \rfloor \geq \log_7 6$. In $\mathbb{F}_7[x_1, x_2, x_3, x_4, x_5]^{S_5}$, $\{s_i^{(5)} \mid i = 1, 2, 3, 4\}$ is a minimal separating subset.*

In view of the above results it seems natural to formulate the following problems:

Problem 4.10. Is it true that for any prime p and any $k \in \{1, 2, \dots, p-1\}$ there exists a positive integer n such that $s_k^{(n)}$ is irreplaceable over \mathbb{F}_p ?

Problem 4.11. Given a prime p and positive integers $k \leq n$, determine the minimal fields \mathbb{F}_q of characteristic p such that $s_k^{(n)}$ is irreplaceable over \mathbb{F}_q .

4.6. A bound for the distance from being minimal

We saw above that the separating set given in Theorem 1.1 is not always minimal. Moreover, even when it is minimal, it may not be of minimal possible cardinality. For example, for $n = 9$ and $q = 3$, the minimal cardinality of a separating set in $\mathbb{F}_3[x_1, \dots, x_9]^{S_3}$ is $\lceil \log_3 \binom{9+2}{2} \rceil = 4$ by [7, Theorem 1.1], whereas the minimal separating set given in Corollary 4.9 has cardinality $|\lceil 9 \rceil_3| = |\{1, 2, 3, 6, 9\}| = 5$. Our aim here is to point out that however, for $q = p$ and for large n , the separating set given in Theorem 1.1 is not much bigger than a separating set of minimal cardinality.

Denote by $d_q(n)$ the difference of $|[n]_q|$ (the number of elements in the separating subset of $\mathbb{F}_q[x_1, \dots, x_n]^{S_n}$ given in Theorem 1.1) and the number of elements in a separating subset of $\mathbb{F}_q[x_1, \dots, x_n]^{S_n}$ of minimal possible cardinality.

Proposition 4.12. *For any prime p we have the inequality $d_p(n) \leq p-2$. Consequently, one can get a minimal (with respect to inclusion) separating set in $\mathbb{F}_p[x_1, \dots, x_n]^{S_n}$ by removing at most $p-2$ elements from the separating set given in Theorem 1.1.*

Proof. By [7, Theorem 1.1] the minimal cardinality of a separating set in $\mathbb{F}_p[x_1, \dots, x_n]^{S_n}$ is the upper integer part of the logarithm with base p of the number of S_n -orbits in \mathbb{F}_p^n . The number of S_n -orbits in \mathbb{F}_p^n is $\binom{n+p-1}{p-1}$. There exists an $m \in \{1, \dots, p-1\}$ and $k \in \mathbb{Z}_{\geq 0}$ with

$$mp^k \leq n < (m+1)p^k.$$

We have

$$\begin{aligned}
 \lceil \log_p \binom{n+p-1}{p-1} \rceil &\geq \lceil \sum_{j=1}^{p-1} \log_p(mp^k + j) - \log_p((p-1)!) \rceil \\
 &> (p-1) \log_p(mp^k) - \log_p((p-1)!) \\
 &= (p-1)k + (p-1) \log_p m - \log_p((p-1)!) \\
 &> (p-1)k + m - (p-1),
 \end{aligned} \tag{18}$$

where the last inequality follows from

$$\frac{(p-1)!}{m^{p-1}} = \frac{p-1}{m} \cdot \frac{p-2}{m} \cdots \frac{m+1}{m} \cdot \frac{m}{m} \cdot \frac{m-1}{m} \cdots \frac{1}{m} < p^{p-1-m}.$$

On the other hand, the number of elements in the separating set given in Theorem 1.1 is $k(p-1) + m$. Taking into account (18) we get

$$d_p(n) < k(p-1) + m - ((p-1)k + m - (p-1)) = p-1. \quad \square$$

5. Multisymmetric polynomials over \mathbb{F}_q

Consider the m -fold direct sum of the representation of S_n on F^n . So the underlying vector space of this representation is $(F^n)^m = F^n \oplus \cdots \oplus F^n$. For $j = 1, \dots, m$ and $i = 1, \dots, n$ denote by $x_i^{(j)}$ the element in the dual space of $(F^n)^m$ mapping an m -tuple $(v^{(1)}, \dots, v^{(m)}) \in (F^n)^m$ of vectors to the i th coordinate of the j th vector component $v^{(j)}$. We get an induced action on the nm -variable polynomial algebra $A_{n,m} := F[x_i^{(j)} \mid i = 1, \dots, n; j = 1, \dots, m]$ given by $\pi \cdot x_i^{(j)} = x_{\pi(i)}^{(j)}$. The corresponding algebra $A_{n,m}^{S_n}$ of polynomial invariants is called *the algebra of multisymmetric polynomials*. Our aim in this section is to give a separating set of multisymmetric polynomials, where a subset T of $A_{n,m}^{S_n}$ is said to be *separating* if for any $v, w \in (F^n)^m$ with different S_n -orbits there is a polynomial $f \in T$ with $f(v) \neq f(w)$.

