# Subgraph densities in Markov spaces

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November 13, 2023

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	*Rese	arch supported by ERC Consolidator Grant 648017			

<sup>†</sup>Research supported by ERC Synergy Grant No. 810115.

<sup>‡</sup>Research was partially supported by the NKFIH "Élvonal" KKP 133921 grant.

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#### Abstract

We generalize subgraph densities, arising in dense graph limit theory, to Markov spaces (symmetric measures on the square of a standard Borel space). More generally, we define an analogue of the set of homomorphisms in the form of a measure on maps of a finite graph into a Markov space. The existence of such homomorphism measures is not always guaranteed, but can be established under rather natural smoothness conditions on the Markov space and sparseness conditions on the graph. This continues a direction in graph limit theory in which such measures are viewed as limits of graph sequences.

## 1 Introduction

Dense graph limit theory is arguably the most complete graph limit theory: There is a rather satisfactory duality between the local and global points of view; subgraph densities and large scale structures (such as Szemerédi partitions) are connected via the counting lemma and the inverse counting lemma; limit objects, called graphons, are well known. Furthermore, the problem of soficity does not arise: every potential limit object is the limit of finite graphs.

Substantial work has been done pushing these things into the sparse regime. Graphons are bounded functions on  $[0, 1]^2$ , and a natural next step is to explore the regime of "unbounded graphons". Borgs, Chayes, Cohn and Zhao [5] extended various results in dense graph limit theory to " $L^p$ graphons" (symmetric functions in  $L^p([0, 1]^2)$ ); here only degree-restricted simple graphs were guaranteed to have finite densities. The authors [16] introduced a very general framework that was among other things meant to encode homomorphism convergence of multigraphs, and allows for finite densities for all decorated graphs in all limit objects. In the simplest case, this corresponds to symmetric functions in  $L^{\omega} = \bigcap_{p \in [1,\infty)} L^p([0,1]^2)$ , which is the largest function space in which all elements have finite densities for all simple graphs. Other work in this direction includes [3, 4, 23, 12].

To go beyond unbounded graphons, the authors [17] developed a limit theory for not necessarily dense graphs, in which limit objects are symmetric measures on  $[0, 1]^2$  called "s-graphons". (The [0, 1] interval can be replaced by any standard Borel space.) Backhausz and Szegedy [1] developed a stronger convergence theory with similar limit objects, which they call "graphops". While these approaches have the potential to unify various branches of graph limit theory, both of them are based on convergence notions which could be called "global convergence" or "right convergence". The local point of view seems to be lost: subgraph densities and subgraph distributions in general have not been defined in symmetric measures on  $[0, 1]^2$ .

Graphons or more generally unbounded graphons correspond to measures on  $[0, 1]^2$  that are absolutely continuous with respect to the uniform measure. The main purpose of this paper is to study local aspects of graph limit theory for singular measures. Our results show that this is possible as long as the measure has certain smoothness properties, whilst the graph to be mapped has certain sparseness properties. The smoother the measure, the more finite graphs will have well-defined densities in them. This leads to a remarkable hierarchical viewpoint on graph limit theory, where smoothness of limit objects corresponds to certain sparsity properties of graph sequences. At the top of this hierarchy are the bounded and the  $L^{\omega}$ -graphons of [16] as the smoothest objects.  $L^p$ -graphons from [5] form the next level of smoothness.

The hope that one may extend local properties to singular measures has already emerged in a previous work by the authors of the present paper. In [18] we investigated random orthogonal representations of finite graphs by vectors on *n*-dimensional unit sphere  $S^n$ . As it turns out, such representations can also be viewed as random homomorphisms into a singular measure defined on  $S^n \times S^n$ : namely, the uniform distribution  $\eta_n$  on orthogonal pairs of vectors in  $S^n$ . Quite surprisingly, for every finite graph H and sufficiently large natural number n one can introduce a robust notion for the density of H in  $\eta_n$ . Moreover one can introduce a measure on copies of H in  $\eta_n$ ; if the total measure is finite, then one can normalize it to a probability measure. See Section 3.2 and also [18] as source of concrete, illustrative examples supporting the more general and abstract content of the present paper.

To keep our treatment relatively simple, in this paper we address a special case of s-graphons, which we call Markov spaces and (in their bipartite version) bi-Markov spaces. A Markov space is a standard Borel sigma-algebra  $(J, \mathcal{B})$  endowed with a probability measure  $\eta$  on  $\mathcal{B} \times \mathcal{B}$ . We restrict our attention to symmetric measures on  $\mathcal{B} \times \mathcal{B}$ . We will denote the marginal distribution of  $\eta$  on J by  $\pi$ . For a (finite) graph G, the uniform distribution on E(G) defines a Markov space. Markov spaces are essentially equivalent to reversible Markov chains with a specified stationary distribution. Graphops and s-graphons can be obtained by adding a probability measure on the points, generalizing the uniform distribution on the nodes of a graph. See also Remark 2.6.

We address the following three questions:

(i) How to define a reasonable notion of the density of a graph G = (V, E)in a Markov space  $(J, \mathcal{B}, \eta)$ ?

Subgraph densities play a crucial role in graph limit theory, in the definition of local convergence, extremal graph theory and graph property testing, just to name a few applications; they also arise as Feynman integrals in quantum physics (see e.g. [13], Section 8.2). Subgraph densities can be viewed as analogues of the moments of functions defined on product spaces (cf. [21] and [19], Appendix A4). We can calculate densities of finite graphs in analytic objects representing graph limits such as graphons and graphings. More general Markov spaces are also known to represent limits of finite graphs, but the right notion for subgraph density is still missing. Examples can be given showing that subgraph densities satisfying reasonable conditions cannot be defined in full generality.

(ii) How to define the homomorphism set  $\text{Hom}(G,\eta)$  where G is a finite graph and  $(J, \mathcal{B}, \eta)$  is a Markov space?

If H is a simple finite graph then Hom(G, H) is a subset of the set  $V(H)^V$  of all maps from V = V(G) to V(H). However, if we consider

an edge-weighted graph H, then there is no general, natural way to interpret  $\operatorname{Hom}(G, H)$  as a subset of  $V(H)^V$ . Rather, the edge weights induce a function on  $V(H)^V$ , the function value being the product of the edge weights of the images of the edges of G under the corresponding vertex map. More generally, if  $(J, \mathcal{B}, \pi)$  is a probability space and  $W: J^2 \to [0, 1]$  is a graphon, then our interpretation of  $\operatorname{Hom}(G, W)$  is a measure  $\eta^G$  on  $J^V$  whose density function (Radon–Nikodym derivative) with respect to  $\pi^V$  is the function

$$W^{G}(x_{1}, x_{2}, \dots, x_{n}) := \prod_{(i,j) \in E(G)} W(x_{i}, x_{j}),$$
(1)

where  $V = \{1, 2, ..., n\}$ . With this definition, the total measure  $\eta^G(J^V)$  is the familiar homomorphism density t(G, W). If we apply this definition to a graphon that represents a finite graph H by its adjacency function  $V(H) \times V(H) \to \{0, 1\}$ , then we obtain the counting measure on  $\operatorname{Hom}(G, H)$  normalized by the number  $|V(H)|^{|V(G)|}$  of all maps from V(G) to V(H).

Our goal is to introduce similar measures representing  $\operatorname{Hom}(G,\eta)$  for Markov spaces. The fact that generalized homomorphism sets are represented by measures and not by sets is perfectly in line with the fact that the "edge set" of a Markov space is not a set either: It is represented by the measure  $\eta$  which tells us how to choose a random edge. Unfortunately, the product formula in (1) does not make sense if  $\eta$  is singular with respect to  $\pi^2$ , and so we have to use different methods to define  $\eta^G$ .

Our main approach relies on axiomatizing the properties of homomorphism measures. We introduce some relatively simple and natural properties (related to, but different from, the notion of a Markov random field; see Appendix 9.2) that are strong enough to uniquely define the measures  $\eta^G$ . This also allows us to define the subgraph density

$$t(G,\eta) = \eta^G(J^V),$$

answering (i) in this case. We warn that  $t(G, \eta)$  can be infinite. However, if the total measure  $t(G, \eta)$  is finite, then we can turn the measure  $\eta^G$  into a probability measure by normalizing it. These normalized versions can then be used to define random copies of G in  $\eta$ .

(iii) Can Markov spaces be approximated by a sequence  $G_1, G_2, \ldots$  of finite graphs, so that the density of every finite graph F in  $G_n$  (suitably normalized) tends to the density of F in the limit space?

A Markov space that can be approximated this way will be called *sofic*. In the case of dense graphs, the limit objects (graphons) are sofic; this takes an easy construction via sampling. In the case of bounded-degree graphs, soficity of the limit objects (involution-invariant distributions or graphings) is the famous Aldous–Lyons conjecture, which is stronger than the soficity problem for finitely generated groups.

We offer two approaches (and their combination) to these problems.

(a) The first approach builds on the fact that the generalization of (i) to (ii) allows for a recursive definition of these "Hom-measures". The "axioms" for these measures enable us to build up the measure corresponding to a graph G recursively from the measures pertaining to smaller graphs, by attaching their nodes one-by-one. The independence of the construction from the order in which the graph is built up is the main difficulty of this approach, and in fact it does not hold in general (see Example 3.1 below). We can prove this independence for triangle-free graphs (under smoothness assumptions on the measure  $\eta$ ).

(b) In the second approach, we consider approximations of  $\eta$  by sequences of graphons. By considering the densities of subgraphs within each graphon of the sequence, and taking their limit, one naturally obtains a notion of subgraph densities (more generally, homomorphism measures) in  $\eta$ . However, in order to obtain a robust, well-defined notion through this approach, we have to make sure that subgraph densities in these approximating sequences have a limit, and that this limit is independent of the sequence considered. This independence also hinges on certain smoothness properties of  $\eta$ .

Soficity is clearly related to our second approach, the discretization of the Markov space, which can be used to produce a sofic approximation.

The equivalence of these two approaches is a nontrivial problem that is also addressed in this paper. As remarked above, our methods do not work in full generality; very likely there is some theoretical limitation on how far one can go with defining  $\eta^G$  in arbitrary Markov spaces. However, the full analysis of this problem is left as an important open question.

In the next part of the introduction we will state our main definitions and results more precisely. We start with our definition of discretized Markov spaces. Let  $\mathcal{P} = \{J_1, J_2, \ldots, J_n\}$  be a finite measurable partition of the space J such that every partition class has positive  $\pi$ -measure. If the Markov space is given by the measure  $\eta$  on  $J \times J$ , then it makes sense to "project"  $\eta$  to  $\mathcal{P}$ . When restricted to a product set  $J_i \times J_j$ , the new measure  $\eta_{\mathcal{P}}$  is a scaled version of  $\pi^2$  such that  $\eta_{\mathcal{P}}(J_i \times J_j) = \eta(J_i \times J_j)$  holds. In other words, the Radon–Nikodym derivative W of  $\eta_{\mathcal{P}}$  with respect to  $\pi^2$  is a graphon whose value on  $J_i \times J_j$  is constant  $\eta(J_i \times J_j)\pi(J_i)^{-1}\pi(J_j)^{-1}$ .

We call a sequence of partitions of J a generating partition sequence, if the partition classes are Borel, have positive  $\pi$ -measure, each partition is a refinement of the previous one, and the partition classes generate all Borel sets. If we are more interested in generating the measure algebra rather than the Borel sets proper, i.e., we only require the partition classes to generate a sigma-algebra whose  $\pi$ -completion contains all Borel sets, we obtain the slightly more general class of *exhausting partition sequences* (see Section 2.6).

Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence. We can then try to define  $\eta^G$  as the limit of homomorphism measures of G in the graphons  $\eta_{\mathcal{P}_i}$ . The existence of these limits and the independence from the chosen partition sequence is nontrivial and not always true.

The definition of what we mean by the convergence  $\eta_{\mathcal{P}_i}^G \to \eta^G$  should also be clarified. Let  $(J^m, \mathcal{B}^m)$  be the product sigma-algebra of m copies of  $(J, \mathcal{B})$ . A set of the form  $C = B_1 \times \cdots \times B_m$ , where  $B_i \in \mathcal{B}$ , will be called a *box*. For measures  $(\mu, \mu_1, \mu_2, \ldots)$  on  $\mathcal{B}^m$ , the relation  $\mu_n \to \mu$  on *boxes* means that  $\mu_n(C) \to \mu(C)$  on every box C. This is a rather weak notion of convergence, and in fact it is equivalent to weak convergence if we put a compact topology on  $(J, \mathcal{B})$ , and all of the measures  $\mu_n$  as well as  $\mu$  have the same marginals – this is left as an exercise to the reader.

If  $\eta_{\mathcal{P}_i}^G \to \eta^G$  on boxes for every exhausting partition sequence, then we say that  $\eta^G$  is *partition approximable*. Note that this in particular means that  $t(G, \eta_{\mathcal{P}_i}) \to t(G, \eta)$ .

Since graphons can be approximated by finite graphs via sampling, and the homomorphism measures of these finite graphs approximate the homomorphism measure of the graphon, the results on partition approximability of  $\eta^G$  can be interpreted as a partial answer to the soficity problem (iii).

Now we turn to the definitions needed to generalize homomorphism sets. As we mentioned above, our goal is to construct measures  $\eta^{G[S]}$  on  $\mathcal{B}^S$  for each induced subgraph G[S] of a graph G = (V, E). This measure should depend on the induced subgraph G[S] only<sup>1</sup>. Intuitively, the measure  $\eta^G$ represents some kind of normalized homomorphism counting of G in  $\eta$ .

To motivate our approach, consider a graphon  $W: J^2 \to \mathbb{R}_+$ , representing  $\eta$ . Then  $\eta^G$  is the measure on  $J^V$  whose Radon–Nikodym derivative with respect to  $\pi^V$  is equal to  $W^G$  (see (1)). These measures satisfy a certain logmodularity property, which relates  $\eta^G$  to measures corresponding to smaller graphs. Assume that  $V(G) = V = U \cup T$  such that there is no edge between  $U \setminus S$  and  $T \setminus S$ , where  $S = U \cap T$ . Then  $W^G W^{G[S]} = W^{G[U]} W^{G[T]}$ . We can rewrite this equation:

$$W^G/W^{G[S]} = (W^{G[U]}/W^{G[S]})(W^{G[T]}/W^{G[S]}).$$
 (2)

<sup>&</sup>lt;sup>1</sup>More formally, if  $f: S_1 \to S_2$  is an isomorphism between  $G[S_1]$  and  $G[S_2]$ , then it contra-variantly induces a function  $f^{\#}: J^{S_2} \to J^{S_1}$  by  $f^{\#}(x)_v := x_{f(v)}$ , and we require the pushforward measure  $f_*^{\#} \mu^{G[S_2]} := \mu^{G[S_2]} (f^{\#})^{-1}$  to be equal to  $\mu^{G[S_1]}$ .

Note that these quotients have a natural meaning even if W is allowed to vanish. For example

$$(W^G/W^{G[S]})(x_1, x_2, \dots, x_n) = \prod_{(i,j)\in E(G)\setminus E(S)} W(x_i, x_j).$$

One of the key observations is that equation (2) has a measure theoretic interpretation that allows us to extend it to singular measures. The function  $W^G/W^{G[S]}$  can be interpreted as a disintegration of the measure  $\eta^G$  with respect to  $\eta^{G[S]}$  (see Proposition 2.1). The only condition that we need for this type of disintegration is that the marginal of  $\eta^G$  on  $J^S$  be absolutely continuous with respect to  $\eta^{G[S]}$ . We will call this property of the family of measures *decreasing*.

For a Markov space  $\eta$  for which the homomorphism measures  $\eta^{G[S]}$  are defined and have the decreasing property, disintegration yields a family of measures  $\nu_{S,T,x}$ , where  $x \in J^S$  ( $S \subseteq T \subseteq V$ ), and  $\nu_{S,T,x}$  is a measure on  $J^{T \setminus S}$ .

In particular, when  $U \subseteq V$  is such that  $S = U \cap T$ ,  $V = U \cup T$  and there are no edges between  $T \setminus U$  and  $U \setminus T$ , the equation (2) translates to the condition

$$\nu_{S,V,x} = \nu_{S,U,x} \times \nu_{S,T,x}.$$
(3)

In the special case when  $S = \emptyset$ , this means that the measure assigned to the disjoint union of two graphs is the product of the measures assigned to them. We will call (3) the *Markovian property* of the family of the measures. As stated above, this type of Markovian property is not simply a property of a measure  $\eta^G$  by itself, but instead it describes how various measures  $\eta^G$ corresponding to a graph and its induced subgraphs are related to each other.

In addition, we impose the natural condition that the family of measures  $(\eta^{G[S]}: S \subseteq V)$  is normalized, in the sense that  $\eta^{K_2} = \eta$  for a single edge  $K_2$  and  $\eta^{K_1} = \pi$  for a single node  $K_1$ . We say that G is well-measured in  $\eta$  if there is a family of measures  $(\eta^{G[S]}: S \subseteq V)$  that is normalized, decreasing and Markovian. It will be an important additional property that  $\eta^G$  is finite. This easily implies that all other measures  $\eta^{G[S]}$  are finite. In this case we say that G is well-measured in  $\eta$  with finite density.

This Markovian property concept (3) allows for a recursive construction of measures for a graph utilizing measures of smaller graphs, decomposing G along a cutset. To initialize the construction, we need the normalized property. To apply it, we need a proper cutset of nodes in G; this is not available for complete graphs, and this is our main reason for having to exclude triangles.

This recursive construction has important consequences.

**Theorem 1.1** If G is a triangle-free graph, and there is a normalized, decreasing and Markovian family of measures on its induced subgraphs, then this family is uniquely determined.

This implies, in particular, that  $\eta^{G[S]}$  depends on the induced subgraph G[S] only. More precisely, if G[S] and G[T] are isomorphic induced subgraph and  $\xi : S \to T$  is an isomorphism, then the pushforward of  $\eta^{G[S]}$  to  $J^T$  is  $\eta^{G[T]}$ .

We will use this construction to prove the existence of such families of measures, but it will be a nontrivial question under what conditions are the measures independent of the choice of the particular way of building up the graph.

To guarantee the decreasing property for our measures thus constructed we will have to assume the decreasing property for small stars, which translates to the following "smoothness" property of the Markov space  $\eta$ . Choose a point x from the stationary distribution  $\pi$ , and make k independent single steps  $y_1, \ldots, y_k$  each starting from x so that  $(x, y_i) \sim \eta$ , and the  $y_i$ 's are conditionally mutually independent given x. Let  $\sigma_k$  denote the joint distribution of  $(y_1, \ldots, y_k)$ . We say that the Markov space is k-loose, if  $\sigma_k$  is absolutely continuous with respect to  $\pi^k$ . We shall also make use of a further refinement of this notion: a k-loose Markov space is (k, p)-loose, if the Radon–Nikodym derivative  $d\sigma_k/d\pi^k$  is in  $L^p(\pi^k)$ . This technical condition will turn out to be equivalent to the property that the complete bipartite graph  $K_{k,p}$  is well-measured in  $\eta$  with finite density (see Corollary 6.17).

The following result (see Section 5.4) will allow us to define the measure  $\eta^G$  (which is not a finite measure in general, see Example 5.10).

**Theorem 1.2** Let G = (V, E) be a triangle-free graph, and let  $\mathbf{M} = (J, \mathcal{B}, \eta)$ be a Markov space such that every complete bipartite subgraph  $K_{a,b}$  of G is well-measured in  $\mathbf{M}$ . Then G is well-measured in  $\mathbf{M}$ .

The condition implies that the Markov chain is k-loose, where k is the maximum degree of G. If the graph contains no 4-cycles, then the stars are the only complete bipartite subgraphs; hence we can state the following corollary.

**Corollary 1.3** Let  $(J, \mathcal{B}, \eta)$  be a k-loose Markov space. Then every graph of girth at least 5 and with all degrees at most k is well-measured in  $\eta$ .

It will turn out that densities of bipartite graphs are much better behaved, and we have more transparent formulas for them. Using these formulas, we will prove the following (see Sections 6.5 and 6.6). **Theorem 1.4** Let  $(J, \mathcal{B}, \eta)$  be an (a, b)-loose Markov space. Let G = (V, E) be a bipartite graph with bipartition  $V = U \cup W$  such that  $\deg(w) \leq a$  for all  $w \in W$  and  $\deg(u) \leq b$  for all  $u \in U$ . Then G is well-measured in  $\eta$  with finite density, and  $\eta^G$  is partition approximable.

Densities of cycles are particularly interesting because of their connection with operator theory. Every Markov space  $(J, \mathcal{B}, \eta)$  acts naturally as a bounded operator  $\mathbf{A}_{\eta}$  on  $L^2(J, \pi)$ . We prove the next theorem (see Theorem 7.5).

**Theorem 1.5** If the k-th Schatten norm of  $\mathbf{A}_{\eta}$  is finite for some  $k \in \mathbb{N}$ , then  $C_k$  is well-measured in  $\eta$  with finite density,  $\eta^{C_k}$  is partition approximable, and  $t(C_k, \eta) = \operatorname{tr}(\mathbf{A}_{\eta}^k)$ .

Note that the finiteness of the k-th Schatten norm implies that  $\mathbf{A}_{\eta}$  is a compact operator with eigenvalues  $\{\lambda_i\}_{i=1}^{\infty}$  such that the series  $\sum_{i=1}^{\infty} \lambda_i^k$ is absolutely convergent, and we have  $\operatorname{tr}(\mathbf{A}_{\eta}^k) = \sum_{i=1}^{\infty} \lambda_i^k$ . This suggests a third way to define the density of a graph in Markov spaces using spectral approximations provided that the operator  $\mathbf{A}_{\eta}$  is compact. This direction, however, is not explored in this paper.

### **2** Preliminaries

#### 2.1 Notation

All graphs considered are finite and simple. A *bipartite graph* is a graph that is 2-colorable. A *bigraph* is a bipartite graph with a fixed bipartition, where the order of bipartition classes is also specified. Formally, a bigraph is a triple G = (U, W, E), where  $E \subseteq U \times W$ . Let  $K_{a,b}$  denote the complete bipartite graph with bipartition  $U \cup W$ , where |U| = a and |W| = b. With a slight abuse of notation, we also denote the bigraph  $(U, W, U \times W)$  by  $K_{a,b}$ .

For a map  $x \in J^V$ , we denote by  $x_V$  the image of V under this map, as a labeled set  $(x_v: v \in V)$ .

For a measurable space  $(I, \mathcal{A})$ , we denote by  $\mathfrak{M}(I, \mathcal{A})$  (or simply by  $\mathfrak{M}(\mathcal{A})$ ) the set of all finite measures on  $\mathcal{A}$ . If  $\mu \in \mathfrak{M}(\mathcal{A})$  and  $f \in L^1(\mu)$ , then we define the measure  $f \cdot \mu$  and the number  $\mu(f)$  by

$$(f \cdot \mu)(B) = \int_B f \, d\mu, \qquad \mu(f) = (f \cdot \mu)(J) = \int_J f \, d\mu.$$

If  $(I, \mathcal{A}, \pi)$  is a probability space and  $f, g : I \to \mathbb{R}$  are measurable functions, then we define

$$\langle f,g \rangle = \langle f,g \rangle_{\pi} = \int_{J} f(x)g(x) \, d\pi(x)$$

If V is a finite set,  $\emptyset \neq S \subseteq V$ , and  $\mu$  is a measure on  $\mathcal{A}^V$ , then we denote by  $\mu^S$  the marginal of  $\mu$  on  $\mathcal{A}^S$ .

#### 2.2 Derivative and disintegration

Let  $\mu$  and  $\nu$  be two measures on the same Borel space  $(J, \mathcal{B})$ . We say that a function  $f: J \to [0, \infty]$  is the *Radon–Nikodym derivative* of  $\nu$  with respect to  $\mu$ , denoted by  $f = d\nu/d\mu$ , if  $\nu = f \cdot \mu$ . Note that we allow infinite values for f, under the convention  $\int_B \infty d\mu = 0$  whenever  $\mu(B) = 0$ . The existence of the Radon–Nikodym derivative is usually stated for two sigma-finite measures, but we'll need a slightly more general fact (see Appendix 9.1).

Let  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  be measurable spaces, and let  $\Phi = (\mu_x : x \in I)$  be a family of sigma-finite measures on  $(J, \mathcal{B})$ . We say that  $\Phi$  is a *measurable* family, if  $\mu_x(B)$  is a measurable function of x for every  $B \in \mathcal{B}$ . Note that if  $\Phi$  is a family of probability measures, then this essentially corresponds to a Markov kernel (see Subsection 2.4).

We need the following version of the Disintegration Theorem (which is usually stated for the case when  $\alpha = \gamma^1$ ); see e.g. [8] or [2], Section 10.6.

**Proposition 2.1** Let  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  be standard Borel spaces. Let  $\alpha$  be a sigma-finite measure on  $\mathcal{A}$ , and let  $\gamma$  be a sigma-finite measure on  $\mathcal{A} \times \mathcal{B}$ . Then there is a measurable family  $\Phi$  of measures on  $\mathcal{B}$  such that for every bounded measurable function  $f : I \times J \to [0, \infty)$ ,

$$\int_{I \times J} f(x, y) d\gamma(x, y) = \int_{I} \int_{J} f(x, y) d\mu_x(y) d\alpha(x).$$
(4)

if and only if the marginal  $\gamma^1$  of  $\gamma$  on  $\mathcal{A}$  is absolutely continuous with respect to  $\alpha$ . Furthermore, the measurable family  $\Phi = (\mu_x : x \in I)$  is uniquely determined up to changing  $\mu_x$  for x in a zero  $\alpha$ -measure subset of I.

We say that the measurable family  $\Phi$  is a *disintegration* of the measure  $\gamma$  with respect to the measure  $\alpha$ .

A key component of our constructions will be a "reverse" of the disintegration, essentially integrating a measurable family  $\Phi$  with respect to a suitable measure  $\alpha$  to obtain a sigma-finite measure on the product space. Indeed, consider a measurable family  $\Phi$  of finite measures on  $\mathcal{B}$ . For  $\alpha \in \mathfrak{M}(\mathcal{A})$ , define

$$\alpha[\Phi](A \times B) = \int_{A} \mu_x(B) \, d\alpha(x) \qquad (A \in \mathcal{A}, \ B \in \mathcal{B}).$$
(5)

If the measures  $\mu_x$  are finite and uniformly bounded, this extends to a finite measure  $\alpha[\Phi]$  on  $\mathcal{A} \times \mathcal{B}$ , whose disintegration with respect to  $\alpha$  is trivially  $\Phi$ .

The marginal of  $\alpha[\Phi]$  on  $\mathcal{A}$  is the measure  $g \cdot \alpha$ , where  $g(x) = \mu_x(J)$ . The marginal of  $\alpha[\Phi]$  on  $\mathcal{B}$  is the mixture of  $\Phi$  by  $\alpha$ . The definition also implies that if all of the  $\mu_x$ , as well as  $\alpha$ , are probability distributions, then so is  $\alpha[\Phi]$ , and  $\alpha$  is the marginal of  $\alpha[\Phi]$  on  $\mathcal{A}$ . Conversely, if  $\alpha[\Phi]$  and  $\alpha$  are probability distributions, then  $\mu_x$  is a probability distribution for  $\alpha$ -almost all  $x \in J$ .

An important example of this construction will be the family of distributions of transition probabilities in a Markov chain; see Section 2.4 below.

The sigma-finite extension also goes through in case we drop the uniform boundedness condition, by partitioning I into countably many measurable parts corresponding to the level sets [k, k+1)  $(k \in \mathbb{N}_0)$  of the total measure function  $\mu_x(J)$ . On each such set uniform boundedness is satisfied, and the above applies.

Note, however, that for general families  $\Phi$  of sigma-finite measures, one quickly encounters technical difficulties with the extension. Fortunately, as shown in the following lemma, a family  $\Phi$  that arises from a disintegration is well-behaved.

**Lemma 2.2** Let  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  be standard Borel spaces. Let  $\alpha_1, \alpha_2$  be sigma-finite measures on  $\mathcal{A}$  with  $\alpha_2$  absolutely continuous with respect to  $\alpha_1$ , and let  $\gamma$  be a sigma-finite measure on  $\mathcal{A} \times \mathcal{B}$  such that the marginal  $\gamma^1$  of  $\gamma$ on  $\mathcal{A}$  is absolutely continuous with respect to  $\alpha_1$ . Let  $\Phi = (\mu_x : x \in I)$  be disintegration of  $\gamma$  with respect to  $\alpha_1$ . Then  $\alpha_2[\Phi]$  defined via (5) extends to a sigma-finite measure on  $\mathcal{A} \times \mathcal{B}$ .

**Proof.** As  $\alpha_2$  is absolutely continuous with respect to  $\alpha_1$ , and both are sigma-finite, we may write I as a countable disjoint union  $\bigcup_k I_k$  of measurable sets with  $\alpha_j(I_k) < \infty$  for all k and j = 1, 2, and it suffices then to prove the existence of the appropriate extension on each  $I_k \times J$ . We may therefore without loss of generality restrict our attention to the case of both  $\alpha_1$  and  $\alpha_2$  being finite.

As  $\gamma$  is sigma-finite, consider a partition of  $I \times J$  into a countable disjoint union  $\bigcup_k G_k$  of measurable sets with  $\gamma(G_k)$  finite. Let  $\gamma_{G_k}$  be the restriction of  $\gamma$  to  $G_k$ , and  $\Phi_k = (\mu_{k,x} : x \in I)$  its disintegration with respect to  $\alpha_1$ . Note that

$$\int_{I} \mu_{k,x}(J) d\alpha(x) = \int_{I} \int_{J} 1 d\mu_{k,x}(y) d\alpha(x) = \int_{I \times J} 1 d\gamma_{G_k}(x,y) = \gamma_{G_k}(I \times J)$$

is finite, hence we have that  $\mu_{k,x}$  is finite for  $\alpha_1$ -a.e.  $x \in I$ .

Since the  $G_k$ 's are disjoint, we have that  $\gamma_{G_k} \perp \gamma_{G_\ell}$  for any  $k \neq \ell$ , and thus also for  $\alpha_1$ -a.e.  $x \in I$ ,  $\mu_{k,x} \perp \mu_{\ell,x}$ . Consequently, we have that  $\mu_x = \sum_k \mu_{x,k}$ and it is sigma-finite for  $\alpha_1$ -a.e.  $x \in I$ . Since  $\alpha_2$  is absolutely continuous with respect to  $\alpha_1$ , for product sets  $A \times B$  ( $A \in \mathcal{A}, B \in \mathcal{B}$ ), we by (5) clearly have

$$\alpha_2[\Phi](A \times B) = \sum_k \alpha_2[\Phi_k](A \times B).$$

Since each  $\alpha_2[\Phi_k]$  extends to a sigma-finite measure on  $\mathcal{A} \times \mathcal{B}$ , so does their countable sum  $\alpha_2[\Phi]$ .

**Remark 2.3** By the above, if  $\Phi$  is the disintegration of a sigma-finite measure  $\gamma$  with respect to  $\alpha$ , we have  $\gamma = \alpha[\Phi]$ .

### 2.3 Markov property

Let  $(J, \mathcal{B})$  be a standard Borel space, and let V be a finite set. Let  $\mathcal{M} = (\mu_S \in \mathfrak{M}(\mathcal{B}^S) : S \subseteq V)$  be a family of measures. We say that  $\mathcal{M}$  is *decreasing*, if for  $S \subseteq T \subseteq V$ , the marginal  $(\mu_T)^S$  is absolutely continuous with respect to  $\mu_S$ . A trivial example of such a family is  $\mu_S = \phi^S$  for any  $\phi \in \mathfrak{M}(\mathcal{B}^V)$ , which we call the *marginal family* defined by  $\phi$ .

If  $\mathcal{M} = (\mu_S \in \mathfrak{M}(\mathcal{B}^S) : S \subseteq V)$  is a decreasing family of sigma-finite measures, then for  $S \subseteq T$ , the Disintegration Theorem (Proposition 2.1) gives a measurable family of measures  $N_{S,T} = (\nu_{S,T,x} : x \in J^S)$  on  $\mathcal{B}^{T \setminus S}$  such that

$$\mu_S[N_{S,T}] = \mu_T. \tag{6}$$

This definition implies that the Radon–Nikodym derivative  $d(\mu_T)^S/d\mu_S$  exists and it can be expressed as

$$\frac{d(\mu_T)^S}{d\mu_S}(x) = \nu_{S,T,x}(J^{T\setminus S}) \tag{7}$$

for  $\mu_S$ -almost all  $x \in J^S$ .

We can informally think of  $\nu_{S,T,x}$  as the measure on extensions of x from S to T. This motivates the following "chain rule". For  $S, T \subseteq V$  with  $S \cap T = \emptyset$  and  $x \in J^S$ ,  $y \in J^T$ , we denote by  $xy \in J^{S \cup T}$  the union of the maps x and y. Then for  $S \subseteq T \subseteq U \subseteq V$ , we can first extend  $x \in J^S$  to an  $xy \in J^T$ , and then extend xy to U. Defining  $N_{T,U,x} = (\nu_{S,T,xy} : y \in J^{T \setminus S})$ , we can write this as

$$\nu_{S,U,x} = \nu_{S,T,x}[N_{T,U,x}] \tag{8}$$

for  $\mu_S$ -almost all  $x \in J^S$ . Indeed, for every  $A \in \mathcal{B}^S$ ,  $B \in \mathcal{B}^{T \setminus S}$  and  $C \in \mathcal{B}^{U \setminus T}$ , using (4),

$$\int_{A} \nu_{S,U,x}(B \times C) \, d\mu_S(x) = \mu_U(A \times B \times C) = \int_{A \times B} \nu_{T,U,xy}(C) \, d\mu_T(xy)$$
$$= \int_{A} \int_{B} \int_{B} \nu_{T,U,xy}(C) \, d\nu_{S,T,x}(y) \, d\mu_S(x).$$

This holds for every  $A \in \mathcal{B}^S$ , which proves (8).

Let G = (V, E) be a finite simple graph. Let  $\mathcal{M} = (\mu_S : S \subseteq V)$  be a family of sigma-finite measures, with the corresponding disintegrations  $N_{S,T}$ . We say that  $\mathcal{M}$  is *Markovian*, or has the *Markov property* (with respect to G), if it is decreasing, and for any two sets  $U, W \subseteq V$  and  $S = U \cap W$  such that no edge connects  $U \setminus S$  and  $W \setminus S$ , and for  $\mu_S$ -almost all  $x \in J^S$ , we have

$$\nu_{S,U\cup W,x} = \nu_{S,U,x} \times \nu_{S,W,x}.$$
(9)

**Lemma 2.4** A decreasing family  $\mathcal{M} = (\mu_S : S \subseteq V)$  of sigma-finite measures has the Markov property with respect to a graph G if and only if

 $\nu_{U,U\cup W,xy} = \nu_{U\cap W,W,x}$ 

holds for all  $U, W \subseteq V$  with no edges connecting  $U \setminus W$  and  $W \setminus U$ , for  $\mu_{U \cap W}$ -almost all  $x \in J^{U \cap W}$  and for  $\nu_{U \cap W, U, x}$ -almost all  $y \in J^{U \setminus W}$ .

In particular, the measure on the left is independent of y almost everywhere.

**Proof.** To prove the necessity of the condition, let U and W be as in the lemma, and set  $S = U \cap W$ . Suppose that (9) holds, then for all  $B \in \mathcal{B}^{U \setminus W}$  and  $C \in \mathcal{B}^{W \setminus U}$ , and  $\mu_S$ -almost all  $x \in J^S$  we have

$$\nu_{S,U\cup W,x}(B\times C) = \nu_{S,U,x}(B)\nu_{S,W,x}(C),$$

but also by (8) and the chain rule,

$$\nu_{S,U\cup W,x}(B \times C) = \nu_{S,U,x}[N_{U,U\cup W,x}](B \times C)$$
$$= \int_{B} \nu_{U,U\cup W,xy}(C) \, d\nu_{S,U,x}(y).$$

It follows that

 $\nu_{U,U\cup W,xy}(C) = \nu_{S,W,x}(C)$ 

must hold for all  $C \in \mathcal{B}^{W \setminus S}$ ,  $\mu_S$ -almost all  $x \in J^S$  and  $\nu_{S,U,x}$ -almost all  $y \in J^{U \setminus S}$ . This proves the necessity of the condition in the Lemma. The reverse implication follows by a similar computation.

Markovian measure families are related to, but different from, Markov random field on graphs. See Appendix 9.2 for the details of this connection (which we don't use in this paper).

#### 2.4 Markov spaces, graphons and bigraphons

A Markov space consists of a sigma-algebra  $(J, \mathcal{B})$ , together with a probability measure  $\eta$  on  $(J \times J, \mathcal{B} \times \mathcal{B})$  whose marginals are equal. In this paper, we assume that  $(J, \mathcal{B})$  is a standard Borel sigma-algebra. In the probability literature,  $\eta$  is often called the *ergodic flow*, or *ergodic circulation*, and its marginals  $\pi(A) = \eta(A \times J) = \eta(J \times A)$  are the *stationary distribution* of the Markov space  $(\mathcal{B}, \eta)$ . A Markov space is *symmetric*, if  $\eta(A \times B) = \eta(B \times A)$ for all  $A, B \in \mathcal{B}$ . We note already here that beyond Remark 2.5, all Markov spaces will be assumed to be symmetric unless stated otherwise.

Markov spaces are intimately related to Markov chains. A Markov chain is usually defined on a sigma-algebra  $(J, \mathcal{B})$ , specifying a probability measure  $P_u$  on  $\mathcal{B}$  for every  $u \in J$ , called the *step distributions*. One assumes that for every  $A \in \mathcal{B}$ , the value  $P_u(A)$  is a measurable function of  $u \in J$ . The map  $u \mapsto P_u$  is called a *Markov scheme* or *Markov kernel*. To get a Markov space, we also have to assume that the Markov chain has a *stationary distribution*  $\pi$  on  $\mathcal{B}$  satisfying

$$\int_{J} P_u(A) d\pi(u) = \pi(A) \tag{10}$$

for all  $A \in \mathcal{B}$ , and we fix such a distribution. (A Markov scheme may have none or more than one stationary distributions.) Then

$$\eta(A \times B) = \int_{A} P_u(B) \, d\pi(u) \qquad (A, B \in \mathcal{B}) \tag{11}$$

defines a Markov space. Conversely, every Markov space arises from an essentially unique Markov scheme this way; this can be constructed by disintegrating  $\eta$  with respect to  $\pi$  (see Section 2.2). The Markov scheme is time-reversible precisely when this Markov space is symmetric.

As a generalization of the notion of bigraphs, we define a *bi-Markov space* as a quintuple  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$ , where  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  are standard Borel spaces, and  $\eta$  is a probability measure on  $\mathcal{A} \times \mathcal{B}$ . We denote the marginals of  $\eta$  on I and J by  $\pi_I$  and  $\pi_J$ , respectively. While a bi-Markov space does not directly define a Markov chain, the disintegration of  $\eta$  according to  $\pi_I$ still makes sense, and gives a measurable family  $(P_u : u \in I)$  of measures on  $(J, \mathcal{B})$  such that

$$\eta(A \times B) = \int_{A} P_u(B) \, d\pi_I(u) \tag{12}$$

for  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ , similarly to the symmetric case. However, from a point  $u \in I$  you step to a point  $w \in J$ , so the step cannot be repeated.

In a bigraph G = (U, W, E), we can interchange the bipartition classes to obtain another bigraph  $G^* = (W, U, E^*)$ , which is isomorphic to G as an undirected graph. Similarly, for every bi-Markov space  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$ , we can construct the reverse bi-Markov space  $\mathbf{M}^* = (J, I, \mathcal{B}, \mathcal{A}, \eta^*)$ .

**Remark 2.5** If we identify the Borel spaces  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  (which is usually possible), we get an (asymmetric) Markov space, which is a generalization of directed graphs (digraphs). A symmetric Markov space is a generalization of undirected graphs, and a bi-Markov space is a generalization of bigraphs. If we identify  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  and also assume that  $\pi_I = \pi_J$ , then we look at a generalization of Eulerian digraphs; these are also equivalent to (not necessarily reversible) Markov chains with a fixed stationary distribution.

Extending our results to digraphs (Eulerian or not) would be interesting, but in this paper we only deal with Markov spaces generalizing undirected graphs and bigraphs: symmetric Markov spaces and bi-Markov spaces. For the rest of this paper, we drop the adjective "symmetric".

Let  $(J, \mathcal{B}, \pi)$  be a standard Borel probability space, and let  $W : J^2 \to \mathbb{R}_+$ be a graphon, a symmetric integrable function with respect to  $\pi$ . In the theory of dense graph limits, graphons are assumed to be bounded by 1, but since then, much of the theory has been extended to the unbounded case [16, 5]. If a graphon is bounded, then it can be scaled to a 1-bounded graphon. We call W 1-regular, if  $\int_I W(x, y) d\pi(y) = 1$  for all x.

Every 1-regular graphon W determines a Markov space  $\eta_W = W \cdot (\pi \times \pi)$ . Trivially,  $\eta_W$  is absolutely continuous with respect to  $\pi^2$ . Conversely, if we have a Markov space for which  $\eta$  is absolutely continuous with respect to  $\pi \times \pi$ , then the Radon–Nikodym derivative  $W = d\eta/d\pi^2$  is a corresponding 1-regular graphon.

Let  $(I, \mathcal{A}, \pi_I)$  and  $(J, \mathcal{B}, \pi_J)$  be standard Borel probability spaces. A *bigraphon* is a bounded measurable function  $W : I \times J \to \mathbb{R}_+$ . The bigraphon is 1-regular, if

$$\int_{I} W(x, \cdot) d\pi_{I}(x) = \int_{J} W(\cdot, y) d\pi_{J}(y) = 1.$$
(13)

Every 1-regular bigraphon defines a bi-Markov space by

$$\eta = W \cdot (\pi_I \times \pi_J).$$

#### 2.5 Graphops and linear functionals

Let us survey some notions related to Markov spaces with a functional analysis flavor; these were introduced in the theory of *action convergence* [1].

Every Markov space defines an operator  $\mathbf{A} = \mathbf{A}_{\eta}$ :  $L^{1}(\pi) \to L^{1}(\pi)$  by

$$(\mathbf{A}_{\eta}f)(x) = \mathsf{E}(f(x')) = P_x(f) = \int_J f(y) \, dP_x(y),$$

where x' is the point obtained by a random step from x. The integral on the right is well-defined for  $\pi$ -almost-all  $x \in J$ . We call **A** the *adjacency operator* of the Markov space. This operator is contractive with respect to any  $L^{p}$ -norm  $(p \geq 1)$ . Hence it maps every subspace  $L^{p}(\pi)$  into itself, and  $\|\mathbf{A}\|_{p\to p} = 1$  for every  $p \in [1, \infty]$ . The adjacency operator is monotone, self-adjoint, and 1-regular (which means that  $\mathbb{1}_{J}$  is an eigenfunction with eigenvalue 1). A monotone and self-adjoint bounded linear operator  $L^{\infty}(\pi) \to L^{1}(\pi)$  is called a *graphop*, so the adjacency operator, restricted to  $L^{\infty}(\pi)$ , is a 1-regular graphop.

We also note that for every  $B \in \mathcal{B}$  and  $\pi$ -almost-all x,

$$(\mathbf{A}\mathbb{1}_B)(x) = P_x(B),\tag{14}$$

since for every  $A \in \mathcal{B}$ ,

$$\int_{A} (\mathbf{A} \mathbb{1}_{B})(x) d\pi(x) = \int_{A} \int_{J} \mathbb{1}_{B}(y) dP_{x}(y) d\pi(x) = \int_{J^{2}} \mathbb{1}_{A}(x) \mathbb{1}_{B}(y) d\eta(x, y)$$
$$= \eta(A \times B) = \int_{A} P_{x}(B) d\pi(x).$$

Theorem 6.3 in [1] implies that, conversely, every self-adjoint, monotone, 1regular and contractive operator  $\mathbf{A}$ :  $L^p(J,\pi) \to L^p(J,\pi)$   $(p \ge 1)$  is the adjacency operator of a Markov space with stationary measure  $\pi$ .

It is clear that the k-th power of the adjacency operator is itself an adjacency operator of a Markov space. In the Markov chain setting, this corresponds to considering k consecutive steps as one. The edge measure of this new Markov space will be denoted by  $\eta^k$ .

If a Markov space is defined by an  $L^2$ -graphon (a function in  $L^2(\pi^2)$ ), then its adjacency operator  $\mathbf{A}$  is a Hilbert-Schmidt operator, and hence it is compact. It is well known that for a symmetric operator  $\mathbf{A}$  on a Hilbert space and any integer  $k \geq 1$ ,  $\mathbf{A}^k$  is compact if and only if  $\mathbf{A}$  is compact. Often we'll be concerned with Markov spaces for which a finite power of  $\mathbf{A}_{\eta}$  is defined by a graphon, and so  $\mathbf{A}_{\eta}$  is a compact operator. However, see Example 6.19 for a Markov space with an "almost" compact adjacency operator, to which extensions of our results would be particularly desirable.

**Remark 2.6** The finite version of the probability measure  $\eta$  of a Markov space is the uniform measure on the edges of a finite graph. The marginal  $\pi$ is the stationary distribution of the random walk, where the probability of a vertex is proportional to its degree. It is natural to introduce the uniform measure on the vertices as well. In the general case, this means endowing a Markov space  $(J, \mathcal{B}, \pi)$  with an additional probability measure  $\lambda$  on  $(J, \mathcal{B})$ . This richer structure would then include non-regular graphons, general (not necessarily 1-regular) graphops, and s-graphons as defined in [17]. Putting it in a slightly sloppy form,

reversible Markov chain + stationary distribution  $\cong$  Markov space

and

Markov space + vertex distribution  $\cong$  graphop  $\cong$  s-graphon.

Extending the results of this paper to the case when a vertex-measure is present is an important task for further research.

For bi-Markov spaces, the operator  $\mathbf{A}$  can be defined just as above, except that  $\mathbf{A}$  will not be self-adjoint.

#### 2.6 Partitions

Let  $(J, \mathcal{B}, \pi)$  be a standard probability space, and let  $\mathcal{P} = \{J_1, \ldots, J_n\}$  be a finite, measurable, non-degenerate partition of J (this means that  $J_i \in \mathcal{B}$  and

 $\pi(J_i) > 0$ ). Let  $\widehat{\mathcal{P}}$  denote the (finite) set algebra generated by the partition classes in  $\mathcal{P}$ . We denote by  $\mathcal{P}^k$  the partition of  $J^k$  whose classes are the product sets  $J_{i_1} \times \cdots \times J_{i_k}$ .

**Definition 2.7** Let  $\mathcal{R}$  be a countable family of Borel sets. Let  $\sigma(\mathcal{R})$  denote the sigma-algebra generated by  $\mathcal{R}$ . We say that  $\mathcal{R}$  is *generating*, if  $\sigma(\mathcal{R}) = \mathcal{B}$ . We say that  $\mathcal{R}$  is *exhausting* with respect to a measure  $\pi$  on  $(J, \mathcal{B})$ , if for every  $A \in \mathcal{B}$  there is a set  $B \in \sigma(\mathcal{R})$  such that  $\pi(A \triangle B) = 0$ . Clearly every generating family is exhausting.

A partition sequence is a sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$  of finite measurable nondegenerate partitions of  $(J, \mathcal{B}, \pi)$  such that  $\mathcal{P}_{i+1}$  is a refinement of  $\mathcal{P}_i$ . We associate with every partition sequence the set families  $\mathcal{R} = \bigcup_i \mathcal{P}_i$  and  $\widehat{\mathcal{R}} = \bigcup_i \widehat{\mathcal{P}}_i$ . We say that a partition sequence is generating [exhausting], if the family  $\mathcal{R}$  is generating [exhausting].

It is easy to see that every set in  $\widehat{\mathcal{R}}$  is a finite union of disjoint members of  $\mathcal{R}$ . The family  $\widehat{\mathcal{R}}$  is closed under finite union, finite intersection, and complementation, so it is a set algebra.

We note that there is not much difference between talking about exhausting or generating partition sequences: every exhausting partition sequence can be transformed in a generating one by changing partitions on a  $\pi$ -null-set (see Appendix 9.3).