The algebra $A_{n,m}$ is $\mathbb{Z}_{\geq 0}^m$ -graded: the multihomogeneous component $A_{n,m,\alpha}$ of $A_{n,m}$ of multidegree $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$ is spanned by polynomials all of whose non-zero terms have total degree α_j in the variables $x_1^{(j)}, \dots, x_n^{(j)}$ for each $j = 1, \dots, m$. The S_n -action preserves this multigrading, hence the algebra of multisymmetric polynomials is also multigraded: we have $A_{n,m}^{S_n} = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^m} A_{n,m,\alpha}^{S_n}$. Denote by $s_{k,\alpha}^{(n)}$ the component of multidegree α of $s_k^{(n)}(\sum_{j=1}^m x_1^{(j)}, \dots, \sum_{j=1}^m x_n^{(j)})$. Clearly $s_{k,\alpha}^{(n)}$ is non-zero only if $\alpha_1 + \cdots + \alpha_m = k$. The multisymmetric polynomials $s_{k,\alpha}^{(n)}$ are called the *polarizations of the elementary symmetric polynomials*. They generate $A_{n,m}^{S_n}$ when the characteristic of the base field F is greater than n or is 0 (in other words, when $\text{char}(F)$ does not divide the order of the group S_n , see for example [3] and the relevant references therein). However, when $0 < \text{char}(F) \leq n$, the polarizations of the elementary symmetric polynomials are

not sufficient to generate the algebra of multisymmetric polynomials in general. In fact the maximal degree of an element in a minimal homogeneous generating system of $A_{n,m}^{S_n}$ tends to infinity together with m , see [6]. For the modular case generating systems of $A_{n,m}^{S_n}$ are given in [13], [3]. A minimal homogeneous generating system is obtained in [5] for the case $F = \mathbb{F}_2$, and in [11] for an arbitrary base ring F .

Proposition 5.1. *For any field F , the following is a separating set in $A_{n,m}^{S_n}$:*

$$\left\{ \sum_{\alpha_2+2\alpha_3+\dots+(m-1)\alpha_m=d} s_{k,\alpha}^{(n)} \mid k=1,\dots,n; \quad d=0,1,\dots,(m-1)n \right\}.$$

Proof. The elementary symmetric polynomials generate $F[x_1, \dots, x_n]^{S_n}$, therefore by [4, Theorem 3.4] their “cheap polarizations” form a separating set. It is easy to see that the cheap polarizations of the elementary symmetric polynomials are the polynomials given in the statement. \square

Let us recall another class of multisymmetric polynomials. For any $\alpha \in \mathbb{Z}_{\geq 0}^m$ set

$$s_k^{(n)}(x^\alpha) := s_k^{(n)}\left(\prod_{j=1}^m x_1^{(j)\alpha_j}, \dots, \prod_{j=1}^m x_n^{(j)\alpha_j}\right).$$

Note that the formulae from [1] were used in [3, Equation (6)] to express the polynomials $s_{k,\alpha}^{(n)}$ in terms of the polynomials $s_k^{(n)}(x^\gamma)$. Write $|\alpha| := \sum_{j=1}^m \alpha_j$, and denote by $\gcd(\alpha)$ the greatest common divisor of $\alpha_1, \dots, \alpha_m$.

Proposition 5.2. *For any field F the elements $s_k^{(n)}(x^\alpha)$ with $\alpha \in \mathbb{Z}_{\geq 0}^m$, $k|\alpha| \leq n$, $\gcd(\alpha) = 1$ constitute a separating set in $A_{n,m}^{S_n}$.*

Proof. Proposition 5.1 implies in particular that the elements of $A_{n,m}^{S_n}$ with degree at most n form a separating subset. It follows that the elements of degree at most n from any homogeneous system of generators of the algebra $A_{n,m}^{S_n}$ form a separating set. Applying this for the generating system given in [3, Corollary 5.3] we obtain the desired statement. \square

Theorem 5.3. *For the field $F = \mathbb{F}_q$ of q elements the following is a separating set in $A_{n,m}^{S_n}$:*

$$\{s_{j^p}^{(n)}(x^\alpha) \mid j \in \{1, \dots, q-1\}, \quad \alpha \in \mathbb{Z}_{\geq 0}^m, \quad |\alpha| \leq n, \quad \gcd(\alpha) = 1, \quad (19)$$

$$\alpha_j \leq q-1 \text{ for } j = 1, \dots, m, \quad k \in \{0, 1, \dots, \lfloor \log_p \frac{n}{|\alpha|} \rfloor\}\}.$$