## 3 Subgraph densities: known cases

We recall a couple of special classes of Markov spaces where subgraph densities have been introduced and studied.

#### 3.1 Graphons

Subgraph densities (or, to be more exact, homomorphism densities) can be defined for bounded graphons. In fact, all densities are still finite if we extend our attention to unbounded symmetric functions  $W : [0,1]^2 \to \mathbb{R}_+$  in  $L^{\omega}$ , see [16]. If the degrees of the graphs mapped into the graphon are bounded by p, then subgraph densities can actually be defined for all of  $L^p$ -graphons [5]. Subgraph densities can also be defined in graphings, but this seems to be rather different from the dense case. It is possible that this notion cannot be extended to all Markov spaces; but we will be able to do so for Markov spaces which are sufficiently rich. For the question to make sense in more general situations, we modify the normalization of subgraph densities. Recall that for a graphon  $W: J^2 \rightarrow [0, 1]$ , the density of a graph G = (V, E) in W is defined by the integral

$$t(G,W) = \pi^{V}(W^{G}) = \int_{J^{V}} W^{G}(x) \, d\pi^{V}(x), \tag{15}$$

where

$$W^G(x) = \prod_{ij \in E(G)} W(x_i, x_j) \qquad (x \in J^V).$$

$$(16)$$

If  $W = W_H$  is the graphon associated with a graph H, then

$$t(G, W_H) = t(G, H) = \frac{\hom(G, H)}{|V(H)|^{|V(G)|}}$$

is the homomorphism density of  $G \to H$ . In this paper we use the normalization

$$t^*(G, W) = \frac{t(G, W)}{t(K_2, W)^{|E(G)|}}.$$
(17)

Note that the right hand side of (17) is invariant under scaling the function W. If

$$t(K_2, W) = \int_{J^2} W \, d\pi^2 = 1,$$

(in particular, if W is 1-regular) we have  $t^*(G, W) = t(G, W)$  for every G.

It will be very useful to consider the measure  $W^G \cdot \pi^V$  with density function  $W^G$  on  $J^V$ . This measure has nice properties, for example, it is Markovian. We call this the *density measure* of G in W. This construction will be particularly useful when we generalize the above formulas to the case when W is not bounded. Then the density (15) may be infinite, but we still obtain a sigma-finite measure  $W^G \cdot \pi^V$  on maps  $V \to J$ . See Section 5.3 for a detailed discussion of this generalization.

For a bigraph G = (S, T, E) and a bigraphon  $\mathbf{M} = (I, \mathcal{A}, J, \mathcal{B}, \pi_I, \pi_J, W)$ , there is a natural version of the subgraph density:

$$t(G,W) = \int_{I^S} \int_{J^T} \prod_{ij \in E} W(x_i, y_j) d\pi_J^T(y) d\pi_I^S(x).$$
(18)

Clearly  $t(G, W) = t(G^*, W^*)$ .

#### 3.2 Orthogonality spaces

Consider the Borel sets in the (d-1)-dimensional unit sphere  $S^{d-1}$ , and let  $\eta$  be the uniform measure on orthogonal pairs of vectors in  $S^{d-1}$ . This class of Markov spaces was studied in detail in [18]. Maps  $V \to S^{d-1}$  that map edges onto orthogonal pairs are called *ortho-homomorphisms*.

**Example 3.1** The case of complete bipartite graphs will be important. Consider the complete bigraph  $K_{a,b} = (U, W, U \times W)$ , where |U| = a, |W| = b, and a + b = d + 1. The first relevant example is mapping the 4-cycle into  $S^2$ . Let  $x \in (S^{d-1})^V$  be an ortho-homomorphism. Since the image of U spans a subspace that is orthogonal to the subspace spanned by the image of W, one of the color classes must be mapped onto linearly dependent vectors. If this degenerate color class is U, then W can be mapped freely into  $x(U)^{\perp}$ . So every homomorphism is degenerate, and if  $a, b \geq 2$ , then there are two possible degenerations. This means that there is no "natural" or "canonical" way of defining a measure on ortho-homomorphisms. It is also easy to observe that the trouble is caused by the fact that making d random single steps each starting from a given point of  $S^{d-1}$ , we obtain d linearly dependent points, so the joint distribution of these d points is singular.

To motivate some of our later arguments, let us try to construct an orthohomomorphism of the 4-cycle into  $S^3$  by mapping the nodes one-by-one. The first three nodes can be mapped in an arbitrary order (taking care of the orthogonality of images of edges). Almost surely the neighbors of the fourth node will be neither equal nor antipodal, and so this node must be mapped either on the image of its non-neighbor, or on its antipodal. Leaving instead one of its neighbors for last, the other pair of non-neighbors will be parallel, so we obtain a totally different distribution.

It was shown in [18] that a canonical "nice" Markovian sigma-finite measure on the ortho-homomorphisms into  $S^{d-1}$  can be defined for every graph G not containing  $K_{a,b}$  with a + b = d + 1. Furthermore, the density of G in  $\eta_d$  can also be defined (it may be infinite). The construction followed the same lines as our treatment in Section 5 below, providing explicit formulas in this special case.

### 4 Trees

The case of mapping trees into Markov spaces is easy, but it will be a very useful starting point for the more general case. For a tree F, we denote by L(F) the set of its leaves and by M(F) the set of its interior nodes. In the

case of a tree denoted by F, we will set L = L(F) and M = M(F). So  $F \setminus L = F[M]$  is the subtree induced by the internal nodes of F.

Let  $S_n$  and  $P_n$  denote the star and the path with n edges, respectively. Unless stated otherwise, we label  $V(S_k) = \{0, 1, \ldots, k\}$  with 0 in the center. The tree consisting of a single edge uv can be viewed either as a path  $P_1$ , or as a star  $S_1$ . We distinguish them by letting  $P_1$  have two leaves, so  $L(P_1) = \{u, v\}$  and  $M(P_1) = \emptyset$ , and designating one of the nodes of  $S_1$ (say u) as its center, and the other one as its leaf, so that  $L(S_1) = \{v\}$  and  $M(S_1) = \{u\}$ . It will be convenient to consider the tree  $S_0$  with a single node u, where we have  $L(S_0) = \emptyset$  and  $M(S_0) = \{u\}$ .

Let F be a tree and  $uv \in E(F)$ . The subtree  $F_1$  of F induced by u and all nodes separated from u by the edge uv is called a *branch of* F *attached at* u. We denote by  $F \setminus F_1$  the subtree obtained by deleting from F the nodes in  $V(F_1) \setminus \{u\}$ .

#### 4.1 Random mappings of trees

Our first step is to show that a random mapping of a tree into a Markov space can be defined in a robust (and, as we shall see, useful) way. This simple construction is well-known (branching Markov chains etc.), but we need some special properties of it.

**Definition 4.1** Let F = (V, E) be tree, and  $(J, \mathcal{B}, \eta)$ , a Markov space. We define a random homomorphism of F into  $\eta$  as a random map  $\mathbf{h} : V(F) \to J$ , recursively as follows. If |V(F)| = 1, then we define  $\mathbf{h}$  as a random point from  $\pi$ . If |V(F)| > 1, then let u be a leaf of F, incident with a single edge uv. The random map  $\mathbf{h}' : V(F \setminus u) \to J$  is already constructed. We let  $\mathbf{h}|_{V\setminus u} = \mathbf{h}'$ , and we define  $\mathbf{h}(u)$  by making a Markov step from the point  $\mathbf{h}'(v)$ . We denote the distribution of  $\mathbf{h}$  by  $\eta^F$ . In formula, for  $W \in \mathcal{B}^{V\setminus u}$  and  $A \in \mathcal{B}$ ,

$$\eta^F(W \times A) = \int\limits_W P_{x_u}(A) \, d\eta^{F'}(x). \tag{19}$$

We can also describe this construction slightly differently. Let  $(v_1, \ldots, v_n)$  be a search order of V(F), i.e. an ordering for which every node  $v_i$  different from the "root"  $v_1$  is adjacent to exactly one earlier node  $v_{i'}$   $(1 \le i' < i)$ . We select  $\mathbf{h}(v_1)$  from  $\pi$ , and for  $i = 2, \ldots, n$  we generate  $\mathbf{h}(v_i)$  by making a Markov step from  $\mathbf{h}(v_{i'})$ . We call this the *sequential construction* of the random map.

**Lemma 4.2** The recursive definition (19) gives a distribution  $\eta^F$  that is independent of the leaf chosen. Equivalently, if constructed sequentially, it is independent of the search order chosen.

**Proof.** We proceed by induction on the number of vertices in F. If |V(F)| = 1, then clearly  $\eta^F = \pi$ , and if  $F = P_1$ , then we have

$$\eta^{P_1}(A_1 \times A_2) = \eta^{S_1}(A_1 \times A_2) = \int_{A_1} P_{x_1}(A_2) \, d\pi(x_1) = \eta(A_1 \times A_2),$$

which remains the same when the indices are interchanged by symmetry. Now suppose that |V(F)| > 2, and let u, w be two leaves, with neighbors vand z, respectively (v = z is possible). Then u and w are not adjacent, and so  $F'' = F \setminus u \setminus w$  is a tree. We have

$$\int_{\prod(A_i:\ i\in V\setminus u)} P_{x_v}(A_u) \, d\eta^{F'}(x) = \int_{\prod(A_i:\ i\in V\setminus\{u,w\})} P_{x_v}(A_u) P_{x_z}(A_w) \, d\eta^{F''}(x).$$
(20)

We get the same if the roles of u and w are interchanged.

4.2 Marginals and conditioning on trees

We need some properties and associated constructions for the measure  $\eta^F$ , where F is a tree. The marginal  $(\eta^F)^U$  on a set  $U \subseteq V$  is particularly simple when  $U = V(F_1)$  for a subtree  $F_1$ , since then we can start a search order of F with a search order of  $F_1$ , which implies that

$$(\eta^F)^{V(F_1)} = \eta^{F_1}.$$
(21)

Another simple but useful fact about node sets U of subtrees is that we can condition on any map  $x \in J^U$ , since a random extension of it can be constructed in a well-defined way.

In the case when U = L is the set of leaves of F, we will denote the marginal  $(\eta^F)^L$  by  $\sigma^F$ .

We also need conditioning on maps  $z \in J^U$ , where  $U \subseteq V$  is a general subset. This is not straightforward, since the measure of a singleton z according to the marginal  $\alpha := (\eta^F)^U$  is typically zero. However, we can use disintegration: Using the marginal  $\alpha = (\eta^F)^U$ , Proposition 2.1 implies that there is a measurable family  $\Theta = \Theta^{F,U} = (\theta_z^F : z \in J^U)$  of distributions on  $\mathcal{B}^{V\setminus U}$  such that  $\eta^F = \alpha[\Theta]$ , or explicitly

$$\int_{A} \theta_z^F(B) \, d\alpha(z) = \eta^F(A \times B) \tag{22}$$

for all  $A \in J^{V \setminus U}$  and  $B \in J^U$ . It will be convenient to define  $\Theta^{S_0,\emptyset}$  (recall that  $S_0$  is the tree with a single node, no leaves) by  $\theta_{\emptyset} = \pi$  for the empty sequence  $\emptyset$ .

We can (informally) think of  $\theta_z$  as the distribution of a random copy of F, conditional on the set U being mapped by z. Note, however, that  $\theta_z$  is determined only up to an  $\alpha$ -nullset of mappings z. This fact (and that  $\theta_z$  is only implicitly defined) make this construction useless without some smoothness condition on  $\eta$ .

It is easy to extend the definition of  $\eta^F$  to forests F, by taking the product measure over the connected components. This way we have a measure  $\eta^{F[S]}$ for every  $S \subseteq V(F)$ . This family of measures, however, does not have the decreasing property: for example, the marginal of  $\eta^F$  on the set L of leaves is not necessarily absolutely continuous with respect to  $\pi^L$ . In the next subsection we introduce properties of the Markov chain that fixes this (and will play a crucial role for more general graphs as well.)

#### 4.3 Looseness

We start with one of our main definitions.

**Definition 4.3** We say that the tree F is *loose* in the Markov space  $\mathbf{M} = (J, \mathcal{B}, \eta)$ , if  $\sigma^F$  is absolutely continuous with respect to  $\pi^{L(F)}$ . In this case we can define the Radon–Nikodym derivative

$$s^{F}(z) = \frac{d\sigma^{F}}{d\pi^{L}}(z) \tag{23}$$

(determined for  $\pi^L$ -almost all  $z \in J^L$ ).

For the tree  $F = S_0$  with a single node u, we define  $s^F(z_0) = 1$  for the empty sequence  $z_0$ . The edge  $F = S_1$  (with one endpoint in L) is loose in every Markov space, since  $\sigma^F = \pi$ , and so  $s^F(z) \equiv 1$ . The edge  $F = P_1$ (with both endpoints in L) is loose in  $\eta$  if and only if  $\eta$  is induced by some (possibly unbounded) graphon W; we have then  $s^{P_1}(x_1, x_2) = W(x_1, x_2)$ . If  $\eta$  is induced by a graphon, then every tree is loose in  $\eta$  (cf. Section 5.3).

If  $s^F$  exists, then

$$\int_{J^L} s^F(z) \, d\pi^L(z) = \sigma^F(J^L) = \eta^F(J^V) = 1, \tag{24}$$

and hence  $s^F(z)$  is finite for  $\pi^L$ -almost all z.

If F is loose in **M**, then we can disintegrate  $\eta^F$  with respect to  $\pi^L$ , to get a measurable family  $\Psi^F = (\psi_z^F : z \in J^L)$  of measures on  $\mathcal{B}^{M(F)}$  such that

$$\pi^L[\Psi^F] = \eta^F. \tag{25}$$

It is easy to see that these measures relate to those obtained by disintegrating with respect to  $\sigma^F$  by the equation

$$\psi_z^F = s^F(z)\theta_z^F. \tag{26}$$

We note that the measures  $\psi_z^F$  are finite for almost all z, but they are not probability measures in general. In fact,

$$\psi_z^F(J^{M(F)}) = s^F(z)\theta_z^F(J^{M(F)}) = s^F(z).$$
(27)

The measures  $\psi_z^F$  are not necessarily absolutely continuous with respect to  $\pi^M$  or  $\eta^{F \setminus L}$ , but we can state the following simple lemma:

**Lemma 4.4** If F is loose in  $\eta$ , then for every set  $B \in \mathcal{B}^M$  with  $\eta^{F \setminus L}(B) = 0$ , we have  $\psi_z^F(B) = 0$  for  $\pi^L$ -almost all  $z \in J^L$ .

**Proof.** Indeed,  $\eta^{F \setminus L}(B) = 0$  implies that  $\eta^F(A \times B) = 0$  for every  $A \in \mathcal{B}^L$  (just start a search order of F with M(F)). In particular

$$\int_{J^L} \psi_z^F(B) \, d\pi^L(z) = \eta^F(J^L \times B) = 0,$$

thus  $\psi_z^F(B) = 0$  for  $\pi^L$ -almost all  $z \in J^L$ .

The property of looseness is *not* inherited by subtrees; in fact, for the two most important special trees, monotonicity goes in different directions. It is easy to see that if the star  $S_k$   $(k \ge 2)$  is loose in  $\eta$ , then so is  $S_j$  for j < k. On the other hand, if a path  $P_k$   $(k \ge 1)$  is loose in  $\eta$ , then so is  $P_j$  for j > k.

**Theorem 4.5** Let  $(I, \mathcal{A}, \eta)$  be a Markov space, let F be a tree, let  $F_1$  be a branch of F, and let  $F_2$  be obtained from F by removing this branch. If both trees  $F_1$  and  $F_2$  are loose in  $\eta$ , then so is F.

**Proof.** Let  $F_1$  be attached at u, and let e be the edge of  $F_1$  incident with u. Let  $L_i = L \cap V(F_i)$ , then  $L(F_1) = L_1 \cup \{u\}$ , and  $L(F_2)$  is either  $L_2$  or  $L_2 \cup \{u\}$ . Let  $F' = F \setminus e$ , then F' is a forest with two components  $F'_1 = F_1 \setminus u$  and  $F_2$ . Let  $\tau = \pi \times (\sigma^{F'_1})^{L_1}$ , which is a distribution on  $J^{L(F_1)}$ , where the first factor corresponds to u.

Let  $\lambda_x$  denote the marginal of  $\eta^{F_2}$  on  $L_2$  conditioned on  $u \mapsto x$ , and let  $\Lambda = (\lambda_x : x \in J)$ . A random map from  $\lambda_x$  can be generated by using a search order of  $F_2$  starting with u. We can also denote  $\lambda_x$  by  $\lambda_z$  for  $z \in J^{L(F_1)}$ , simply ignoring the coordinates other than  $z_u$ . Then

$$\sigma^F = \sigma^{F_1}[\Lambda]$$
 and  $(\sigma^{F'})^L = \tau[\Lambda]$ 

By Lemma 9.3, we have  $\sigma^{F_1} \ll \tau$ , and hence by Lemma 9.2,

$$\sigma^F = \sigma^{F_1}[\Lambda] \ll \tau[\Lambda] = (\sigma^{F'})^L.$$
(28)

Clearly  $\eta^{F'} = \eta^{F'_1} \times \eta^{F_2}$ . Using that  $(\eta^{F'_1})^{L_1} = (\eta^{F_1})^{L_1}$ , we have

$$(\sigma^{F'})^{L} = (\eta^{F'})^{L} = (\eta^{F'_{1}} \times \eta^{F_{2}})^{L} = (\eta^{F'_{1}})^{L_{1}} \times (\eta^{F_{2}})^{L_{2}} = (\eta^{F_{1}})^{L_{1}} \times (\eta^{F_{2}})^{L_{2}} = (\sigma^{F_{1}})^{L_{1}} \times (\sigma^{F_{2}})^{L_{2}}.$$

By hypothesis,  $\sigma^{F_i} \ll \pi^{L(F_i)}$  and hence  $(\sigma^{F_i})^{L_i} \ll \pi^{L_i}$ . This implies that  $(\sigma^{F'})^L \ll \pi^L$ , and combined with (28), we are done.

The notion of k-looseness defined in the Introduction is the special case of looseness of the tree  $F = S_k$  (the star with k leaves). We have  $\sigma^{S_k} = \sigma_k$ ; note that  $\sigma_1 = \pi$ . For every k,  $\sigma_k$  is a measure on k-tuples of points of J (ordered, but  $\sigma_k$  is invariant under permuting the nodes). If  $\eta$  is k-loose, then we can define the function

$$s_k(x_1, \dots, x_k) = s_k^{\eta}(x_1, \dots, x_k) = \frac{d\sigma_k}{d\pi^k}(x_1, \dots, x_k).$$
 (29)

Also recall that  $\eta$  is (k, p)-loose, if the function  $s_k$  is not only in  $L^1(\pi^k)$  (which follows by the definition) but in  $L^p(\pi^k)$ .

With this notion, we have the following corollary to Theorem 4.5.

**Corollary 4.6** For any  $k \ge 2$  and tree F, if the maximum degree satisfies  $2 \le \Delta(F) \le k$ , then F is loose in every k-loose Markov space.

**Proof.** The proof is by induction on the size of F. As previously mentioned,  $P_{\ell}$  is 2-loose for all  $\ell \geq 2$ . Also, note that any tree F with maximum degree between 2 and k is either a star (and thus loose by definition), a path of length  $\geq 2$ , or we can split off a branch  $F_1$  such that both it and the remainder  $F_2 = F \setminus F_1$  have at least 3 vertices, in which case we are done by induction and Theorem 4.5.

**Looseness in bi-Markov spaces.** We don't define looseness of a general tree for bi-Markov spaces, we define k-looseness only. Let  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$ 

be a bi-Markov space, and let  $(P_x : x \in I)$  be the disintegration of  $\eta$  defined in (12). Select a point  $u \in I$  from  $\pi_I$ , and select k independent points  $x_1, \ldots, x_k \in J$  from the distribution  $P_u$ . We say that **M** is k-loose from I, if the joint distribution  $\sigma_{I,k}$  of  $(x_1, \ldots, x_k)$  is absolutely continuous with respect to  $\pi_J^k$ . If this is the case, we can define the Radon–Nikodym derivative

$$s_{k,I}(x_1, \dots, x_k) = \frac{d\sigma_{I,k}}{d\pi_J^k}(x_1, \dots, x_k).$$
(30)

We define k-looseness from J analogously. We also define (k, p)-looseness from I and from J analogously. (Note that k-looseness from I does not imply k-looseness from J in general.)

## 5 Random mapping by tree decomposition

#### 5.1 Sequential tree decomposition

A sequential tree decomposition<sup>2</sup> of a graph G is a sequence  $(F_1, \ldots, F_m)$  of edge-disjoint trees, so that  $G = \bigcup_i F_i$ , and  $V(F_i) \cap V(F_1 \cup \cdots \cup F_{i-1}) = Z_i$  is the set of leaves of  $F_i$ , for  $i = 1, \ldots, m$ . In particular,  $F_1$  is a singleton tree.

Let us list some special constructions of sequential tree decompositions.

Edge decomposition. A trivial construction is to start with singleton trees for each node, and continue with attaching  $P_1$ 's to get the edges.

**Star decomposition.** A less trivial decomposition is the following. Let  $V = (v_1, \ldots, v_n)$  be any ordering of V. For each node i, we construct the star  $F_i$  centered at i, with edges connecting i to earlier nodes. This decomposition will be particularly well-behaved if G is bipartite, and the ordering starts with singleton trees for the nodes in one bipartition class, and continues with the full stars of the nodes in the other class.

Subdivision decomposition. Another useful example is obtained when G is a subdivision of a graph H with any number of new nodes on each edge. The sequence starts with the nodes in U = V(H) as singleton trees, and then it continues with the paths replacing the original edges (in any order).

**Double star decomposition.** Select an edge ij in a bipartite graph G; then ij and the edges adjacent to it form a tree  $F_{ij}$  (a double star). The graph G arises from  $G' = G \setminus \{i, j\}$  by attaching the tree  $F_{ij}$ . Continuing this with G' instead of G, we get a sequential tree-decomposition of G (in backwards order).

<sup>&</sup>lt;sup>2</sup>Not to be confused with "tree decomposition" in the theory of graph minors.

**Open ear decomposition.** An ear decomposition into paths is a further example (this will not concern us here).

#### 5.2 Sequential construction of measures

Let  $\mathbf{M} = (J, \mathcal{B}, \eta)$  be a Markov space, and let  $\mathcal{F} = (F_1, \ldots, F_m)$  be a sequential tree decomposition of the graph G. We construct a random mapping  $x: V \to J$  as follows. We select  $x(F_1)$  from distribution  $\pi$ . Assuming that the nodes in  $F_1 \cup \cdots \cup F_{i-1}$  have been mapped  $(i \leq m)$ , we choose the image of  $M(F_i)$  from the conditional distribution  $\theta_{x(L(F_i))}^{F_i}$  (defined in Section 4.2). The distribution of this random map will be denoted by  $\rho_{\mathcal{F}}$ .

There are two major problems with this construction:

— First, the disintegration  $\theta_z^{F_i}$  is determined only up to a set of  $\sigma^{F_i}$ measure zero, and there is no guarantee that the construction will not produce an image of  $L(F_i)$  that falls in a zero-set of  $\sigma^{F_i}$  with positive probability. As a trivial example, an edge decomposition has this problem if  $\eta$  is not absolutely continuous with respect to  $\pi \times \pi$ .

— Second, even if this does not happen, the distribution we construct may depend on the specific decomposition into trees. This problem actually occurs even in the case of the star decomposition of bipartite graphs; see Example 3.1. One of our main results (Theorem 1.2) says that in a sense these are the only bad examples.

Both problems can be handled by making an appropriate looseness assumption about  $\eta$  and sparseness assumption about G. To describe these remedies, suppose that a graph G = (V, E) has a sequential tree decomposition  $\mathcal{F} = (F_1, \ldots, F_m)$  such that every tree  $F_i$  is loose in  $\mathbf{M}$ . Set  $L_i = L(F_i)$ and  $M_i = M(F_i)$ . Define the functions  $s^{F_i}$  by (23) and let

$$f_{\mathcal{F}}(x) = \prod_{i=1}^{m} s^{F_i}(x_{L_i}).$$
(31)

Let  $\rho = \rho_{\mathcal{F}}$  be the distribution on  $\mathcal{B}^V$  constructed above, and define the measure

$$\eta_{\mathcal{F}} = f_{\mathcal{F}} \cdot \rho_{\mathcal{F}}.\tag{32}$$

It is clear from this definition that  $\eta_{\mathcal{F}}$  is sigma-finite.

It will be useful to express this definition in a recursive way. The sequence  $\mathcal{F}' = (F_1, \ldots, F_{m-1})$  is a sequential tree decomposition of the graph  $G' = (V', E') = F_1 \cup \cdots \cup F_{m-1}$ . We use the measurable family  $\Psi^{F_m} = (\psi_z^{F_m} : z \in \mathbb{C})$ 

 $J^{L_m}$ ) defined in (25). With some abuse of notation, sometimes it is useful to consider  $\Psi_m$  as indexed by vectors  $z \in J^{V'}$  (nodes in  $V' \setminus L_m$  considered as dummies). Then by definition

$$\pi^{L_m}[\Psi^{F_m}] = \eta^{F_m},\tag{33}$$

and it is easy to check that

$$\eta_{\mathcal{F}} = \eta_{\mathcal{F}'}[\Psi^{F_m}]. \tag{34}$$

We can use (33) and (34) as a recursive definition of  $\eta_{\mathcal{F}}$ . We also define the "density of G in  $\eta$ " as

$$t_{\mathcal{F}}(G,\eta) = \eta_{\mathcal{F}}(J^V) = \int_{J^V} \prod_{i=1}^k s^{F_i}(x_{L_i}) \, d\pi_J(x).$$
(35)

Let us note that (34) implies that

$$(\eta_{\mathcal{F}})^{V'} \ll \eta_{\mathcal{F}'}.\tag{36}$$

To address the first problem described above, let us note the following. Assume that  $\eta_{\mathcal{F}'}$  is already given. Note that the measures  $\psi_z^{F_m}$  are determined by (33) up to a set of indices  $z \in J^{L_m}$  of  $\pi^{L_m}$ -measure 0. If a set  $Z \subset J^{L_m}$ satisfies  $\pi^{L_m}(Z) = 0$  but  $(\eta_{\mathcal{F}'})(Z \times J^{V' \setminus L_m}) = (\eta_{\mathcal{F}'})^{L_m}(Z) > 0$ , then changing  $\Psi^{F_m}$  for these indices  $z \in Z$  will change the right hand side of (34), and we are in trouble. So for the recursive construction to work, we need that  $(\eta_{\mathcal{F}'})^{L_m} \ll \pi^{L_m}$ .

**Definition 5.1** Let us say that the sequential tree decomposition  $\mathcal{F} = (F_1, \ldots, F_m)$  is *smooth* in **M**, if  $(\eta_{(F_1, \ldots, F_{i-1})})^{L_i} \ll \pi^{L_i}$  for  $i = 1, \ldots, m$ .

We are going to show that star-decompositions are smooth in many triangle-free graphs, and all tree decompositions are smooth in graphons.

Our main special case will be star-decompositions. Let G be a graph with maximum degree at most k. Let  $\mathcal{F}_p = (F_1, \ldots, F_n)$  be a sequential star decomposition of G, determined by an ordering  $p = (v_1, \ldots, v_n)$  of the nodes, where  $v_i$  is the center of  $F_i$ . We set  $\eta_p = \eta_{\mathcal{F}_p}$ .

#### 5.3 Unbounded graphons

Our first application of the general scheme described above is the case of Markov spaces  $\mathbf{M} = (J, \mathcal{B}, \eta)$  with the property that  $\eta$  is absolutely continuous with respect to  $\pi \times \pi$ . It is convenient to represent such Markov

spaces by the Radom-Nikodym derivative  $W = d\eta/d(\pi \times \pi)$ , which is a nonnegative, symmetric measurable function  $W: J \times J \to \mathbb{R}$  with the property that  $\int_x W(x, y) d\pi = 1$  holds for every  $y \in J$ . In particular we have that the  $L^1$  norm of W is 1. We call measurable functions with this property 1regular graphons. Note that every 1-regular graphon W uniquely determines a Markov space  $\mathbf{M}_W = (J, \mathcal{B}, \eta_W)$  where

$$\eta_W = W \cdot \pi^2 \tag{37}$$

In the rest of this section we are going to omit the subscript W wherever no confusion can arise.

The 1-regularity of the graphon implies that the transition probabilities for this Markov space are given by

$$P_x(A) = \int\limits_A W(x, y) \, d\pi(y). \tag{38}$$

**Lemma 5.2** Every tree is loose in  $\eta$ . In other words, the measure  $\eta$  is k-loose for every natural number k.

**Proof.** The identity  $\eta^F = W^F \cdot \pi^{V(F)}$  is easily checked for trees F = (V, E), using (38). This implies that  $\eta^F \ll \pi^V$ , and hence

$$\sigma^F = (\eta^F)^{L(F)} \ll (\pi^V)^{L(F)} = \pi^{L(F)}$$

Thus F is loose in  $\eta$ , proving the lemma.

A convenient special property of such Markov spaces comes from the fact that the function W can be directly used to produce homomorphism measures for every finite graph G:

**Theorem 5.3** Let W be a 1-regular graphon, and let  $\mathcal{F} = (F_1, \ldots, F_m)$  be a sequential tree-decomposition of a graph G = (V, E). Then  $\mathcal{F}$  is smooth in  $\eta$ , and

$$\eta_{\mathcal{F}} = W^G \cdot \pi^V. \tag{39}$$

In particular, it follows that  $\eta_{\mathcal{F}}$  is independent of the decomposition and  $\eta^G = W^G \cdot \pi^V$  is well-defined.

**Proof.** We express the measures in the construction of  $\eta_{\mathcal{F}}$  as integrals of W. First, let F = (V, E) be a tree. It is easy to see that, by the definition of  $\eta^F$  and by (38), that

$$\eta^F = W^F \cdot \pi^V. \tag{40}$$

This implies that for  $A \in \mathcal{B}^L$ ,

$$\sigma^F(A) = \eta^F(A \times J^M) = \int_{A \times J^M} W^F \, d\pi^V \tag{41}$$

and for  $B \in \mathcal{B}^M$  and  $z \in J^L$ ,

$$\psi_{z}^{F}(B) = \int_{B} W^{F}(z, y) \, d\pi^{M}(y).$$
(42)

Now let  $\mathcal{F} = (F_1, \ldots, F_m)$  be a sequential tree-decomposition of a graph G = (V, E). We are going to prove by induction on m that this decomposition satisfies (39). This will imply that the decomposition is smooth.

Let  $\mathcal{F}' = (F_1, \ldots, F_{m-1})$  and  $G' = (V', E') = F_1 \cup \cdots \cup F_{m-1}$ . To prove that G satisfies (39), we use the recurrence (34), along with (39) for G' and (42). Let  $A \in \mathcal{B}^{V'}$  and  $B \in \mathcal{B}^{M_m}$ , then

$$\eta_{\mathcal{F}}(A \times B) = \int_{A} \left( \int_{B} W^{F_m}(z_{L_m}, y) \, d\pi^{M_m}(y) \, W^{G'}(z) \right) \, d\pi^{V'}(z, w)$$
$$= \int_{A \times B} W^G \, d\pi^V$$

(here w is the vector of dummy variables in  $J^{V'\setminus L_m}$ ). This proves (39).

To prove that  $\mathcal{F}$  is smooth, it suffices to note that (39) implies that  $\eta_{\mathcal{F}'} \ll \pi^{V'}$ , and hence  $(\eta_{\mathcal{F}'})^{L(F_m)} \ll \pi^{L(F_m)}$ . This holds for all other prefixes of  $\mathcal{F}$  by the same argument.

A direct application of Theorem 5.3 implies that the formalism of this paper is a consistent extension of earlier results in bounded graphon theory.

**Corollary 5.4** Let W be a 1-regular graphon and let G = (V, E) be a finite graph. Then

$$t(G,\eta) = t(G,W) = \eta^G(J^V) = \int_{J^V} W^G \ d\pi^V.$$

Note, however, that this value may be infinite (see Example 5.8).

If W has stronger properties, then we can strengthen the k-looseness property of graphons to (k, p)-looseness.

**Lemma 5.5** Let k, p be natural numbers. For  $x \in J$  let f(x) denote the  $L^p$ -norm of the function  $y \mapsto W(x, y)$ . If  $f \in L^k(\pi)$ , that is,  $\int (\int_J W(x, y)^p d\pi(y))^{k/p} d\pi(x) < \infty$ , then  $\eta$  is (k, p)-loose.

**Proof.** For  $x \in J$  let  $H_x : J^k \to \mathbb{R}$  denote the function defined by  $H_x(y_1, y_2, \ldots, y_k) := W(x, y_1)W(x, y_2) \ldots W(x, y_k)$ . It is easy to see that the  $L^p$ -norm of  $H_x$  on  $(J^k, \pi^k)$  is equal to  $f(x)^k$ . Thus by the convexity of  $L^p$ -norm we have that the  $L^p$ -norm of  $H := \int_J H_x d\pi(x)$  is at most  $\int_x f(x)^k d\pi$  and so the condition of the lemma implies that the  $L^p$ -norm of H is finite. This implies that  $\eta$  is (k, p)-loose.

This lemma has two immediate corollaries.

**Corollary 5.6** Let W be a 1-regular  $L^p$ -graphon for some natural number p > 1. Then  $\eta$  is (p, p)-loose.

**Corollary 5.7** Let W be a 1-regular graphon such that for some  $c \in \mathbb{R}$  we have that  $\int_{y} W(x,y)^{p} d\pi \leq c$  holds for every x. Then W is (k,p)-loose for every natural number k.

Our next two examples show that 1-regular graphons can be rather wild objects in terms of spectral properties and subgraph densities.

**Example 5.8** Let  $W : [0,1]^2 \to \mathbb{R}^2$  be the function whose value is defined by  $W(x,y) = 2^k$  whenever  $x, y \in I_k = (2^{-k}, 2^{-(k-1)})$ , and 0 otherwise. We define  $\pi$  as the Lebesgue measure on [0,1]. It is clear that for every natural number  $k \ge 1$ , the indicator function of  $I_k$  is an eigenvector of W with eigenvalue 1. Thus the eigenspace of W with eigenvalue 1 is infinite dimensional. This implies that W is not a compact operator. Direct calculation shows that if a connected graph G = (V, E) is not a tree, then  $t(G, W) = \infty$ . More precisely, since G is connected,  $W^G = 0$  unless all nodes are mapped into the same interval  $I_k$ . Hence

$$t(G,W) = \sum_{k=1}^{\infty} \int_{I_k^V} W^G(x) \, dx = \sum_{k=1}^{\infty} 2^{k(|E(G)| - |V(G)|)},$$

which is equal to 1 if |E(G)| = |V(G)| - 1 (i.e., G is a tree) and  $\infty$  otherwise.

Note that in the preceding example the graphon is an  $L^1$  function, whereas any graphon in  $L^2$  would at least have finite cycle densities (as the max degree is 2). Changing the parameters we can obtain a family of examples in  $L^p$  $(1 \le p < 2)$  that get arbitrarily close to being Hilbert-Schmidt kernels, yet still have infinite densities for all non-trees. **Example 5.9** Let  $\varepsilon \in [0,1)$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of positive reals such that  $\sum_{n \in \mathbb{N}} a_n = 1$  and  $\sum_{n \in \mathbb{N}} a_n^{1-\varepsilon} < \infty$ . Let  $(J_n : n = 1, 2, ...)$  be a measurable partition of J with  $\pi(J_n) = a_n$ . Define the unbounded kernel W by

$$W(x,y) = \begin{cases} 1/a_n, & \text{if } x, y \in J_n, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that W is 1-regular, and

$$\|W\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{n \in \mathbf{m}N} a_n^2 / a_n^{1+\varepsilon} = \sum_{n \in \mathbb{N}} a_n^{1-\varepsilon} < \infty.$$

On the other hand, the density of any connected graph G in W can be obtained as the sum of the densities in each of the diagonal blocks, i.e.,

$$t(G,W) = \sum_{n \in \mathbb{N}} a_n^{|V(G)|} / a_n^{|E(G)|}.$$

The sum is equal to 1 for trees, and infinite for all other graphs G.

Although the above construction with an infinite number of independent blocks seems to suggest that the key to infinite densities is non-compactness, this is not quite the case. Indeed, the next example shows that compactness of the operator defined by W is by itself not enough to guarantee that subgraph densities behave any better.

**Example 5.10** Let  $f: I = [-1, 1] \to \mathbb{R}$  be a function with the following properties:  $f \ge 0$ ; f(-x) = f(x) for all  $x \in I$ ;  $\int_I f(x) dx = 1$ ; f is convex and monotone decreasing for x > 0. Define a graphon by

$$W(x,y) = f(x-y) \qquad (x,y \in I),$$

where f is extended periodically modulo 2. Clearly W is symmetric and 1-regular. As a kernel operator, W is positive semidefinite and compact as  $L^2(\mu) \to L^2(\mu)$ . In the special case

$$f(x) = \frac{1}{|x|(2 - \ln(|x|))^2},$$

no operator power of W has finite trace. So  $t(C_n, W) = \infty$  for all n (see Appendix 9.4 for details).

#### 5.4 Triangle-free graphs

In this section we concentrate on sequential star decompositions. We need a simple combinatorial lemma.

**Lemma 5.11** Let  $F : S_V \to X$ , where  $S_V$  is the set of permutations of the node set of a triangle-free graph G = (V, E), and X is any set. Assume that F has the following two invariance properties for every permutation  $p = (v_1, \ldots, v_n)$ :

(i) If  $v_k v_{k+1} \notin E$ , then interchanging  $v_k$  and  $v_{k+1}$  in p does not change F(p);

(ii) If every node in  $A = \{v_1, \ldots, v_a\}$  is connected to every node in  $B = \{v_{a+1}, \ldots, v_{a+b}\}$ , then interchanging the blocks A and B in p does not change F(p).

Then F is constant.

Note that in (ii), A and B must be independent node sets as G is trianglefree, so  $A \cup B$  induces a complete bipartite graph.

**Proof.** We use induction on n. For a fixed  $v \in V$ , the function  $F_v(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_{n-1}, v)$  satisfies the conditions in the lemma, so by the induction hypothesis, it is constant. This means that there is a function  $f: V \to X$  such that  $F(x_1, \ldots, x_n) = f(x_n)$ .

Let  $u, v \in V$  be nonadjacent. Considering any permutation  $(v_1, \ldots, v_{n-2}, u, v)$ , we see that

$$f(v) = F(v_1, \dots, v_{n-2}, u, v) = F(v_1, \dots, v_{n-2}, v, u) = f(u).$$

Now let  $u, v \in V$  be adjacent. If there is a path in the complement  $\overline{G}$  connecting u and v, then applying the previous observation repeatedly we get that f(u) = f(v). If there is no such path, then there is a partition  $V = A \cup B$  so that  $u \in A$ ,  $v \in B$ , and every edge between A and B is present. Since G is triangle-free, it follows that G is a complete bipartite graph. Let  $A = \{u_1, \ldots, u_a = u\}$  and  $B = \{v_1, \ldots, v_b = v\}$ , then

$$f(v) = F(u_1, \dots, u_a, v_1, \dots, v_b) = F(v_1, \dots, v_b, u_1, \dots, u_a) = f(u).$$

So f is constant, and then so is F.

Let G be a triangle-free graph with maximum degree k, and let  $\mathbf{M} = (J, \mathcal{B}, \eta)$ . For a sequential star decomposition  $\mathcal{F} = (F_1, \ldots, F_n)$  of G, determined by an ordering  $p = (v_1, \ldots, v_n)$  of the nodes, let  $\eta_p = \eta_{\mathcal{F}}$  denote the measure on  $J^V$  defined by (34). In general,  $\eta_p$  will depend on the ordering

p and also on the measure families  $\Psi^{F_i} = (\psi_z^{F_i} : z \in J^{L(F_i)})$ , which are determined only up to a set of indices  $z \in J^{L(F_i)}$  of  $\pi^{L(F_i)}$ -measure zero.

Now we are ready to prove Theorem 1.2.

**Theorem 1.2** Let G = (V, E) be a triangle-free graph, and let  $\mathbf{M} = (J, \mathcal{B}, \eta)$ be a Markov space such that every complete bipartite subgraph  $K_{a,b}$  of G is well-measured in  $\mathbf{M}$ . Then G is well-measured in  $\mathbf{M}$ .

**Proof of Theorem 1.2** We prove the theorem by induction on n. The condition is clearly inherited by induced subgraphs of G, so we may assume that every proper induced subgraph of G is well-measured in  $\mathbf{M}$ .

First we prove that for every ordering  $p = (v_1, \ldots, v_n)$  of the nodes of G, the measure  $\eta_p$  does not depend on the choice of the measure families  $\Psi^{F_i}$ . We know by induction that  $G' = G \setminus v_n$  is well-measured in  $\mathbf{M}$ , so  $\eta^{G'}$  does not depend on these choices. Consider the measures  $(\psi_z : z \in J^{N(v_n)})$ . Two different choices of the measures  $\psi_z$  can differ on a set  $Z_0 \in \mathcal{B}^{L_n}$  of maps z with  $\pi^{L_n}(Z_0) = 0$ . By the definition of well-measurability, we have  $(\eta_{\mathcal{F}'})^{L_n} \ll \eta^{G[L_n]}$ , where  $\mathcal{F}' = (F_1, \ldots, F_{n-1})$ . Since G is triangle-free,  $L_n$  is an independent set of nodes, so  $\eta^{G[L_n]} = \pi^{L_n}$  and hence  $(\eta_{\mathcal{F}'})^{L_n} \ll \pi^{L_n}$ . Thus  $\eta_{\mathcal{F}}$  is uniquely determined by (34).

To prove that for any two orderings p and q of the nodes of G, we have  $\eta_p = \eta_q$ , we use Lemma 5.11. For a permutation  $p \in S_V$ , let  $F(p) = \eta_p$ . Condition (i) is trivial, and condition (ii) is also easy: if the first a + b nodes induce a complete bipartite subgraph, then the sequential construction up to the first a + b nodes results in the same measure by the hypothesis of the theorem, and the completion of the construction does not depend on the order of these a + b nodes.

So the sequential construction provides a measure  $\eta^G$  independent of the ordering. Recall that the measures  $\eta^{G[S]}$ , where  $S \subset V$ , are also given by induction. This family of measures is trivially normalized and, as remarked before, sigma-finite. The decreasing property is easy: we can start the sequential construction by any given set S, and the  $(\eta^G)^S \ll \eta^{G[S]}$  follows by repeated application of (36). To prove the Markov property, let  $V = U \cup T$  such that there is no edge between  $U \setminus S$  and  $T \setminus S$  where  $S = U \cap T$ . Consider an ordering p of V starting with S. Recall (34), describing the recursive definition of  $\eta^G$ . It follows that the disintegration  $(\mu_{U,V,z} : z \in J^U)$  of  $\eta^{G[V]}$  by  $\eta^{G[U]}$  has the property that  $\mu_{U,V,z}$  depends only on  $z|_S$ , and we have a similar property with U and T interchanged. Hence for every  $x \in J^S$ ,

 $\mu_{S,V,x} = \mu_{S,U,x} \times \mu_{S,T,x},$ 

proving that the measure family  $(\eta^{G[S]}: S \subseteq V)$  is Markovian.

**Remark 5.12** Note that the proof above only uses that the sequential construction of  $\eta^{K_{a,b}}$  gives the same measure if we start with one bipartition class or the other. It is not hard to see, along the lines of the proof of Lemma 5.11, that this is equivalent with  $K_{a,b}$  being well-measured. We will return to the question of which complete bipartite graphs are well-measured in a Markov space in Section 6.5.

**Remark 5.13** As we have mentioned in the Introduction, if G has girth at least 5, then the only complete bipartite subgraphs of G are stars, and the condition means that **M** is k-loose, where k is the maximum degree of G. Also note that the condition on G is inherited by all subgraphs of G.

The condition that all degrees are bounded by k could be relaxed: the construction would work for all graphs that are k-degenerate (i.e., repeatedly deleting nodes with degree at most k, the whole graph can be eliminated). For k = 1 (which imposes no condition on the Markov space), we get the measure  $\eta^F$  for all trees. (Recall, however, that this does not imply that trees are well-measured: the decreasing property fails.) The extension of the considerations in Section 4 is left for further study.

An important example of this more general setup would be the following. There are Markov spaces  $\eta$  whose k-th power  $\eta^k$  (as introduced along with the adjacency operator) is induced by a bounded graphon W, but they themselves are not. For example, the orthogonality space in any dimension has this property. If  $\eta$  has this property and G' is a k-subdivision of a graph G then G' is 2-degenerate. Working with subdivision decompositions of G', we can construct  $\eta^{G'}$ , which will be finite. So we see that  $\eta^G$  exists and  $t(G, \eta) < \infty$  holds for such Markov spaces and for a large set of graphs Gwith no degree bound.

**Remark 5.14** Note that the bi-Markov space analogue of Theorem 1.2 also holds and the proof is essentially the same mutatis mutandis.

#### 5.5 Bigraphs and bi-Markov spaces

The sequential construction of  $\eta_G$  takes a particularly simple form when G is bipartite. Let G = (U, W, E) be a bigraph and  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$ , a bi-Markov space k-loose from J. Our considerations apply, in particular, to k-loose Markov spaces.

To define  $\eta^G$ , we can use an ordering of the nodes that starts with U. Then the nodes in U will be mapped onto independent random points of I from distribution  $\pi_I$ . Furthermore, the points of W will be mapped conditionally independently given the image of U. For this to make sense, it suffices to require that all nodes in W have degree at most k.

For every finite sequence  $(x_1, \ldots, x_d)$  of points  $d \ge 1$  of I, we have a measurable family of measures  $\Psi_d = (\psi_x : x \in I^d)$  on  $\mathcal{B}$  defined by the disintegration

$$\eta^{S_d} = \pi^d_I[\Psi_d]. \tag{43}$$

We can think of  $\psi_x$  informally as the measure on the common neighbors of  $x = (x_1, \ldots, x_d)$ .

For a node  $w \in W$ , let  $F_w$  denote the star formed by the edges incident with w. We define the product measure and the corresponding measurable family by

$$\widehat{\psi}_x = \prod_{w \in W} \psi_{x_{N(w)}}$$
 and  $\widehat{\Psi} = (\widehat{\psi}_x : x \in I^U).$ 

Then we define

$$\eta^G = \pi_I^U[\widehat{\Psi}],\tag{44}$$

or explicitly,

$$\eta^G(A \times B) = \int_A \widehat{\psi}_{x_{N(w)}}(B) \, d\pi^U_I(x) \qquad (A \in \mathcal{A}^U, B \in \mathcal{B}^W). \tag{45}$$

This measure  $\eta^G$  is well-defined, since the measures  $\psi_{x_{N(w)}}$  can be changed on a  $\pi_I$ -nullset only. Note that the definition is more general than our construction in Section 5.4, since no assumption is necessary for the degrees of nodes in U.

Formula (44) makes sense when the disintegration  $\Psi^{F_w}$  in (25) can be defined. By Proposition 2.1, this happens if  $\sigma^{F_w} \ll \pi_I^{N(w)}$ , that is,  $\eta$  is kloose from J, and all degrees of G in W are bounded by k, for some  $k \ge 1$ . If this holds, then the density function  $s_w = s_{w,J} = s^{F_w}$  is well-defined in (29), and

$$\psi_{x_{N(w)}}(J^W) = s_w(x_{N(w)}) = s_{\deg(w)}(x_{N(w)}).$$

In particular, we obtain the following formula for the density of the bigraph G in **M**:

$$t(G,\eta) = \eta^{G}(I^{U} \times J^{W}) = \int_{I^{U}} \prod_{w \in W} s_{\deg(w)}(x_{N(w)}) \, d\pi^{U}_{I}(x).$$
(46)

Formula (44) does not define  $\eta^G$  if  $G_0$  is not a bigraph but only a bipartite graph (so its bipartition classes are not fixed). It may even happen that only one of these measures is well-defined (for example, if the maximum degree in U is larger than k).