Proof. Suppose that $v, w \in (\mathbb{F}_q^n)^m$ belong to different S_n -orbits. By Proposition 5.2 there exist a $t \in \{1, \dots, n\}$, $\alpha \in \mathbb{Z}_{\geq 0}^m$ with $t|\alpha| \leq n$ such that $s_t^{(n)}(v^\alpha) \neq s_t^{(n)}(w^\alpha)$, where v^α (respectively w^α) stands for the vector in \mathbb{F}_q^n whose i th coordinate is $\prod_{j=1}^m v_i^{(j)\alpha_j}$ (respectively $\prod_{j=1}^m w_i^{(j)\alpha_j}$). We may assume that $|\alpha|$ is minimal possible. Then $\alpha_j \leq q-1$, because for all j with $\alpha_j > 0$, denoting by γ_j the unique element of $\{1, \dots, q-1\}$ which is congruent to α_j modulo $q-1$ (and setting $\gamma_j = 0$ if $\alpha_j = 0$), we have $v^\alpha = v^\gamma$ and $w^\alpha = w^\gamma$. Thus by minimality of $|\alpha|$ we must have $\alpha_j \in \{0, 1, \dots, q-1\}$ for all j . Moreover, we must have $\gcd(\alpha) = 1$, since otherwise $s_t^{(n)}(x^\alpha)$ can be expressed as a polynomial of elements of the form $s_u^{(n)}(x^\gamma)$ with $|\gamma| < |\alpha|$ (see [3, Page 517] for explanation), and some $s_u^{(n)}(x^\gamma)$ with $|\gamma| < |\alpha|$ would separate v and w , contrary to the minimality of $|\alpha|$. Finally, it follows by Corollary 3.3 (ii) that there exists a $j \in \{1, \dots, q-1\}$ and $k \in \{0, 1, \dots, \lfloor \log_p t \rfloor\}$ such that $s_{jp^k}^{(n)}(v^\alpha) \neq s_{jp^k}^{(n)}(w^\alpha)$. So we showed that whenever $v, w \in (\mathbb{F}_q^n)^m$ have different S_n -orbits, then v and w can be separated by an element from (19). \square

Remark 5.4. The separating set given in Theorem 5.3 for $F = \mathbb{F}_q$ exploits Theorem 1.1. For fixed q and m and “sufficiently large n ” it is significantly smaller than the separating sets given in Proposition 5.1 or Proposition 5.2 for general F , see Example 5.5 for illustration. On the other hand, in the special case $q = 2$ a stronger result is known, since in [7, Theorem 4.8] a minimal separating subset of $A_{n,m}^{S_n}$ is determined for $F = \mathbb{F}_2$; this separating set is a proper subset of the one given by the special case $q = 2$ of our Theorem 5.3, as it involves a stronger upper bound for the parameter k in the multisymmetric polynomials included in the separating set.

Example 5.5.

- (i) Take $q = 3$, $m = 2$, and $n = 26$. The possible α to consider in the separating set (19) in Theorem 5.3 for $A_{26,2}^{S_{26}}$ are $\alpha = (1, 0), (0, 1), (1, 1), (2, 1), (1, 2)$. The corresponding numbers $\lfloor \log_3 \frac{26}{|\alpha|} \rfloor$ are 2, 2, 2, 1, 1. Thus in this case the separating set (19) has $2 \cdot (3 + 3 + 3 + 2 + 2) = 26$ elements. The separating set of $A_{n,m}^{S_n}$ given in Proposition 5.1 has $n(n(m-1) + 1)$ elements in general, so it has $26 \cdot 27 = 702$ elements in our case $n = 26$, $m = 2$.

On the other hand, the number of S_{26} -orbits in $\mathbb{F}_3^{26} \oplus \mathbb{F}_3^{26}$ is $\binom{34}{8} = 18156204$. Thus by [7, Theorem 1.1], there is a separating set in $A_{26,2}^{S_{26}}$ consisting of $\lceil \log_3 \binom{34}{8} \rceil = 16$ multisymmetric polynomials.

- (ii) Take $q = 3$, $m = 2$, and $n = 8$. The separating set (19) for $A_{8,2}^{S_8}$ in Theorem 5.3 has 16 elements, the separating set given in Proposition 5.2 has 44 elements, whereas the separating set given in Proposition 5.1 has 72 elements. The minimal cardinality of a separating set for $A_{8,2}^{S_8}$ over \mathbb{F}_3 is 9.

6. Comment on lacunary polynomials

Lemma 4.6 has the following reformulation in terms of polynomials: The elementary symmetric polynomial $s_k^{(n)} \in F[x_1, \dots, x_n]$ is irreplaceable if there exist two polynomials $f, g \in F[x]$ having degree at most n and constant term 1 such that both f and g split as a product of linear factors over F , the degree k coefficient of f differs from the degree k coefficient of g , and all the other coefficients of f coincide with the corresponding coefficient of g . This is reminiscent of the topic of the book [9], where lacunary polynomials that split as a product of root factors over \mathbb{F}_q are studied.

In particular, our Corollary 2.2 has similar flavour as the following theorem of Rédei [9, Paragraph 10]: Suppose that the polynomial $f(x) = x^q + g(x)$ splits as a product of root factors over the field \mathbb{F}_q , and the formal derivative of f is non-zero. Then $\deg(g) \geq (q+1)/2$ or $f(x) = x^q - x$. For applications of this result in finite geometry see [12].

Data availability

No data was used for the research described in the article.

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