But assume that both of them are well-defined; is then  $\eta^G = (\eta^*)^{G^*}$  or at least  $t(G^*, \eta^*) = t(G, \eta)$ ? By Theorem 5.3, this is the case when  $\eta$  is defined by a graphon, and by the bi-Markov space analogue of Theorem 1.2 (see Remark 5.14), this also holds true if G contains no quadrilaterals. Further sufficient conditions will be given below. We'll state such a theorem (Theorem 6.14) later. On the other hand, Example 3.1 shows that some condition along these lines is necessary.

One of the difficulties caused by this asymmetry can be partly remedied as follows.

**Lemma 5.15** Let  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$  be a bi-Markov space k-loose from J and let G = (U, W, E) be a bigraph such that every vertex of W has degree at most k in G. Then the marginal of  $\eta^G$  on U is absolutely continuous with respect to  $\pi_I^U$ . Furthermore the marginal of  $\eta^G$  on any node of W is absolutely continuous with respect to  $\pi_J$ .

**Proof.** It is clear by (45) that if  $\pi_I^U(A) = 0$ , then  $\eta^G(A \times J^W) = 0$ , which implies the first assertion. Similar claim does not follow for a general  $B \in \mathcal{B}^W$ from  $\pi_J^W(B) = 0$  (see Example 3.1); however, if B is a box  $B = \prod_{w \in W} B_w$  $(B_w \in \mathcal{B})$ , then by (45) we have

$$\eta^{G}(A \times B) = \int_{A} \prod_{w \in W} \psi_{x_{N(w)}}(B_{w}) \, d\pi^{U}_{I}(x).$$
(47)

The bi-Markov space analogue of Lemma 4.4 and the fact that **M** is k-loose from J implies that if  $\pi_J(B_w) = 0$  for some  $w \in W$ , then  $\psi_{x_{N(w)}}(B_w) = 0$  for  $\pi_I$ -almost all  $x_{N(w)}$ , and so  $\eta^G(I^U \times J^{W \setminus \{w\}} \times B_w) = 0$ .

## 6 Approximation by graphons

### 6.1 Convergence of graphons to Markov spaces

Suppose that a sequence of graphons  $W_n$  "tends to" a Markov space  $(J, \mathcal{B}, \eta)$ in some sense. Does this imply that for graphs G satisfying suitable conditions, we have  $t(G, W_n) \to t(G, \eta)$ ? We prove two results along these lines. The first was used (implicitly) in [18]; the second will be used later in this paper. Let  $(J, \mathcal{B}, \eta)$  be a k-loose Markov space, and let  $W_n$  (n = 1, 2, ...) be a sequence of 1-regular graphons on  $(J, \mathcal{B}, \pi)$ . We say that  $\eta$  is the k-limit of the sequence  $(W_n)$ , if

$$s_k^{W_n}(x) \to s_k^{\eta}(x)$$

for  $\pi^k$ -almost all  $x \in J^k$ , and there is a constant  $C = C(\eta, k)$  independent of x and an integer  $n_0 \ge 1$ , such that

$$s_k^{W_n}(x) \le C s_k^{\eta}(x)$$

for every  $n \ge n_0$  and  $\pi^k$ -almost all  $x \in J^k$ .

We say that a (k, p)-loose Markov space  $(J, \mathcal{B}, \eta)$  is the (k, p)-limit of the sequence  $(W_1, W_2, ...)$  of graphons, if  $s_k^{W_n} \to s_k^{\eta}$  in  $L^p(\pi^k)$  (note that there then exists a constant C > 0 such that  $||s_k^{W_n}||_p \leq C$  for every n).

We need an important analytic tool that allows us to bound products of functions in multivariate  $L^p$  spaces, namely a special case of the general, multivariate version of Hölder's inequality, called Finner's theorem ([11, Theorem 2.1]). For the sake of self-containedness, we state this special case, and its main corollary that will be relevant to us.

**Theorem 6.1** Let  $(J, \mathcal{B}, \pi)$  be a probability space, and p, n and m positive integers. Let  $f_k : J^n \to \mathbb{R}$   $(1 \le k \le m)$  be measurable functions, where  $f_k$ depends only a set  $M_k$  of variables. Assume that every variable  $x_i$   $(1 \le i \le m)$  is contained in at most p sets  $M_k$ . Then

$$\int_{J^n} \prod_{k=1}^m f_k \, d\pi^n \le \prod_{k=1}^m \|f_k\|_p.$$
(48)

By a standard telescopic decomposition argument, this yields the following convergence result.

**Corollary 6.2** Let  $(J, \mathcal{B}, \pi)$  be a probability space, and p, n and m positive integers. Let  $f_k, f_{k,\ell} : J^{M_k} \to \mathbb{R}$   $(1 \le k \le m, \ell \in \mathbb{N})$  be measurable functions, where  $f_k$  and  $f_{k,\ell}$  depend only on a set  $M_k$  of variables. Assume that every variable  $x_i$   $(1 \le i \le m)$  is contained in at most p sets  $M_k$ . Also assume that  $f_k \in L^p(J, \mathcal{B}, \pi)$  and

$$\lim_{\ell \to \infty} \|f_{k,\ell} - f_k\|_p = 0$$

holds for all  $1 \leq k \leq m$  and  $\ell \in \mathbb{N}$ . Then

$$\lim_{\ell \to \infty} \int_{J^n} \prod_{k=1}^m f_{k,\ell} \, d\pi^n = \int_{J^n} \prod_{k=1}^m f_k \, d\pi^n.$$
(49)

**Theorem 6.3** Let  $(J, \mathcal{B}, \eta)$  be a k-loose Markov space, let  $W_n$  (n = 1, 2, ...) be a sequence of 1-regular graphons on  $(J, \mathcal{B})$  such that  $\eta$  is the k-limit of  $(W_n)$ . Let G = (U, W, E) be a bigraph in which  $\deg(w) \leq k$  for all  $w \in W$ , and assume that  $t(G, \eta) < \infty$ . Then

$$t(G,\eta) = \lim_{n \to \infty} t(G, W_n).$$

The right hand side is invariant under interchanging the bipartition classes of G. Thus if, in addition to the conditions of Theorem 6.3,  $\deg(u) \leq k$  holds for all  $u \in U$ , then  $t(G, \eta) = t(G^*, \eta)$ .

**Proof.** We have

$$t(G,\eta) = \int_{J^U} \prod_{w \in W} s^{\eta}_{\deg(w)}(x_{N(w)}) \, d\pi^U(x).$$

and

$$t(G, W_n) = \int_{J^U} \prod_{w \in W} s_{\operatorname{deg}(w)}^{W_n}(x_{N(w)}) \, d\pi^U(x).$$

Here

$$\prod_{w \in W} s_{\deg(w)}^{W_n}(x_{N(w)}) \to \prod_{w \in W} s_{\deg(w)}^{\eta}(x_{N(w)})$$

almost everywhere, and

$$\prod_{w \in W} s_{\deg(w)}^{W_n}(x_{N(w)}) \le C^{|W|} \prod_{w \in W} s_{\deg(w)}^{\eta}(x_{N(w)}).$$

Since the function on the right is integrable by the condition that  $t(G, \eta) < \infty$ , the theorem follows by Lebesgue's Dominated Convergence Theorem.  $\Box$ 

We state an analogous theorem under the stronger assumption of (k, p)looseness. Recall that a Markov space  $(J, \mathcal{B}, \eta)$  is (k, p)-loose  $(k, p \in \mathbb{N})$ , if it is k-loose and  $||s_k^{\eta}||_p$  is finite. Quite surprisingly it will turn out that (k, p)-looseness of Markov spaces is a symmetric notion: a Markov space  $(J, \mathcal{B}, \eta)$  is (k, p)-loose if and only if it is (p, k)-loose. (For bi-Markov spaces this symmetry property no longer holds, however.)

**Theorem 6.4** Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space, let  $W_n$  (n = 1, 2, ...) be a sequence of 1-regular graphons on  $(J, \mathcal{B})$  such that  $\eta$  is the (k, p)-limit of  $(W_n)$ . Let G = (U, W, E) be a bigraph, and assume that  $\deg(u) \leq p$  for  $u \in U$  and  $\deg(w) \leq k$  for  $w \in W$ . Then

$$t(G,\eta) = \lim_{n \to \infty} t(G, W_n) < \infty.$$

**Proof.** Let  $W = \{v_1, ..., v_m\}$ . Then

$$t(G,\eta) = \int_{J^U} \prod_{v \in W} s^{\eta}_{\operatorname{deg}(v)}(x_{N(v)}) \, d\pi^U(x)$$

and

$$t(G, W_n) = \int_{J^U} \prod_{v \in W} s^{W_n}_{\operatorname{deg}(v)}(x_{N(v)}) \, d\pi^U(x).$$

Each variable  $x_u$  ( $u \in U$ ) occurs in at most p factors, and so Corollary 6.2 implies the theorem.

### 6.2 **Projection onto stepfunctions**

A natural approximation of a Markov space  $(J, \mathcal{B}, \eta)$  is the following. Let  $\mathcal{P} = \{P_1, \ldots, P_n\}$  be a finite, measurable, non-degenerate partition. For a function  $f \in L^1(\pi)$ , we define

$$f_{\mathcal{P}} = \frac{1}{\pi(P_i)} \int_{P_i} f \, d\pi \qquad (x \in P_i).$$

We generalize this to every k-variable function  $h: J^k \to \mathbb{R}$  by

$$h_{\mathcal{P}} = h_{\mathcal{P}^k}.$$

In particular, for a graphon W we have

$$W_{\mathcal{P}}(x,y) = \frac{1}{\pi(P_i)\pi(P_j)} \int_{P_i \times P_j} W(x,y) \, d\pi(x) \, d\pi(y) \qquad (x \in P_i, \ y \in P_j).$$

The linear operator  $\mathbb{E}_{\mathcal{P}}$ :  $f \mapsto f_{\mathcal{P}}$  (called a "stepping operator" in [19]) is a bounded linear operator  $L^1(J, \mathcal{A}, \pi) \to L^{\infty}(J, \mathcal{B}, \pi)$ . If we consider it as an operator  $L^2(J, \mathcal{A}, \pi) \to L^2(J, \mathcal{B}, \pi)$ , then it is self-adjoint and idempotent.

One property of the stepping operator that will be important for us is that it is contractive with respect to most "everyday" norms [19, Proposition 14.13], in particular, with respect to all  $L^p$ -norms ( $p \in [1, \infty]$ ):

$$\|f_{\mathcal{P}}\|_p \le \|f\|_p \tag{50}$$

for all  $f \in L^p(J, \mathcal{B}, \pi)$ .

We can extend this construction to Markov spaces, where its image is a bounded graphon  $W = W_{\eta_{\mathcal{P}}}$ , defined by

$$W_{\eta_{\mathcal{P}}}(x,y) = \frac{\eta(P_i \times P_j)}{\pi(P_i)\pi(P_j)} \qquad (x \in P_i, \ y \in P_j).$$

The edge measure associated with this graphon is

$$\eta_{\mathcal{P}} = \sum_{i,j=1}^{k} \frac{\eta(P_i \times P_j)}{\pi(P_i)\pi(P_j)} \left( (\mathbb{1}_{P_i}\pi) \times (\mathbb{1}_{P_j}\pi) \right).$$

Note that the marginals of  $\eta_{\mathcal{P}}$  are  $\pi$ , and so  $W_{\eta_{\mathcal{P}}}$  is 1-regular.

In terms of the adjacency operator  $\mathbf{A}$  of the Markov space, the operator  $\mathbf{A}_{\mathcal{P}}$  associated with  $\eta_{\mathcal{P}}$  can be expressed as the operator product  $\mathbf{A}_{\mathcal{P}} = \mathbb{E}_{\mathcal{P}} \mathbf{A} \mathbb{E}_{\mathcal{P}}$ .

We will also need the stepping operator for bi-Markov spaces. Let  $(I, J, \mathcal{A}, \mathcal{B}, \eta)$  be a bi-Markov space, and let  $\mathcal{P} = \{P_1, \ldots, P_k\}$  and  $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$  be finite, measurable, nondegenerate partitions of  $(I, \mathcal{A}, \pi)$  and  $(J, \mathcal{B}, \pi_J)$ , respectively. We define the following measures on  $\mathcal{A} \times \mathcal{B}$ :

$$(\mathbb{E}_{\mathcal{P}}\eta)(S\times T) = \sum_{i=1}^{k} \frac{\pi(S\cap P_i)}{\pi(P_i)} \eta(P_i \times T),$$

and

$$(\eta \mathbb{E}_{\mathcal{Q}})(S \times T) = \sum_{j=1}^{l} \frac{\pi_J(T \cap Q_j)}{\pi_J(Q_j)} \eta(S \times Q_j).$$

We can also partition both sigma-algebras, to obtain

$$(\mathbb{E}_{\mathcal{P}}\eta\mathbb{E}_{\mathcal{Q}})(S\times T) = \sum_{i=1}^{k}\sum_{j=1}^{l}\frac{\pi(S\cap P_{i})}{\pi(P_{i})}\frac{\pi_{J}(T\cap Q_{j})}{\pi_{J}(Q_{j})}\eta(P_{i}\times Q_{j}).$$

For a bi-Markov space, we also have a (non-self-adjoint) operator  $\mathbf{A}$ , and then the measures  $\mathbb{E}_{\mathcal{P}}\eta$ ,  $\eta\mathbb{E}_{\mathcal{Q}}$  and  $\mathbb{E}_{\mathcal{P}}\eta\mathbb{E}_{\mathcal{Q}}$  are associated with the (non-selfadjoint) operators  $\mathbb{E}_{\mathcal{P}}\mathbf{A}$ ,  $\mathbf{A}\mathbb{E}_{\mathcal{Q}}$  and  $\mathbb{E}_{\mathcal{P}}\mathbf{A}\mathbb{E}_{\mathcal{Q}}$ , respectively. Clearly all three of these measures have the same marginals  $\pi_I$  and  $\pi_J$  as  $\eta$ .

**Lemma 6.5** For every bi-Markov space  $(I, J, \mathcal{A}, \mathcal{B}, \eta)$  and finite, measurable, nondegenerate partitions  $\mathcal{P}$  and  $\mathcal{Q}$  of I and J, respectively, the measures  $\mathbb{E}_{\mathcal{P}}\eta, \eta \mathbb{E}_{\mathcal{Q}}$  and  $\mathbb{E}_{\mathcal{P}}\eta \mathbb{E}_{\mathcal{Q}}$  are absolutely continuous with respect to  $\pi_I \times \pi_J$ , with a bounded density function. **Proof.** Checking this for  $\mathbb{E}_{\mathcal{P}}\eta$ , let  $S \in \mathcal{A}$  and  $T \in \mathcal{B}$ . Then

$$(\mathbb{E}_{\mathcal{P}}\eta)(S \times T) \le \sum_{i=1}^{k} \frac{\pi_I(S)}{\pi_I(P_i)} \eta(I \times T) = \Big(\sum_{i=1}^{k} \frac{1}{\pi_I(P_i)}\Big) \pi_I(S) \pi_J(T),$$

which implies that  $\mathbb{E}_{\mathcal{P}}\eta$  is absolutely continuous with respect to  $\pi_I \times \pi_J$ , and its density function is bounded by  $\sum_i 1/\pi_I(P_i)$ . The argument for  $\eta \mathbb{E}_Q$  is symmetric, and the result for  $\mathbb{E}_{\mathcal{P}}\eta \mathbb{E}_Q$  follows from the previous two, the fact that  $\mathbb{E}_{\mathcal{P}}\eta \mathbb{E}_Q = (\mathbb{E}_{\mathcal{P}}\eta)\mathbb{E}_Q$  and the fact that the marginals of  $\mathbb{E}_{\mathcal{P}}\eta$  are also  $\pi_I$ and  $\pi_J$ .

### 6.3 Stepfunction approximation

Let W be a bounded graphon and let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence (see Subsection 2.6). The Martingale Convergence Theorem implies that  $W_{\mathcal{P}_i} \to W$  almost everywhere on  $J^2$ , and hence  $(W_{\mathcal{P}_i})^G \to W^G$  almost everywhere on  $J^V$  for every graph G. It is easy to check that the sequence  $W_{\mathcal{P}_i}^G$  is uniformly integrable, and hence  $W_{\mathcal{P}_i}^G \to W^G$  in  $L^1$ , which implies that the corresponding measures also converge. In particular,

$$t(G, W_{\mathcal{P}_i}) \to t(G, W) \tag{51}$$

How far does this fact extend beyond graphons? Under what conditions on G and  $\eta$  does  $\lim_{i\to\infty} t(G, \eta_{\mathcal{P}_i})$  exist for every exhausting partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$ ? Is the limit value independent of the sequence of partitions?

Recall that we say that  $\eta^G$  is partition approximable if  $\eta^G_{\mathcal{P}_i} \to \eta^G$  on boxes for every exhausting partition sequence. Our goal in the next sections is to establish that  $\eta^G$  is partition approximable for reasonably large classes of graphs G and Markov spaces  $\eta$ . To motivate this goal, let us state a simple consequence about the normalized density  $t^*$  (see Equation (17)).

**Proposition 6.6** Let  $(J, \mathcal{B}, \eta)$  be a Markov space. Then there is a sequence of simple graphs  $(H_i)_{i=1}^{\infty}$  such that

$$\lim_{i \to \infty} t^*(G, H_i) = t(G, \eta)$$

for every graph G such that  $\eta^G$  is partition approximable.

**Proof.** Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence. For every  $i \geq 1$ , there is an appropriate number  $c_i > 0$  such that  $c_i \eta_{\mathcal{P}_i}$  is a graphon (with values in [0, 1]), and so by dense graph limit theory, there is a sequence of graphs

 $H_{i,1}, H_{i,2}, \ldots$  such that  $t(G, H_{i,j}) \to t(G, c_i \eta_{\mathcal{P}_i}) = c_i^{|E|} t(G, \eta_{\mathcal{P}_i})$  for any G. In particular,  $t(K_2, H_{i,j}) \to c_i t(K_2, \eta_{\mathcal{P}_i})$ , and hence  $t^*(G, H_{i,j}) \to t(G, \eta_{\mathcal{P}_i})$ . Furthermore,  $t(G, \eta_{\mathcal{P}_i}) \to t(G, \eta)$   $(i \to \infty)$  if  $\eta^G$  is partition approximable. Since there are countably many graphs G to be considered, a standard diagonalization argument completes the proof. Note that two diagonalizations should happen: one to get rid of the partitions  $\mathcal{P}_i$  and one to make a single sequence for every G.

### 6.4 Weakly norming graphs

A graph G is called *weakly norming* if

$$||W||_G := t(G, |W|)^{1/|E(G)|}$$

is a norm on symmetric bounded measurable functions  $W: I^2 \to \mathbb{R}$ . This property was introduced by Hatami [14]. It is easy to see that all weakly norming graphs are bipartite; main examples are even cycles, hypercubes and complete bipartite graphs.

Since the operator  $W \mapsto W_{\mathcal{P}}$  is contractive with respect to a large class of norms, including all norms defined by graphs (see e.g. Proposition 14.13 in [19]), weakly norming graphs satisfy the inequality

$$t(G, W_{\mathcal{P}}) \le t(G, W) \tag{52}$$

for every graphon W and every finite, measurable, non-degenerate partition  $\mathcal{P}$ . This property is closely related to the well-known Sidorenko-Simonovits conjecture, which says that  $t^*(G, H) \geq 1$  for every bipartite graph G and every graph H. This is equivalent to saying that  $t^*(G, W) \geq 1$  for every bipartite graph G and every graphon W. For the trivial partition  $\mathcal{P}_0 = \{J\}$  we have  $t(G, W_{\mathcal{P}_0}) = t(K_2, W)^{|E(G)|}$ , and hence every graph G satisfying (52) satisfies the Sidorenko conjecture.

Property (52) of a graph G, required for every graphon W and every finite, measurable, non-degenerate partition  $\mathcal{P}$ , was introduced in [15], and called the *step Sidorenko property*. It was proved in [9] that this property is equivalent to being weakly norming.

For us, however, the inequality (52) is relevant only for 1-regular graphons. Then it holds for more graphs besides weakly norming ones, for example, for all trees. Therefore we name it the *weak step Sidorenko property*. It is easy to see that only bipartite graphs can have this property. As far as we can see, it might even hold for all bipartite graphs. If the graph G has the weak step Sidorenko property, then the convergence in (51) is monotone. **Remark 6.7** These considerations motivate the following version of density, which we call *partition-density*:

$$t_{\text{part}}(G,\eta) = \sup_{\mathcal{P}} t(G,\eta_{\mathcal{P}}),\tag{53}$$

where  $(J, \mathcal{B}, \eta)$  is a Markov space, and  $\mathcal{P}$  ranges over all finite, measurable, non-degenerate partitions of J. Partition density may be different from density even for ordinary graphs in place of  $\eta$ . For example, if H is bipartite and G is not, and H has at least one edge, then for the trivial (indiscrete) partition  $\mathcal{P}$ , we have  $(\eta_H)_{\mathcal{P}} = c(\pi \times \pi)$ , and so t(G, H) = 0 but  $t_{\text{part}}(G, \eta_H) > 0$ .

On the other hand, the monotonicity from (52) and the Martingale Convergence Theorem applied to  $(W^G)_{\mathcal{P}_i}$  along any exhaustive partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$  implies that  $t_{\text{part}}(G, W) = t(G, W)$  for every weakly norming graph G and every graphon W. It could be interesting to explore further properties of the partition-density.

**Remark 6.8** The weakly norming property, the step Sidorenko property and its weak version can be defined, mutatis mutandis, for bi-Markov spaces, and the above considerations remain valid. In particular, even cycles, complete bigraphs and hypercubes remain weakly norming, and hence have the step Sidorenko property.

### 6.5 Partition approximation of (k, p)-loose spaces

While our main goal is to prove results about Markov spaces, we study (k, p)looseness in bi-Markov spaces first. We address the issues of approximability by step functions. It turns out that for (k, p)-loose Markov spaces and bi-Markov spaces,  $\eta^G$  is partition approximable for a large class of (bipartite and bi-) graphs.

We start with discussing the total measure of  $\eta^G$ . For a (k, p)-loose Markov space, we can define the quantity

$$\|\eta\|_{k,p} := \|s_k^{\eta}\|_p^{1/k}.$$
(54)

For a bi-Markov space (k, p)-loose from (say) I, we define similarly

$$\|\eta\|_{I,k,p} := \|s_{I,k}^{\eta}\|_{p}^{1/k}.$$
(55)

If  $p \geq 2$  and  $\eta$  is not (k, p)-loose, then we define  $\|\eta\|_{k,p}$  to be infinite. If k = 1, then  $s_1^{\eta} = d\sigma_1/d\pi = 1$ , so  $\|\eta\|_{k,p} = 1$  by (54). When p = 1, it may happen that  $\eta$  is not k-loose and thus  $s_k^{\eta}$  is not defined. However, the  $L^1$  norm of a Radon–Nikodym derivative being the same as the total measure,

we can extend the above definition to also encompass the non-k-loose cases and define  $\|\eta\|_{k,1} := \sigma_k (J^k)^{1/k} = 1$  for any  $\eta$ .

Note that

$$\|\eta\|_{k,p} = \|s_k^{\eta}\|_p^{1/k} = t(K_{k,p},\eta)^{1/(kp)}$$
(56)

by (46). We will show that for Markov spaces  $\|\eta\|_{k,p} = \|\eta\|_{p,k}$ , or in other words,  $t(K_{k,p},\eta) = t(K_{p,k},\eta)$ .

As cited above, Hatami [14] proved that

$$||W||_{k,p} = t(K_{k,p}, W)^{1/(kp)}$$
(57)

is a norm on (not necessarily symmetric) bounded measurable functions  $W: I \times J \to \mathbb{R}$ . Clearly  $t(K_{k,p}, W) = t(K_{p,k}, W^*)$  holds for every bounded measurable function W, and so

$$\|W\|_{k,p} = \|W^*\|_{p,k}.$$
(58)

In particular,  $||W||_{k,p} = ||W||_{p,k}$  if I = J and W is symmetric. It is easy to check that if  $(J, \mathcal{B}, \eta_W)$  is a Markov space defined by a 1-regular graphon W, then

$$\|\eta_W\|_{k,p} = \|W\|_{k,p}.$$

Formally the same equation holds for a bi-Markov space defined by a 1-regular bigraphon.

Consider a bi-Markov space  $(I, J, \mathcal{A}, \mathcal{B}, \eta)$ . Let  $\mathcal{P}$  and  $\mathcal{Q}$  be finite, measurable, non-degenerate partitions of I and J, respectively. By Lemma 6.5, the measures  $\mathbb{E}_{\mathcal{P}}\eta$  and  $\eta^*\mathbb{E}_{\mathcal{P}}$  are represented by bounded measurable functions  $W_1, W_2$ , where trivially  $W_1^* = W_2$ . Hence (58) implies that

$$\|\mathbb{E}_{\mathcal{P}}\eta\|_{k,p} = \|W_1\|_{k,p} = \|W_2\|_{p,k} = \|\eta^*\mathbb{E}_{\mathcal{P}}\|_{p,k}.$$
(59)

Similarly we have  $\|\eta \mathbb{E}_{\mathcal{Q}}\|_{k,p} = \|\mathbb{E}_{\mathcal{Q}}\eta^*\|_{p,k}$ . For an exhausting partition sequence, in the limit, we have more:

**Lemma 6.9** Let  $(I, J, \mathcal{A}, \mathcal{B}, \eta)$  be a bi-Markov space, and let  $(\mathcal{P}_i)_{i=1}^{\infty}$  and  $(\mathcal{Q}_j)_{i=1}^{\infty}$  be exhausting partition sequences of I and J, respectively. Then

$$\lim_{i \to \infty} \|\eta \mathbb{E}_{\mathcal{Q}_i}\|_{k,p} = \lim_{i \to \infty} \|\mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{Q}_i}\|_{k,p} = \lim_{i \to \infty} \|\mathbb{E}_{\mathcal{P}_i} \eta\|_{k,p} = \|\eta\|_{k,p}.$$

**Proof.** We start with the first equality. Since  $K_{k,p}$  is weakly norming, it follows by the step Sidorenko property (52) that both limits exist, and also that  $\|\eta \mathbb{E}_{Q_i}\|_{k,p} \geq \|\mathbb{E}_{\mathcal{P}_i}\eta \mathbb{E}_{Q_i}\|_{k,p}$ . Hence we obtain that

$$\lim_{i \to \infty} \|\eta \mathbb{E}_{\mathcal{Q}_i}\|_{k,p} \ge \lim_{i \to \infty} \|\mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{Q}_i}\|_{k,p}$$

Let  $j \in \mathbb{N}$  be an arbitrary fixed number. Since  $W = \eta \mathbb{E}_{Q_j}$  is a bounded measurable function, the uniformly bounded measurable functions  $\mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{Q_j} = \mathbb{E}_{\mathcal{P}_i}W$  converge to W in  $L^1$  as  $i \to \infty$ , and thus by (57) we get

$$\lim_{i\to\infty} \|\mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{\mathcal{Q}_j}\|_{k,p} = \|\eta\mathbb{E}_{\mathcal{Q}_j}\|_{k,p}.$$

Again by the step Sidorenko property (52) we have that for i > j,

$$\|\mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{\mathcal{Q}_i}\|_{k,p} \ge \|\mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{\mathcal{Q}_i}\mathbb{E}_{\mathcal{Q}_j}\|_{k,p} = \|\mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{\mathcal{Q}_j}\|_{k,p}$$

and so by taking limit on both sides,

$$\lim_{i \to \infty} \|\mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{Q}_i}\|_{k,p} \ge \|\eta \mathbb{E}_{\mathcal{Q}_j}\|_{k,p}$$

This holds for every j, which proves the first equality. The second follows by interchanging the coordinates.

Finally, we prove that

$$\lim_{j \to \infty} \|\eta \mathbb{E}_{\mathcal{Q}_j}\|_{k,p} = \|\eta\|_{k,p}$$

If p = 1 then the statement is trivial since all terms are 1. Assume that p > 1. We have two cases. If  $\eta$  is k-loose from I, then  $s_{I,k}^{\eta}$  is in  $L^1(J^k, \pi_J^k)$ , and so

$$s_{I,k}^{\eta \mathbb{E}_{\mathcal{Q}_j}} = \mathbb{E}(s_{I,k}^{\eta} | \mathcal{Q}_j^k).$$

By Lemma 9.6, we have that  $(\mathcal{Q}_j^k)_{i=1}^{\infty}$  is an exhausting partition sequence for  $\pi_J^k$  and so the (potentially infinite)  $L^p$ -norm of  $\mathbb{E}(s_{I,k}^{\eta}|\mathcal{Q}_j^k)$  converges to the  $L^p$ -norm of  $s_{I,k}^{\eta}$  as  $j \to \infty$ .

Assume now that  $\eta$  is not k-loose from I. We have that  $\sigma_{I,k}$  is not absolutely continuous with respect to  $\pi_J^k$  and so there is a measurable set  $U \subset J^k$  such that  $\pi_J^k(U) = 0$  but  $c = \sigma_{I,k}(U) > 0$ . By Lemma 9.6, for every  $\epsilon > 0$  and large enough j, there is a set U' that is the union of  $\mathcal{Q}_j^k$  partition sets such that  $\pi_J^k(U') \leq \epsilon$  and  $\sigma_{I,k}(U') > c - \epsilon$ . For such a j,

$$\int_{U'} s_{I,k}^{\eta \mathbb{E}_{\mathcal{Q}_j}} d\pi_J^k = \int_{U'} s_{I,k}^{\eta} d\pi_J^k = \sigma_{I,k}(U')$$

Hölder's inequality implies that

$$\int_{U'} \left( s_{I,k}^{\eta \mathbb{E}_{\mathcal{Q}_j}} \right)^p d\pi_J^k \ge \frac{\left( \int_{U'} s_{I,k}^{\eta \mathbb{E}_{\mathcal{Q}_j}} d\pi_J^k \right)^p}{\pi_J^k (U')^{p-1}} \ge \epsilon^{1-p} (c-\epsilon)^p.$$

Applying this for every  $\epsilon > 0$  we obtain that  $\|s_{I,k}^{\eta \mathbb{E}_{Q_j}}\|_p \to \infty$  as  $j \to \infty$ .  $\Box$ 

From the previous lemma we obtain the next theorem.

**Theorem 6.10** Let  $(J, \mathcal{B}, \eta)$  be a Markov space and  $p, k \in \mathbb{N}$ . Then

$$\|\eta\|_{p,k} = \|\eta\|_{k,p} = t_{\text{part}}(K_{k,p},\eta)^{1/(pk)}$$

**Proof.** Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an arbitrary exhausting partition sequence. To see the first equality, observe that by Lemma 6.9 and (59),

$$\|\eta\|_{p,k} = \lim_{i \to \infty} \|\eta \mathbb{E}_{\mathcal{P}_i}\|_{p,k} = \lim_{i \to \infty} \|\mathbb{E}_{\mathcal{P}_i}\eta\|_{k,p} = \|\eta\|_{k,p}.$$

For the second equality, by (57),

$$\lim_{i \to \infty} t(K_{k,p}, \eta_{\mathcal{P}_i})^{1/(pk)} = \lim_{i \to \infty} \|\eta_{\mathcal{P}_i}\|_{k,p} = \|\eta\|_{k,p}.$$

However, since  $K_{k,p}$  has the step Sidorenko property (52), this yields

$$\|\eta\|_{k,p} = \lim_{i \to \infty} t(K_{k,p}, \eta_{\mathcal{P}_i})^{1/(pk)} = \sup_{i \in \mathbb{N}} t(K_{k,p}, \eta_{\mathcal{P}_i})^{1/(pk)}.$$

Since any partition can appear in an exhausting sequence, we obtain the desired equality.  $\hfill \Box$ 

**Lemma 6.11** Let  $(J, \mathcal{B}, \mu)$  be a probability space, and  $p \geq 1$ . Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition system. Assume that a sequence of  $L^p$  functions  $(f_i)_{i=1}^{\infty}$  and another  $L^p$  function f on  $(J, \mathcal{B}, \mu)$  satisfy

- 1.  $\lim_{i\to\infty} \|\mathbb{E}(f_i|\mathcal{P}_j) \mathbb{E}(f|\mathcal{P}_j)\|_1 = 0$  for every j
- 2.  $\lim_{i \to \infty} \|f_i\|_p = \|f\|_p$ .

Then  $\lim_{i\to\infty} ||f_i - f||_p = 0.$ 

**Proof.** Let  $\epsilon > 0$ . Then there is  $\delta > 0$  such that  $||1_U f||_p \leq \epsilon$  holds for every measurable set U with  $\mu(U) \leq \delta$ . We can choose  $j_0$  with the property that  $||f - \mathbb{E}(f|\mathcal{P}_j)||_p \leq \epsilon$  holds for every  $j \geq j_0$ . Then

$$\|1_U \mathbb{E}(f|\mathcal{P}_j)\|_p \le \|1_U f\|_p + \|1_U (f - \mathbb{E}(f|\mathcal{P}_j))\|_p \le 2\epsilon$$

$$\tag{60}$$

hold for every  $j \geq j_0$  and measurable set U with  $\mu(U) \leq \delta$ . For sufficiently big  $i_0$  we can also guarantee that  $|||f_i||_p - ||f||_p| \leq \epsilon$  holds for every  $i \geq i_0$ . For an arbitrary  $i \geq i_0$  we can choose  $j \geq j_0$  such that both  $|f_i - \mathbb{E}(f_i|\mathcal{P}_j)| \leq \epsilon/2$ and  $|\mathbb{E}(f_i|\mathcal{P}_j) - \mathbb{E}(f|\mathcal{P}_j)| \leq \epsilon/2$  holds on a set V of measure at least  $1 - \delta$ . It follows that  $|f_i - \mathbb{E}(f|\mathcal{P}_j)| \leq \epsilon$  holds on V. This implies that

$$\|1_V f_i - 1_V \mathbb{E}(f|\mathcal{P}_j)\|_p \le \epsilon.$$
(61)

Let U be the complement of V. Using (60) we have that

 $\|1_V \mathbb{E}(f|\mathcal{P}_j)\|_p \ge \|\mathbb{E}(f|\mathcal{P}_j)\|_p - \|1_U \mathbb{E}(f|\mathcal{P}_j)\|_p \ge \|f\|_p - 3\epsilon$ 

and thus by (61)

 $||1_V f_i||_p \ge ||f||_p - 4\epsilon.$ 

Using the above inequalities we obtain

$$\|1_U f_i\|_p^p = \|f_i\|_p^p - \|1_V f_i\|_p^p \le (\|f\|_p + \epsilon)^p - (\|f\|_p - 4\epsilon)^p =: g(\epsilon).$$
(62)

From (61), (62) and  $f_i = 1_U f_i + 1_V f_i$  we get that

$$||f_i - 1_V \mathbb{E}(f|\mathcal{P}_j)||_p \le g(\epsilon)^{1/p} + \epsilon$$

By (60), we have

$$\|\mathbb{E}(f|\mathcal{P}_j) - 1_V \mathbb{E}(f|\mathcal{P}_j)\|_p = \|1_U \mathbb{E}(f|\mathcal{P}_j)\|_p \le 2\epsilon$$

and thus

$$||f_i - \mathbb{E}(f|\mathcal{P}_j)||_p \le 3\epsilon + g(\epsilon)^{1/p}.$$

This implies

$$||f_i - f||_p \le 4\epsilon + g(\epsilon)^{1/p}.$$

Since  $\lim_{\epsilon \to 0} g(\epsilon) = 0$  the proof is complete.

**Lemma 6.12** Consider a bi-Markov space  $(I, J, \mathcal{A}, \mathcal{B}, \eta)$ . Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  and  $(\mathcal{Q}_j)_{j=1}^{\infty}$  be exhausting partition sequences of I and J, respectively. Set  $\eta_i = \mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{Q}_i}$ . Then  $s_k^{\eta_i} \to s_k^{\eta}$  in  $L^p$  as  $i \to \infty$ .

**Proof.** We have

$$\lim_{i \to \infty} \|s_k^{\eta_i}\|_p = \lim_{i \to \infty} \|\eta_i\|_{p,k}^k = \|\eta\|_{k,p}^k = \|s_k^{\eta}\|_p,$$

where the second equality is from Lemma 6.9 and the remaining equalities are just definitions. Now according to Lemma 6.11 it suffices to prove that for every  $j \in \mathbb{N}$  we have

$$\lim_{i \to \infty} \mathbb{E}(s_k^{\eta_i} | \mathcal{Q}_j^k) = \mathbb{E}(s_k^{\eta} | \mathcal{Q}_j^k)$$

in  $L_1$ . To see this observe that

$$\mathbb{E}(s_k^{\eta_i}|\mathcal{Q}_j^k) = s_k^{W_{i,j}} \quad \text{and} \quad \mathbb{E}(s_k^{\eta}|\mathcal{Q}_j^k) = s_k^{W_j},$$

where

$$W_{i,j} := \mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{Q}_i} \mathbb{E}_{\mathcal{Q}_j} \quad \text{and} \quad W_j := \eta \mathbb{E}_{\mathcal{Q}_j}.$$

If  $i \geq j$  then  $\mathbb{E}_{Q_i}\mathbb{E}_{Q_j} = \mathbb{E}_{Q_j}$  and so  $W_{i,j} = \mathbb{E}_{\mathcal{P}_i}W_j$ . Since for fixed j we have that  $W_{i,j}$  is a uniformly bounded sequence of measurable functions with  $L_1$ limit  $W_j$  the integral form of  $s_k^{W_{i,j}}$  and  $s_k^{W_j}$  shows the required convergence. More precisely, by abusing the notation, let us identify  $W_{i,j}$  and  $W_j$  with their representations by measurable functions. Then we have

$$s_k^{W_{i,j}}(z_1, z_2, \dots, z_k) = \mathbb{E}_x S_{i,j}(x, z_1, z_2, \dots, z_k)$$

and

$$s_k^{W_j}(z_1, z_2, \ldots, z_k) = \mathbb{E}_x S_j(x, z_1, z_2, \ldots, z_k),$$

where

$$S_{i,j}(x, z_1, x_2, \dots, z_k) := W_{i,j}(x, z_1) W_{i,j}(x, z_2) \cdots W_{i,j}(x, z_k)$$

and

$$S_j(x, z_1, x_2, \dots, z_k) := W_j(x, z_1) W_j(x, z_2) \cdots W_j(x, z_k).$$

Then

$$\begin{aligned} \|s_k^{W_{i,j}} - s_k^{W_j}\|_1 &= \|\mathbb{E}_x(S_{i,j} - S_j)\|_1 \le \|\mathbb{E}_x(|S_{i,j} - S_j|)\|_1 \\ &= \|S_{i,j} - S_j\|_1 \le k \|W_{i,j} - W_j\|_1 \|W_j\|_{\infty}^{k-1}, \end{aligned}$$

where the last inequality follows by changing the terms in the product one by one using the usual telescopic argument and the fact that  $||W_{i,j}||_{\infty} \leq ||W_j||_{\infty}$ . The fact that  $W_{i,j}$  converges to  $W_j$  in  $L_1$  completes the proof.

Now we are ready to state and prove our main theorem in this section.

**Theorem 6.13** Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space, and let G = (U, W, E) be a bigraph such that  $\deg(w) \leq k$  for all  $w \in W$  and  $\deg(u) \leq p$  for all  $u \in U$ . Then  $t(G, \eta) < \infty$ , and for every exhausting partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$ , we have

$$t(G,\eta) = \lim_{n \to \infty} t(G,\eta_{\mathcal{P}_n}).$$

**Proof.** The proof is a consequence of Lemma 6.12 and Theorem 6.4. Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence of J. Let  $W_i := \mathbb{E}_{\mathcal{P}_i} \eta \mathbb{E}_{\mathcal{P}_i}$ . Then by Lemma 6.12 we have that  $s_k^{W_i}$  converges to  $s_k^{\eta}$  in  $L^p$  as  $i \to \infty$ . Thus  $\eta$  is the (k, p)-limit of the sequence of the 1-regular graphons  $\{W_i\}_{i=1}^{\infty}$ . Theorem 6.4 completes the proof.

Note that a bi-Markov space version of Theorem 6.4 gives a bi-Markov space generalization of Theorem 6.13 is a similar way.

**Theorem 6.14** Let  $\mathbf{M} = (I, J, \mathcal{A}, \mathcal{B}, \eta)$  be a bi-Markov space (k, p)-loose from J. Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  and  $\{\mathcal{Q}_j\}_{j=1}^{\infty}$  be exhausting partition sequences of I and J, respectively. Let G = (U, W, E) be a bigraph such that  $\deg(w) \leq a$  for all  $w \in W$  and  $\deg(u) \leq b$  for all  $u \in U$ . Then  $t(G, \eta) < \infty$ , and

$$t(G,\eta) = \lim_{i,j\to\infty} t(G, \mathbb{E}_{\mathcal{P}_i}\eta\mathbb{E}_{\mathcal{Q}_j}).$$

## 6.6 Partition approximation of homomorphism measures

In this section we investigate an alternative approach to homomorphism measures using finite partitions  $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$  of the ground space, approximating  $\eta$  by the projections  $\eta_{\mathcal{P}}$  as in the previous section. As before, the measure  $\eta_{\mathcal{P}}$  is defined by a graphon, and hence the measures  $\eta_{\mathcal{P}}^G$  are defined (see Section 5.3). It is natural to define homomorphism measures  $\eta^G$  as limits of homomorphism measures  $\eta_{\mathcal{P}_i}^G$  for an exhausting partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$ . This requires an appropriate convergence notion for such measures. There are several notions of convergence we can use: strong (pointwise) convergence; convergence in total variation norm; weak convergence (after putting a compact topology on J) etc. We choose a more technical but more convenient path, requiring convergence on sets in  $\mathcal{P}_i^V$ , where  $\mathcal{P}_i^V$  is the partition of  $J^V$  whose elements are boxes of the form  $\prod_{v \in V} P_v$  where  $P_v \in \mathcal{P}_i$ .

The measure  $\eta^G$ , defined in (45), can be expressed as follows: Let  $A = \prod_{u \in U} A_u$  and  $B = \prod_{w \in W} B_w$ , where  $A_u, B_w \in \mathcal{B}$ . Then

$$\eta^G(A \times B) = \int_A \prod_{w \in W} \psi_{x_{N(w)}}(B_w) \, d\pi^U(x).$$
(63)

A simple but important remark is that changing an  $A_u$  or a  $B_w$  on a set of  $\pi$ -measure zero, the value  $\eta^G(A \times B)$  is not changed. This is trivial for the  $A_u$ , and follows by Lemma 4.4 for  $B_w$ .

**Theorem 6.15** Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space, and let G = (U, W, E) be a bigraph such that  $\deg(w) \leq k$  for all  $w \in W$  and  $\deg(u) \leq p$  for all  $u \in U$ . Then  $\eta^G$  is partition approximable.

**Proof.** Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence. We want to prove that  $\eta_{\mathcal{P}_i}^G(C) \to \eta^G(C)$  for every Borel box  $C = \prod_{v \in V} B_v$  as  $i \to \infty$ . First we prove the assertion in a special case.

Claim 1 Suppose that  $B_v \in \mathcal{P}_j$  for some j and all  $v \in V$ . Then  $\eta_{\mathcal{P}_i}^G(C) \to \eta^G(C)$ .

We may restrict our attention to  $i \geq j$ . We may assume that the partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$  is generating, not only exhausting; by Lemma 9.5, this can be achieved by changing each partition class on a set of measure zero.

We want to mimic the proof of Theorem 6.13, which is a related assertion for the total measure  $\eta^G(J^V)$ . To this end, we express homomorphism measures in terms of homomorphism densities of certain bi-Markov spaces.

For a Markov space  $\eta$  and  $B \in \mathcal{B}$  with  $\pi(B) > 0$ , we introduce a bi-Markov space  $X(\eta, B)$  which is basically the restriction of  $\eta$  to  $J \times B$ . Since  $\eta(J \times B) = \pi(B)$ , we have to multiply the restriction of  $\eta$  with  $\pi(B)^{-1}$ to obtain a proper bi-Markov space  $(J, B, \eta|_{J \times B}/\pi(B))$ , where  $\eta|_{J \times B}(A) :=$  $\eta((J \times B) \cap A)$ . It is clear from the definition that if  $\eta$  is k-loose then  $\pi(B)s_k^{X(\eta,B)} \leq s_k^{\eta}$  almost surely on  $J^k$ . It follows that if  $\eta$  is (k, p)-loose then so is  $X(\eta, B)$  for any subset  $B \in \mathcal{B}$  with positive measure.

Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space, and let G = (U, W, E) be a bigraph such that  $\deg(w) \leq k$  for all  $w \in W$  and  $\deg(u) \leq p$  for all  $u \in U$ . Let  $A = \prod_{u \in U} B_u$  and  $B = \prod_{w \in W} B_w$ . Then

$$\eta^{G}(C) = \eta^{G}(A \times B) = \int_{A} \prod_{w \in W} \psi_{x_{N(w)}}(B_{w}) d\pi^{U}(x)$$
$$= \int_{A} \prod_{w \in W} \pi(B_{w}) s_{\deg(w)}^{X(\eta, B_{w})}(x_{N(w)}) d\pi^{U}.$$

For  $w \in W$ , let  $\mathcal{P}_{i,w}$  denote the restriction of  $\mathcal{P}_i$  to  $B_w$ . Define

$$X_{i,w} := X(\eta_{\mathcal{P}_i}, B_w) = \mathbb{E}_{\mathcal{P}_i} X(\eta, B_w) \mathbb{E}_{\mathcal{P}_i, w}$$

then

$$\eta_{\mathcal{P}_i}^G(C) = \int_A \prod_{w \in W} \pi(B_w) s_{\deg(w)}^{X_{i,w}}(x_{N(w)}) \ d\pi^U(x).$$

Lemma 6.12 shows that

$$\lim_{i \to \infty} s_{\deg(w)}^{X_{i,w}}(x_{N(w)}) = s_{\deg(w)}^{X(\eta, B_w)}(x_{N(w)}),$$

where convergence is in  $L^p$ . This completes the proof of Claim 1 by Corollary 6.2.

Note that this Claim implies immediately that the same conclusion holds if  $B_v \in \widehat{\mathcal{P}}_i$  for all v, since such a box is a finite union of boxes in  $\mathcal{P}_i^{U \cup W}$ .

**Claim 2** For every  $\varepsilon > 0$  there is a  $\delta > 0$  and an  $i_0 \in \mathbb{N}$  such that

$$\eta^G(X \times J^{V \setminus v}) < \varepsilon$$
 and  $\eta^G_{\mathcal{P}_i}(X \times J^{V \setminus v}) < \varepsilon$  (64)

for every  $v \in V$ , every  $X \in \mathcal{B}$  with  $\pi(X) < \delta$ , and every  $i \ge i_0$ .

The first inequality (which is independent of i) is just a restatement of the absolute continuity of the marginal  $(\eta^G)^v$  with respect to  $\pi$  (Lemma 5.15). To prove the second, choose  $\delta$  such that  $\eta^G(X \times J^{V \setminus v}) < \varepsilon/2$  for  $\pi(X) < 2\delta$ . Let  $Y \in \mathcal{B}$  be a set with  $\pi(Y) \leq \delta$  maximizing  $\eta^G_{\mathcal{P}_i}(Y \times J^{V \setminus v})$ . We may assume that every partition class of  $\mathcal{P}_i$  has  $\pi$ -measure at most  $\delta$ . Since the marginal  $(\eta^G_{\mathcal{P}_i})^v$  is proportional to  $\pi$  on every partition class of  $\mathcal{P}_i$ , the maximizing Y will consist of the union of at least one partition class and at most one subset of a partition class. So there is a set  $Z \in \widehat{\mathcal{P}}_i$  such that  $Y \subseteq Z$  and  $\pi(Z) \leq 2\delta$ . Then

$$(\eta_{\mathcal{P}_i}^G)^v(X) \le (\eta_{\mathcal{P}_i}^G)^v(Y) \le (\eta_{\mathcal{P}_i}^G)^v(Z) = \eta_{\mathcal{P}_i}^G(Z \times J^{V \setminus v}).$$

Here the box  $Z \times J^{V \setminus v}$  is the product of sets in the set algebra  $\widehat{\mathcal{P}}_i$ , and so by Claim 1,

$$\eta_{\mathcal{P}_i}^G(Z \times J^{V \setminus v}) \le \eta^G(Z \times J^{V \setminus v}) + \frac{\varepsilon}{2} \le \varepsilon.$$

if i is large enough. Choosing  $i_0$  so that if  $i \ge i_0$ , then this holds for all v, completes the proof of Claim 2.

To complete the proof, let  $C = \prod_{v \in V} B_v$  be any box with  $B_v \in \mathcal{B}$ . Lemma 9.5 implies that there are sets  $\overline{B}_v \in \widehat{\mathcal{P}}_i$  for a sufficiently large *i* such that  $\pi(B_v \triangle \overline{B}_v) \leq \delta$  for all  $v \in V$ , where  $\delta$  is chosen as in Claim 2. Let  $\overline{C} = \prod_{v \in V} \overline{B}_v$ . By Claim 1,

$$\left|\eta_{\mathcal{P}_i}^G(\overline{C}) - \eta^G(\overline{C})\right| \le \varepsilon$$

if i is large enough. Furthermore,

$$C \triangle \overline{C} \subseteq \bigcup_{v \in V} (B_v \triangle \overline{B}_v) \times J^{V \setminus v},$$

and so by Claim 2,

$$\left|\eta_{\mathcal{P}_{i}}^{G}(\overline{C}) - \eta_{\mathcal{P}_{i}}^{G}(C)\right| \leq \sum_{v \in V} \eta_{\mathcal{P}_{i}}^{G}\left((B_{v} \triangle \overline{B}_{v}) \times J^{V \setminus v})\right) \leq |V|\varepsilon,$$

and similarly

$$\left|\eta^{G}(\overline{C}) - \eta^{G}(C)\right| \leq \sum_{v \in V} \eta^{G}\left(\left(B_{v} \Delta \overline{B}_{v}\right) \times J^{V \setminus v}\right)\right) \leq |V|\varepsilon.$$

Summing up,

$$\begin{aligned} \left| \eta_{\mathcal{P}_{i}}^{G}(C) - \eta^{G}(C) \right| \\ &\leq \left| \eta_{\mathcal{P}_{i}}^{G}(\overline{C}) - \eta_{\mathcal{P}_{i}}^{G}(C) \right| + \left| \eta_{\mathcal{P}_{i}}^{G}(\overline{C}) - \eta^{G}(\overline{C}) \right| + \left| \eta^{G}(\overline{C}) - \eta^{G}(C) \right| \\ &\leq (2|V|+1)\varepsilon. \end{aligned}$$

This proves the Theorem.

**Corollary 6.16** Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space. Let G = (U, W, E) be a bigraph such that  $\deg(w) \leq k$  for all  $w \in W$  and  $\deg(u) \leq p$  for all  $u \in U$ . Then  $\eta^G = \eta^{G^*}$ .

**Proof.** Choose an generating partition sequence  $(\mathcal{P}_i)_{i=1}^{\infty}$ . Then  $\eta_{\mathcal{P}_i}$  is a graphon, and so  $\eta_{\mathcal{P}_i}^{G^*} = \eta_{\mathcal{P}_i}^G$ . By Theorem 6.15, we have

$$\eta^G(C) = \lim_{i \to \infty} \eta^G_{\mathcal{P}_i}(C) = \lim_{i \to \infty} \eta^{G^*}_{\mathcal{P}_i}(C) = \eta^{G^*}(C)$$

for every box  $C \in \mathcal{P}_j^V$ . Since the sigma-algebra generated by such sets contains all Borel sets, it follows that  $\eta^G = \eta^{G^*}$ .

**Corollary 6.17** If **M** is an (a,b)-loose Markov space, then  $K_{a,b}$  is wellmeasured in M.

**Proof.** Let p be any ordering of  $V(K_{a,b})$ . Let  $\{u_1, \ldots, u_a\}$  and  $\{v_1, \ldots, v_b\}$  be the color classes of  $K_{a,b}$ , and let  $q = (u_1, \ldots, u_a, v_1, \ldots, v_b)$  and  $r = (v_1, \ldots, v_b, u_1, \ldots, u_a)$ . Similarly as in the proof of Lemma 5.11, we may assume that  $\eta_p$  does not change if we reorder the first a + b - 1 elements, and it does not change if we flip consecutive non-adjacent nodes, so it follows that  $\eta_p = \eta_q$  or  $\eta_p = \eta_r$  (depending on the color class of the last node in p). But by Corollary 6.16,  $\eta^{K_{a,b}} = \eta^{K_{b,a}}$  and so  $\eta_q = \eta_r$ . Thus  $\eta_p$  is independent of p.

Combining Corollary 6.17 with Theorem 1.2, we obtain the following:

**Corollary 6.18** Let  $(J, \mathcal{B}, \eta)$  be a (k, p)-loose Markov space. Let G = (U, W, E) be a bigraph such that  $\deg(w) \leq k$  for all  $w \in W$  and  $\deg(u) \leq p$  for all  $u \in U$ . Then G is well-measured in **M**.

This corollary implies Theorem 1.4.

#### 6.7 Products of graphs

In this section we investigate an interesting construction of a sparse graph sequence, where the limit object is easily guessed, but it is more difficult to tell in what sense do these graphs converge to this limit.

For two edge-weighted graphs  $H_1$  and  $H_2$ , we define their product  $H_1 \times H_2$ as the edge-weighted graph on  $V(H_1) \times V(H_2)$ , where the edge-weight w in the product is defined by

$$w((x_1, x_2), (y_1, y_2)) = w_1(x_1, y_1)w_2(x_2, y_2).$$

If every edge weight in  $H_i$  is  $1/(2|E(H_i)|)$ , then this is just the categorical product of the two graphs, with the edges weighted analogously.

Let  $H_n = (V_n, E_n)$ , n = 1, 2, ... be simple graphs, and let  $p_n = |V(H_n)|$ ,  $q_n = |E(H_n)|$ . Define

$$\widehat{H}_n = H_1 \times \cdots \times H_n.$$

We can also define the product of infinitely many graphs. Indeed, let  $J = V_1 \times V_2 \times \cdots$ , with the Borel  $\sigma$ -algebra  $\mathcal{B}$ . There is a natural graph on J, in which  $(u_1, u_2, \ldots)$  is connected to  $(v_1, v_2, \ldots)$  if and only if each  $u_i$  is connected to  $v_i$  in  $H_i$  for every  $i \in \mathbb{N}$ . We need to define a measure on this edge set. A Markov step from a point  $(v_1, v_2, \ldots) \in J$  is obtained by making a step of the random walk on  $H_i$  from  $v_i$ , independently for different indices i. The measure of a cylinder set  $C = A_1 \times \cdots \times A_n \times E_{n+1} \times \cdots (A_i \subset V_i^2)$  is

$$\eta(C) = \begin{cases} \prod_{j=1}^{n} \frac{|A_j|}{|E_j|}, & \text{if } A_j \subseteq E_j \text{ for all } 1 \le j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this Markov space by  $H_{\infty} = (J, \mathcal{B}, \eta)$ . Let  $\mathbf{A}_{\infty}$  denote the adjacency operator of  $H_{\infty}$ .

In this section we study the question whether  $\widehat{H}_n \to H_\infty$  in any reasonable sense.

Let  $\lambda_1^{(n)} = 1, \lambda_2^{(n)}, \ldots$  be the eigenvalues of the transition matrix of the random walk on  $H_n$ , with corresponding eigenvectors  $w_1^{(n)}, w_2^{(n)}, \ldots$  For

every choice of indices  $1 \leq i_j \leq p_j$ , the transition matrix of the graph  $\hat{H}_n$  has an eigenfunction

$$f_{i_1\dots i_n}(u_1, u_2, \dots, u_n) = \prod_{j=1}^n w_{i_j, u_j}^{(j)}$$
(65)

with eigenvalue

$$\lambda_{i_1\dots i_n} = \prod_{j=1}^n \lambda_{i_j}^{(j)}.$$
(66)

These eigenvalues remain eigenvalues in  $H_{\infty}$ , and so do the corresponding eigenfunctions, if we consider them as defined on  $V(H_{\infty})$  but depending only on the first *n* coordinates. We can also think of this as extending the formulas (65) and (66) to infinite products, but choosing the eigenvalue 1 with eigenfunction identically 1 for all j > n. Let us call these eigenvalues finitary.

We may or may not obtain further nonzero eigenvalues as infinite products with infinitely many nontrivial eigenvalues. This will not happen if and only if the transition matrices of the graphs H have a common eigenvalue gap in the sense that for some c > 0,

$$\mu_n = \max_{j \ge 2} |\lambda_j^{(n)}| \le 1 - c \tag{67}$$

for every n.

Trivially, the multiplicity of a nonzero finitary eigenvalue may be infinite, and these eigenvalues may have accumulation points other than 0. It is easy to see that the eigenvalues have no nonzero accumulation point if and only if

$$\mu_n \to 0 \qquad (n \to \infty). \tag{68}$$

The Markov space  $H_{\infty}$  has a natural partition  $\mathcal{P}_n$  defined by the first n coordinates. More exactly,  $\mathcal{P}_n$  has partition classes  $U_z$  ( $z \in V_1 \times \cdots \times V_n$ ), consisting of all extensions of z. Then  $(H_{\infty})_{\mathcal{P}_n}$  is the graphon associated with the graph  $\widehat{H}_n$ , with edge weights  $1/(q_1 \cdots q_n)$ . Let G be a graph with a nodes and b edges, then

$$t(G, (H_{\infty})_{\mathcal{P}_n}) = t^*(G, \widehat{H}_n) = \prod_{j=1}^n t^*(G, H_j) = \prod_{j=1}^n \frac{\hom(G, H_j) p_j^{2b-a}}{(2q_j)^b}.$$

Let us define

$$t^{\times}(G, H_{\infty}) = \prod_{j=1}^{\infty} t^{*}(G, H_{j}),$$
(69)

provided the product is convergent. With this definition,

$$t^*(G, (H_\infty)_{\mathcal{P}_n}) = t^*(G, \widehat{H}_n) \to t^{\times}(G, H_\infty)$$

When does the product in (69) converge? Is the value  $t^{\times}(G, H_{\infty})$  as defined above also the limit of  $t^{*}(G, (H_{\infty})_{Q_{n}})$  for every exhausting partition sequence  $(Q_{i})_{i=1}^{\infty}$ ? Is  $t^{\times}(G, H_{\infty}) = t(G, H_{\infty})$ ? For the first question we give a reasonably general sufficient condition. The other two remain open.

Let  $(H_n)$  be a sequence of (very dense) simple graphs such that  $p_n \ge n$ . Let  $\overline{H}_n$  denote the complement of  $H_n$ , including all loops at the nodes. Let  $d_n$  denote the maximum degree of  $\overline{H}_n$  and assume that  $d_n = O(1)$ . Let  $r_n = p_n^2 - 2q_n$  be the number of oriented edges of  $\overline{H}$ , then  $r_n \le d_n p_n = O(p_n)$ .

Let G be a simple graph with a nodes and b edges. For  $Y \subseteq E(G)$ , let  $G_Y = (V(G), Y)$ . Then by inclusion-exclusion,

$$\hom(G, H_n) = \sum_{Y \subseteq E(G)} (-1)^{|Y|} \hom(G_Y, \overline{H}_n).$$

Here hom $(G_{\emptyset}, \overline{H}_n) = p_n^a$  and hom $(G_Y, \overline{H}_n) = r_n p_n^{a-2}$  if |Y| = 1. If  $|Y| \ge 2$ , then selecting one node from each connected component of  $G_Y$ , we get a - c points, where  $c \ge 2$ . We can map these points  $p_n^{a-c}$  ways, but the remaining points in at most  $d_n^c$  ways, so we get then

$$\hom(G_Y, \overline{H}_n) \le d_n^c p_n^{a-c} = O(p_n^{a-2}).$$

Hence

$$\hom(G, H_n) = p_n^a - br_n p_n^{a-2} + O(p_n^{a-2}),$$

and so

$$t(G, H_n) = 1 - \frac{br_n}{p_n^2} + O(p_n^{-2}).$$

Clearly  $t(K_2, H_n) = 1 - r_n/p_n^2$ , and so

$$t(K_2, H_n)^b = 1 - \frac{br_n}{p_n^2} + O\left(\frac{r_n^2}{p_n^4}\right) = 1 - \frac{br_n}{p_n^2} + O(p_n^{-2}).$$

Thus

$$t^*(G, H_n) = \frac{1 - br_n/p_n^2 + O(p_n^{-2})}{1 - br_n/p_n^2 + O(p_n^{-2})} = 1 + O(p_n^{-2}).$$

Using that  $d_n = O(1)$  and  $p_n \ge n$ , it follows that the product in (69) is convergent.

It is interesting to consider two special examples.

**Example 6.19 (Powers of a graph)** As remarked before, our methods above work for compact operators only. Here is an example where extension of the results to operators that are "almost" compact would be very useful.

Let H = (V, E) be a *d*-regular graph with *n* nodes, and consider its direct powers  $H^{\times k}$ ,  $k = 1, 2, \ldots$  Let  $\eta$  be the uniform distribution on the edges of *H*, then the marginal of  $\eta$  is the uniform distribution  $\pi$  on *V*, and the stationary distribution on  $V(H^{\times k})$  is  $\pi^k$ .

Going to the limit  $k \to \infty$ , we get a limit object on  $J = V^{\mathbb{N}}$ , with sigmaalgebra generated by sets  $A_1 \times A_2 \times \cdots$  where all but a finite number of factors are V, and stationary measure defined by  $\pi^{\omega}(A_1 \times A_2 \times \cdots) = \pi(A_1)\pi(A_2)\cdots$ . The edge measure  $\eta^{\omega}$  is defined similarly. The edge measure is supported on the set  $E \times E \times \cdots$ , so it is quite singular with respect to  $\pi^{\omega} \times \pi^{\omega}$ .

For a point  $(v_1, v_2, ...) \in V^{\omega}$  of the Markov space  $(V^{\omega}, \eta^{\omega})$ , a Markov step is generated by choosing a random neighbor  $u_i$  of  $v_i$  independently for all i, and moving to  $(u_1, u_2, ...)$ .

The operator **A** associated with the Markov space  $\eta^{\omega}$  is, unfortunately, not compact. Let  $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$  be the eigenvalues of H (normalized by d), with corresponding eigenvectors  $w_1, \ldots, w_n$ . Then for every finite sequence of positive integers  $k_1 < \cdots < k_r$ , and every choice of indices  $2 \leq i_1, \ldots, i_r \leq n$ , **A** has an eigenfunction

$$f_i(u_1, u_2, \dots) = \prod_{j=1}^r w_{i_j, u_{k_j}}$$
(70)

with eigenvalue

$$\prod_{j=1}^{r} \lambda_{i_j}.$$
(71)

The multiplicity of each of these eigenvalues is infinite, since there are a countably infinite number of sequences  $(k_i)$  with the same length. So **A** is not compact. On the other hand, the nonzero eigenvalues of **A** are products

of a finite number of normalized eigenvalues of H, so they form a discrete set with only one accumulation point at 0, so **A** does have some resemblance of compact operators.

Can we define the density of a bipartite graph G in  $\eta^{\omega}$ , and show that this is nonzero? If, in addition, we can prove that  $t^*(G, H^{\times k}) \to t(G, \eta^{\omega})$ , then Sidorenko's conjecture would follow.

**Example 6.20 (Products of complete graphs)** Let  $H_n = K_n$  be the complete *n*-graph (without loops). For the product to be nontrivial, we consider  $K_2 \times K_3 \times \cdots$ . For every graph G with p nodes and q edges,

$$|\hom(G, K_n)| = \chi_G(n),$$

where  $\chi_G$  denotes the chromatic polynomial of G. It follows that for  $n \geq 2$ ,

$$t(K_2, \widehat{H}_n) = \prod_{j=2}^{i} \frac{j(j-1)}{j^2} = \frac{1}{i}$$

showing that  $(\hat{H}_1, \hat{H}_2, ...)$  is a sparse graph sequence. It is well known that  $\chi_G$  is a polynomial of degree p, and it has the form  $x^p + a_1 x^{p-1} + a_2 x^{p-2} + ...$  with  $a_1 = -q$ . Hence

$$t^*(G, H_n) = \frac{\chi_G(n)n^{q-p}}{(i-1)^q} = \frac{i^q - qi^{q-1} + a_2i^{q-2} + \cdots}{i^q - qi^{q-1} + \binom{q}{2}q^2 + \cdots} = 1 + O\left(\frac{1}{n^2}\right),$$

and so the product  $\prod_{j=2}^{\infty} t^*(G, H_j)$  is convergent.

The Markov space  $H_{\infty}$  is (k, p)-loose for every  $k, p \geq 1$ . Indeed, let  $x = (x_2, x_3, \ldots)$  be a random point of  $H_{\infty}$ , and let  $y_1 = (y_{12}, y_{13}, \ldots), \ldots, y_k = (y_{k2}, y_{k3}, \ldots)$  be k random steps from x. Then for n > k, the joint distribution of  $y_{1n}, \ldots, y_{kn}$  is uniform over all k-tuples of points of  $K_n$ ; for  $n \leq k$  it is not uniform, but trivially it has a density function  $F_{kn}$ . Then  $s_k = F_{k2}F_{k3}\cdots F_{k,k}$  (not depending on the coordinates n > k) is the density function of  $(y_1, \ldots, y_k)$ . Trivially  $s_k^p$  is a bounded function, and so  $s_k \in L^p$ .

It follows that for every bipartite graph G we have  $\lim_{i\to\infty} t^*(G, (H_\infty)_{\mathcal{Q}_i}) = t(G, H_\infty)$  for every exhausting partition sequence  $(\mathcal{Q}_i)_{i=1}^{\infty}$  by Theorem 6.13.

# 7 Cycle densities and the spectrum

It is well known that the homomorphism number of the k-cycle  $C_k$  in a graph G is the sum of the k-th powers of the eigenvalues of the adjacency matrix

of G. This can be generalized to graphons and even to bounded symmetric measurable functions  $W : \Omega^2 \to \mathbb{R}$  where  $(\Omega, \mu)$  is a standard probability space. In this case  $t(C_k, W)$  is equal to  $\sum_{i=1}^{\infty} \lambda_i^k$  where the numbers  $\lambda_i$  are the eigenvalues of W as an integral kernel operator. In this section we push this further to operators on  $L^2$  spaces whose k-th Schatten norm is finite for some k. In particular the main result of this section (see Theorem 7.5) implies the following theorem.

**Theorem 7.1** Let k be an integer and assume that the k-th Schatten norm of the adjacency operator **A** of a Markov space  $\mathbf{M} = (J, \mathcal{B}, \eta)$  is finite. Then  $\eta^{C_k}$  is partition approximable, and  $t(C_k, \eta)$  is equal to the sum of the k-th powers of the eigenvalues of **A**.

Let  $\|.\|_p^{\bullet}$  denote *p*-th Schatten norm. Also, given a compact self-adjoint operator A on an infinite dimensional Hilbert space H and an integer  $k \ge 1$ , let  $\lambda_k^+(A)$  be the *k*-th largest (counting multiplicities) positive eigenvalue of A, with the convention  $\lambda_k^+(A) = 0$  if there are less than k such eigenvalues. Similarly let  $\lambda_k^-(A)$  be the *k*-th smallest (counting multiplicities) negative eigenvalue of A, with the convention  $\lambda_k^+(A) = 0$  if there are less than k such eigenvalues. Note that we then have

$$(||A||_p^{\bullet})^p = \sum_{k=1}^{\infty} |\lambda_k^+(A)|^p + \sum_{k=1}^{\infty} |\lambda_k^-(A)|^p.$$

**Lemma 7.2** Let U be a d dimensional subspace in an infinite dimensional Hilbert space  $\mathcal{H}$  and let A be a compact self-adjoint operator such that  $\|A\|_{p}^{\bullet} < \infty$  for some  $p \geq 1$ . Let  $\lambda'_{1} \geq \lambda'_{2} \geq \ldots \geq \lambda'_{d}$  be the eigenvalues of  $P_{U}AP_{U}|_{U}$ . Then  $\lambda_{k}^{+}(P_{U}AP_{U}) = \max\{\lambda'_{k}, 0\}, \ \lambda_{k}^{-}(P_{U}AP_{U}) = \min\{\lambda'_{d+1-k}, 0\}$ and  $\lambda_{k}^{+}(A) \geq \lambda'_{k} \geq \lambda_{d+1-k}^{-}(A)$  for all  $1 \leq k \leq d$ .

**Proof.** The identities follow from the fact that the operator  $P_UAP_U$  is reduced by the subspace U, and is the zero operator on  $U^{\perp}$ . Concerning the inequalities, by the Courant–Fischer–Weyl theorem (or minmax principle, see [10, Excercise 6.34]), we have the following:

$$\lambda_{k}^{+}(A) = \sup_{\dim(W)=k} \min_{\substack{v \in W, \\ \|v\|=1}} \langle Av, v \rangle,$$
  
$$\lambda_{d+1-k}^{-}(A) = \inf_{\dim(W)=d+1-k} \max_{\substack{v \in W, \\ \|v\|=1}} \langle Av, v \rangle,$$
  
$$\lambda_{k}' = \sup_{\substack{\dim(W)=k, \\ W \subset U}} \min_{\substack{v \in W, \\ \|v\|=1}} \langle (P_{U}AP_{U}|_{U})v, v \rangle,$$
  
$$\lambda_{k}' = \inf_{\substack{\dim(W)=d+1-k, \\ W \subset U}} \max_{\substack{v \in W, \\ \|v\|=1}} \langle (P_{U}AP_{U}|_{U})v, v \rangle$$

Now note that then

$$\begin{aligned} \lambda'_k &= \sup_{\substack{\substack{v \in W, \\ W \subset U}}} \min_{\substack{v \in W, \\ \|v\|=1}} \langle (P_U A P_U|_U) v, v \rangle = \sup_{\substack{\dim(W)=k, \\ W \subset U}} \min_{\substack{v \in W, \\ \|v\|=1}} \langle A P_U v, P_U v \rangle \\ &= \sup_{\substack{\dim(W)=k, \\ w \in U}} \min_{\substack{v \in W, \\ \|v\|=1}} \langle A v, v \rangle \leq \sup_{\dim(W)=k} \min_{\substack{v \in W, \\ \|v\|=1}} \langle A v, v \rangle = \lambda_k^+(A), \end{aligned}$$

with the other inequality following by symmetry.

This immediately leads to the following result.

**Corollary 7.3** Let U be a d dimensional subspace in a Hilbert space and let A be a self-adjoint operator such that  $||A||_p^{\bullet} < \infty$  for some  $p \ge 1$ . Then  $||P_UAP_U||_p^{\bullet} \le ||A||_p^{\bullet}$ .

The above can be used to express the  $\ell$ -th Schatten norm of an operator as the limit of that of its finite dimensional approximants.

**Proposition 7.4** Assume A is a bounded, self-adjoint operator on a Hilbertspace  $\mathcal{H}$  with  $||A||_{\ell}^{\bullet} < \infty$ . Assume that  $\{\mathcal{H}_j\}_{j=1}^{\infty}$  is a sequence of finite dimensional subspaces of  $\mathcal{H}$  such that  $\mathcal{H}_j \subseteq \mathcal{H}_{j+1}$  holds for every j and  $\bigcup_{j=1}^{\infty} \mathcal{H}_j$  is dense in  $\mathcal{H}$ . Then

$$\sum_{k=1}^{\infty} \lambda_k^+ (A)^\ell = \lim_{j \to \infty} \sum_{k=1}^{\infty} \lambda_k^+ (P_{\mathcal{H}_j} A P_{\mathcal{H}_j})^\ell,$$
$$\sum_{k=1}^{\infty} \lambda_k^- (A)^\ell = \lim_{j \to \infty} \sum_{k=1}^{\infty} \lambda_k^- (P_{\mathcal{H}_j} A P_{\mathcal{H}_j})^\ell,$$

and also  $||A||_{\ell}^{\bullet} = \lim_{j \to \infty} ||A_j||_{\ell}^{\bullet} < \infty.$ 

**Proof.** For  $j \in \mathbb{N}$ , let  $A_j := P_{\mathcal{H}_j}AP_{\mathcal{H}_j}$ . By Lemma 7.2, we have  $0 \leq \lambda_k^+(A_j) \leq \lambda_k^+(A)$  and  $0 \geq \lambda_k^-(A_j) \geq \lambda_k^-(A)$  for every  $k \geq 1$ . If we can show that for every  $k \geq 1$ ,  $\lim_{j\to\infty} \lambda_k^+(A_j) = \lambda_k^+(A)$  and  $\lim_{j\to\infty} \lambda_k^-(A_j) = \lambda_k^-(A)$  hold, then we are done by the monotone convergence theorem.

Fix  $\varepsilon > 0$  and  $k \ge 1$ , and let  $W \subset \mathcal{H}$  be a k dimensional subspace such that

$$\min_{\substack{v \in W, \\ |v||=1}} \langle Av, v \rangle \ge \lambda_k^+(A) - \varepsilon.$$

Since  $\bigcup_{j=1}^{\infty} \mathcal{H}_i$  is dense in  $\mathcal{H}$ , we have that  $\lim_{j\to\infty} P_{\mathcal{H}_j} = I$  strongly, and so

$$\lim_{j \to \infty} \min_{\substack{v \in W, \\ \|v\|=1}} \langle A_j v, v \rangle = \min_{\substack{v \in W, \\ \|v\|=1}} \langle Av, v \rangle \ge \lambda_k^+(A) - \varepsilon,$$

implying that  $\liminf_j \lambda_k^+(A_j) \geq \lambda_k^+(A) - \varepsilon$ . As  $\lambda_k^+(A_j) \leq \lambda_k^+(A)$  for all j, and  $\varepsilon > 0$  was arbitrary, we obtain  $\lim_{j\to\infty} \lambda_k^+(A_j) = \lambda_k^+(A)$  as desired. By symmetry the same holds for the negative eigenvalues, and we are done.  $\Box$ 

For the next theorem we need some preparation. Let  $\mathbf{A}$  be a bounded, self adjoint operator on  $L^2(\Omega, \nu)$ , where  $(\Omega, \nu)$  is a standard probability space. Assume that  $\mathcal{P}$  is a finite, measurable, non-degenerate partition of  $\Omega$ . Then we have that  $\mathbb{E}_{\mathcal{P}} \mathbf{A} \mathbb{E}_{\mathcal{P}}$  is an integral kernel operator representable by a bounded measurable step-function of the form  $W : \Omega^2 \to \mathbb{R}$ . In this context it makes sense to talk about subgraph densities of the form  $t(H, \mathbb{E}_{\mathcal{P}} \mathbf{A} \mathbb{E}_{\mathcal{P}}) := t(H, W)$ .

**Theorem 7.5** Assume that **A** is a bounded, self-adjoint operator on  $L^2(\Omega, \nu)$ where  $(\Omega, \nu)$  is a standard probability space. Assume that  $\|\mathbf{A}\|_{\ell}^{\bullet} < \infty$  for some  $\ell \in \mathbb{N}$ . Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence of  $\Omega$ . Then

$$\lim_{j \to \infty} t(C_{\ell}, \mathbb{E}_{\mathcal{P}_j} \mathbf{A} \mathbb{E}_{\mathcal{P}_j}) = \sum_{k=1}^{\infty} \lambda_k^+ (A)^{\ell} + \sum_{k=1}^{\infty} \lambda_k^- (A)^{\ell}$$

**Proof.** For  $j \in \mathbb{N}$ , let  $\mathcal{H}_j$  denote the finite dimensional space of  $\mathcal{P}_j$ measurable functions. It is clear that the sequence  $\{\mathcal{H}_j\}_{j=1}^{\infty}$  satisfies the conditions of Proposition 7.4. Note that the operator  $\mathbb{E}_{\mathcal{P}_j}$  is equal to  $P_{\mathcal{H}_j}$ . Since  $A_j = \mathbb{E}_{\mathcal{P}_j} A \mathbb{E}_{\mathcal{P}_j}$  is representable by a step function we have that

$$t(C_{\ell}, A_j) = \sum_{k=1}^{\infty} \lambda_k^+ (A_j)^{\ell} + \sum_{k=1}^{\infty} \lambda_k^- (A_j)^{\ell}.$$

Then Proposition 7.4 completes the proof.

Theorem 7.1 follows from these results: Theorem 7.5 implies that for every exhausting partition sequence  $(\mathcal{P}_i)$  we have  $t(C_k, \eta_{\mathcal{P}_i}) \to t(C_k, \eta) < \infty$ . This in particular implies (2, 2)-looseness, so Theorem 6.15 implies that  $\eta^{C_k}$ is partition approximable.

## 8 Open problems

**Problem 1** Find general conditions under which the measure  $\eta_{\mathcal{F}}$  produced by a tree decomposition  $\mathcal{F}$  of a graph G (not necessarily a star decomposition) is independent of the decomposition.

**Problem 2** Is every k-loose Markov space the k-limit of graphons?

**Problem 3** ((k, p)-profile) Let  $\mathcal{L}(\eta) \subseteq \mathbb{N}^2$  denote the set of pairs (k, p) for which  $\eta$  is (k, p)-loose. Theorem 6.13 expresses subgraph densities in  $\eta$  under appropriate conditions on its "(k, p)-profile"  $\mathcal{L}(\eta)$ . Some properties of the set  $\mathcal{L}(\eta)$  have been established above: it is symmetric in the two coordinates and it is monotone in the sense that if  $(k, p) \in \mathcal{L}(\eta)$  and  $k' \leq k, p' \leq p$ , then  $(k', p') \in \mathcal{L}(\eta)$ . It would be interesting to establish further properties. For example, for the *d*-dimensional orthogonality Markov space  $\eta_d$ , we have  $(k, p) \in \mathcal{L}(\eta_d)$  if and only if  $k + p \leq d$  (see Lemma 3 in [18]). How "wild" can the boundary of the set  $\mathcal{L}(\eta)$  be in general?

**Problem 4** Is every 1-regular  $L^p$ -graphon (p + 1, p)-loose? Perhaps (k, p)-loose for every k? (This is false without the assumption that the graphon is 1-regular, as shown by a construction similar to Example 5.9. We are grateful to the anonymous referee for this remark.)

**Problem 5** Does  $t(G, \eta) = t_{\text{part}}(G, \eta)$  hold for every bipartite graph G and every Markov space  $\eta$ ? Could this be true at least for all graphons?

**Problem 6 (Measure family and partition approximation)** Let  $(J, \mathcal{B}, \eta)$  be a Markov space, let G be a graph, and let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence. Assume that there is a normalized Markovian measure family on the induced subgraphs of G. Does this imply that  $\eta_{\mathcal{P}_n}^G \to \eta^G$  on boxes? This is true if  $G = K_2$ , but even this very special case is not absolutely trivial.

**Problem 7** Theorems 1.2 and 6.15 suggest that a theorem along the following lines should hold: Let  $(J, \mathcal{B}, \eta)$  be a k-loose Markov space, let G be a graph with girth at least 5 and degrees at most k, and let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence. Then  $\eta_{\mathcal{P}_n}^G \to \eta^G$  on boxes.

**Problem 8** Are the definitions of subgraph densities based on approximations and based on various sequential tree decompositions equivalent, under reasonably general conditions?

**Problem 9 (Measures for graphings)** For a graphing **H** and a connected graph G, a measure on homomorphisms  $G \to \mathbf{H}$  can be defined in a natural way: We label a node u of G to get a rooted graph  $G_u$ . For each  $x \in J$ , let  $\psi_{G,x}$  be the counting measure on homomorphisms mapping u onto x (this is a finite set of bounded size for a fixed G). Then  $\Psi_G = (\psi_{G,x} : x \in J)$  is a measurable family, and we can define  $\eta^G = \pi[\Psi_G]$ . It can be shown (using the Mass Transport Principle for graphings) that this measure is independent of the choice of the root. Is there a common generalization with our results?

#### Problem 10 (Compactness, and cycles versus other graphs)

Assume that  $t(C_{2k}, \eta) = \infty$  for even cycles  $C_{2k}$ . Is  $t(G, \eta) = \infty$  for every connected bipartite graph G that is not a tree? If this implication is true, then in particular whenever  $t(G, \eta)$  is finite for at least one connected bipartite graph G besides trees, the operator  $\mathbf{A}_{\eta}$  is of some Schatten-class, and hence compact. A weaker question is therefore whether this compactness is a necessary condition in any well-defined sense for the finiteness of at least one density.

**Problem 11 (Regularity and variance)** In [7], a weak regularity partition of a graphon was constructed as a finite, measurable, non-degenerate partition  $\mathcal{P}$  into a given number of classes for which  $||W_{\mathcal{P}}||_2^2$  is (nearly) maximized. Do partitions  $\mathcal{P}$  for which  $||\eta_{\mathcal{P}}||_2^2$  is maximized have special properties and uses?

**Problem 12 (Regularity and spectral approximation)** It seems that the regularity lemma can be defined inside certain sparsity classes. Assume that we just consider measures such that  $t(C_{2k}, \eta) < c$  for some fixed constant. Then there are at most  $c/\varepsilon^{2k}$  eigenvalues greater than  $\varepsilon > 0$ . The corresponding spectral approximation of the operator  $\mathbf{A}_{\eta}$  (represented by some bounded measurable function) may serve as a regularization of  $\eta$ .

**Problem 13 (Quotient topology vs t)** In [17] we introduced a distance of s-graphons using quotients. How does it relate to subgraph densities? Is there some continuity in any direction, generalizing the Counting Lemma and/or the Inverse Counting Lemma for bounded graphons?

**Problem 14 (Edge coloring model approach)** It was observed and used in dense graph limit theory that spectral sums can be used to rewrite t(G, W) as the value of a certain edge coloring model. As an example, see the proof that forcible finite rank graphons are step functions in [22]. Nothing prevents us from pushing this further to more general compact operators  $\mathbf{A}_{\eta}$ .

**Problem 15 (Limit object)** Assume that for a graph sequence  $\{G_i\}_{i=1}^{\infty}$ , the numerical sequence  $t^*(F, G_i)$  is convergent for every graph F satisfying appropriate sparsity constraints. Is there a limit object in the form of an s-graphon?

**Problem 16 (Existence of limit)** Can it happen that  $\lim_{i\to\infty} t(G, \eta_{\mathcal{P}_i})$  is finite for certain exhausting partition sequences and infinite for other ones? Could it oscillate for a given partition sequence?

Acknowledgement. Our thanks are due to the anonymous referees of the first version of this paper for their very thorough and thoughtful comments, which has lead to the elimination of several errors, and to substantial improvement in the presentation.

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# 9 Appendices

### 9.1 Absolute continuity and Radon-Nikodym derivatives

We collect some measure theory facts that are probably known, but difficult to quote.

**Lemma 9.1** Let  $(J, \mathcal{B})$  be a standard Borel space, and  $\mu, \nu$  two measures on  $\mathcal{B}$  such that  $\nu \ll \mu$  and  $\mu$  is sigma-finite. Then the Radon-Nikodym derivative  $d\nu/d\mu$ :  $J \rightarrow [0, \infty]$  exists, and it is uniquely determined  $\mu$ -almost everywhere.

**Proof.** To prove the existence, we can split J into a countable number of Borel sets with finite  $\mu$ -measure, and apply the lemma to each of these. In other words, we may assume that  $\mu(J)$  is finite.

We claim that there is a set  $U \in \mathcal{B}$  such that  $\nu|_U$  sigma-finite and  $\nu(X) = \infty$  for every  $X \subseteq J \setminus U$  with  $\mu(X) > 0$ . Let  $c = \sup\{\mu(X) : X \in \mathcal{B}, \nu|_X$  sigma-finite}. Let  $Y_n \in \mathcal{B}$  be chosen so that  $\nu|_{Y_n}$  is sigma-finite and  $\mu(Y_n) > c - 1/n$ . Then  $U = \bigcup_n Y_n$  has the properties as desired. Clearly  $\nu|_U$  is sigma-finite, and  $\mu(U) \ge c$ . By the maximality of c, we have  $\mu(U) = c$ , and every set  $X \subseteq J \setminus U$  with  $\nu(X) < \infty$  must have  $\mu(X) = 0$ .

The standard Radon-Nikodym theorem, applied to  $\mu|_U$  and  $\nu|_U$ , gives  $f|_U$ . Defining f as constant  $\infty$  on  $X \setminus M$ , we obtain a measurable  $f : J \to [0, \infty]$  such that  $\nu = f \cdot \mu$ .

Uniqueness of f follows by standard arguments.

**Lemma 9.2** Let  $(I, \mathcal{A})$  and  $(J, \mathcal{B})$  be Borel spaces. Let  $\Phi = (\mu_x : x \in I)$  be a measurable family of measures on  $(J, \mathcal{B})$  and  $\alpha_1, \alpha_2 \in \mathfrak{M}(\mathcal{A})$ . If  $\alpha_1 \ll \alpha_2$ then  $\alpha_1[\Phi] \ll \alpha_2[\Phi]$ .

**Proof.** Suppose that  $\alpha_2[\Phi](R) = 0$  for some  $R \in \mathcal{A} \times \mathcal{B}$ . Let  $R(x) = \{y \in J : (x, y) \in R\}$ . Then

$$\alpha_2[\Phi](R) = \int_I \mu_x(R(x)) \, d\alpha_2(x) = 0$$

implies that  $\alpha_2 \{x \in I : \mu_x(R(x)) > 0\} = 0$ . But then  $\alpha_1 \{x \in I : \mu_x(R(x)) > 0\} = 0$ , implying by the same computation that  $\alpha_1[\Phi](R) = 0$ .

**Lemma 9.3** Let  $\sigma$  be a probability distribution on  $\mathcal{B}^V$ . Suppose that  $\sigma \ll \prod_{v \in V} \sigma^{\{v\}}$ . Then  $\sigma \ll \sigma^S \times \sigma^{V \setminus S}$  for every  $S \subseteq V$ .

**Proof.** Let  $\xi = \prod_{v \in V} \sigma^{\{v\}}$  and  $f = d\sigma/d\xi$ . For any  $S \subseteq V$ , the function

$$f^{S}(y) = \int_{J^{V \setminus S}} f(y, z) \, d\xi^{V \setminus S}(z) \qquad (y \in J^{S}).$$

satisfies  $\sigma^S = f^S \cdot \xi^S$ . Let  $U = \{y : f^S(y) = 0\}$  and  $Z = \{z : f^{V \setminus S}(z) = 0\}$ . Suppose that  $(\sigma^S \times \sigma^{V \setminus S})(X) = 0$ . Then

$$\left(\sigma^{S} \times \sigma^{V \setminus S}\right)(X) = \int_{X} f^{S}(y) f^{V \setminus S}(z) d\xi(y, z)$$

implies that  $X \subseteq (U \times J^{V \setminus S}) \cup (J^S \times Z) \xi$ -almost everywhere. Hence

$$\begin{aligned} \sigma(X) &\leq \int_{U \times J^{V \setminus S}} f(x) \, d\xi(x) + \int_{J^S \times Z} f(x) \, d\xi(x) \\ &= \int_{U} f^S(x) \, d\xi^S(x) + \int_{Z} f^{V \setminus S}(x) \, d\xi^{V \setminus S}(x) = 0. \end{aligned}$$

#### 9.2 Markovian property and Markov random fields

We show that Markovian measure families and Markov random fields on a graph G = (V, E) are related. This latter can be defined as a probability distribution  $\mu$  on  $\mathcal{B}^V$  such that the marginal family ( $\mu^S : S \subseteq V$ ) satisfies the Markovian property for sets  $U, W \subseteq V$  such that  $U \cup W = V$ . More precisely, let  $R_{S,T} = (\rho_{S,T,z} : z \in J^S)$  be a measurable family of measures on  $\mathcal{B}^{T\setminus S}$  such that  $\mu^S[R_{S,T}] = \mu^T$ . Then we require that whenever  $U \cup W = V$ ,  $S = U \cap W$ , and there is no edge between  $U \setminus S$  and  $W \setminus S$ , then

$$\rho_{S,V,x} = \rho_{S,U,x} \times \rho_{S,W,x} \tag{72}$$

for  $\mu^S$ -almost all  $x \in J^S$ .

**Proposition 9.4** If a family  $\mathcal{M} = (\mu_S : S \subseteq V)$  of sigma-finite measures is Markovian with respect to a graph G, and  $\mu = \mu_V$  is a probability distribution, then  $\mu$  is a Markov random field on G.

**Proof.** Recall that  $\nu_{S,T,x}$  is the disintegration of  $\mu_T$  with respect to  $\mu_S$ , and  $\rho_{S,T,x}$  is the disintegration of  $\mu^T$  with respect to  $\mu^S$ . Our first step is to express  $\pi_J$  in terms of  $\nu$ . Let  $S \subseteq T \subseteq V$ . We claim that for all  $B \in \mathcal{B}^{T \setminus S}$ and  $\mu^S$ -almost all  $x \in J^S$ ,

$$\rho_{S,T,x}(B) = \int_{B} \frac{\nu_{T,V,xy}(J^{V\setminus T})}{\nu_{S,V,x}(J^{V\setminus S})} \, d\nu_{S,T,x}(y).$$
(73)

First note that by (7), we have

$$\mu^{S}\left(\left\{x: \nu_{S,V,x}(J^{V\setminus S}) = 0\right\}\right) = \int_{\left\{x: \nu_{S,V,x}(J^{V\setminus S}) = 0\right\}} \nu_{S,V,x}(J^{V\setminus S}) \, d\mu_{S}(x) = 0,$$

hence the right hand side is well-defined for  $\mu^S$ -almost all  $x \in J^S$ . To prove (73), we integrate both sides on  $A \in \mathcal{B}^S$  with respect to  $\mu^S$ . The left hand side turns into

$$\int_{A} \rho_{S,T,x}(B) \, d\mu^{S}(x) = \mu^{T}(A \times B),$$

whereas, using (7), the right hand side becomes

$$\int_{A} \int_{B} \frac{\nu_{T,V,xy}(J^{V\setminus T})}{\nu_{S,V,x}(J^{V\setminus S})} d\nu_{S,T,x}(y) d\mu^{S}(x)$$

$$= \int_{A} \int_{B} \frac{1}{B} \nu_{T,V,xy}(J^{V\setminus T}) d\nu_{S,T,x}(y) d\mu_{S}(x)$$

$$= \int_{A\times B} \nu_{T,V,xy}(J^{V\setminus T}) d\mu_{T}(xy) = \int_{A\times B} 1 d\mu^{T}(xy) = \mu^{T}(A \times B).$$

This proves (73). Hence for  $B \in \mathcal{B}^{U \setminus S}$  and  $C \in \mathcal{B}^{W \setminus S}$ ,

$$\rho_{S,V,x}(B \times C) = \int_{B \times C} \frac{1}{\nu_{S,V,x}(J^{V \setminus S})} d\nu_{S,V,x}(y) = \frac{1}{\nu_{S,V,x}(J^{V \setminus S})} \nu_{S,V,x}(B \times C).$$

Using Lemma 2.4,

$$\rho_{S,U,x}(B) = \int_{B} \frac{\nu_{U,V,xy}(J^{V\setminus U})}{\nu_{S,V,x}(J^{V\setminus S})} d\nu_{S,U,x}(y) = \int_{B} \frac{\nu_{S,W,x}(J^{V\setminus U})}{\nu_{S,V,x}(J^{V\setminus S})} d\nu_{S,U,x}(y)$$
$$= \frac{\nu_{S,W,x}(J^{W\setminus S})}{\nu_{S,V,x}(J^{V\setminus S})} \nu_{S,U,x}(B).$$

Using a similar expression for  $\rho_{x,W}(C)$ , we get

$$(\rho_{S,U,x} \times \rho_{S,W,x})(B \times C) = \frac{\nu_{S,W,x}(J^{W \setminus S})\nu_{S,U,x}(J^{U \setminus S})}{\nu_{S,V,x}(J^{V \setminus S})^2}\nu_{S,U,x}(B)\nu_{S,W,x}(C)$$
$$= \frac{1}{\nu_{S,V,x}(J^{V \setminus S})}\nu_{S,V,x}(B \times C) = \rho_{S,V,x}(B \times C).$$

This proves Proposition 9.4.

### 9.3 Partition sequences

We prove the following basic facts about exhausting partition sequences.

**Lemma 9.5** Let  $(J, \mathcal{B}, \pi)$  be a standard Borel probability space, and let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be a partition sequence, with  $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{P}_i$ . Then the following are equivalent:

(i)  $\mathcal{R}$  is exhausting with respect to  $\pi$ , i.e., for every  $X \in \mathcal{B}$  there is a set  $Y \in \overline{\mathcal{R}}$  such that  $\pi(X \triangle Y) = 0$ .

(ii) For every  $X \in \mathcal{B}$  and every  $\varepsilon > 0$  there is a set  $Y \in \widehat{\mathcal{R}}$  such that  $\pi(X \triangle Y) < \varepsilon$ .

(iii) There is a generating partition sequence  $(\mathcal{Q}_i)_{i=1}^{\infty}$  and a Borel set U with  $\pi(U) = 0$  such that  $\mathcal{P}_i|_{J\setminus U} = \mathcal{Q}_i|_{J\setminus U}$  for all *i*.

(iv)  $\bigcup_{i=1}^{\infty} L^1(J, \overline{\mathcal{P}}_i, \pi)$  is dense in  $L^1(J, \mathcal{B}, \pi)$ .

**Proof.** (i) $\Rightarrow$ (ii): Let  $\mathcal{S}$  be the family of sets  $X \in \mathcal{B}$  for which for every  $\varepsilon > 0$  there is a set  $Y \in \widehat{\mathcal{R}}$  such that  $\pi(X \triangle Y) < \varepsilon$ . Then  $\mathcal{S}$  is closed under complementation (trivially), and under finite union and finite intersection (almost trivially). It follows that it is closed under countable union. Indeed, let  $\varepsilon > 0$ ,  $X = X_1 \cup X_2 \cup \cdots$ , where  $X_i \in \mathcal{S}$ , and  $X'_i = X_i \setminus (X_1 \cup \cdots \cup X_{i-1})$ . Since the  $X'_i$  are disjoint, we have  $\sum_{i=N+1}^{\infty} \pi(X'_i) < \varepsilon/2$  for an appropriate N. Since  $X'_i \in \mathcal{S}$ , there are  $Y_i \in \widehat{\mathcal{R}}$  such that  $\pi(X'_i \triangle Y_i) < \varepsilon/(2N)$ . Let  $Y = \bigcup_{i=1}^N Y_i \in \widehat{\mathcal{R}}$ , then

$$\pi(X \triangle Y) \le \sum_{i=1}^{N} \pi(X'_i \triangle Y_i) + \sum_{i=N+1}^{\infty} \pi(X'_i) < \varepsilon.$$

So  $\mathcal{S}$  is a sigma-algebra. Trivially  $\mathcal{R} \subseteq \mathcal{S}$ , so  $\overline{\mathcal{R}} \subseteq \mathcal{S}$ . By (i), for every  $S \in \mathcal{B}$  there is a set  $Z \in \overline{\mathcal{R}}$  such that  $\pi(Z \triangle S) = 0$ , and then  $Z \in \mathcal{S}$  implies that there is a set  $Y \in \widehat{\mathcal{R}}$  for which  $\pi(Z \triangle Y) < \varepsilon$ . Then  $\pi(Y \triangle X) < \varepsilon$ .

(ii) $\Rightarrow$ (i): Let  $X \in \mathcal{B}$ , and for  $k \geq 1$ , let  $Y_k \in \widehat{\mathcal{R}}$  be a set such that  $\pi(X \triangle Y_k) < 2^{-k}$ . Consider the sets

$$Z_n = \bigcap_{k=n}^{\infty} Y_k$$
, and  $Z = \bigcup_{n=1}^{\infty} Z_n$ 

Trivially  $Z \in \overline{\mathcal{R}}$ . Furthermore,

$$\pi(Z_n \setminus X) \le \liminf_k \pi(Y_k \setminus X) \le \liminf_k \pi(Y_k \triangle X) = 0,$$

and

$$\pi(X \setminus Z_n) \le \sum_{k=n}^{\infty} \pi(X \setminus Y_k) \le \sum_{k=n}^{\infty} \pi(X \triangle Y_k) < 2^{1-n}.$$

Using this, a similar computation gives that  $\pi(X \triangle Z) = 0$ .

(i) $\Rightarrow$ (iii): Let  $B_1, B_2, \ldots$  be a countable generating set of  $\mathcal{B}$ . For each i, there is a set  $C_i \in \overline{\mathcal{R}}$  such that  $\pi(B_i \triangle C_i) = 0$ . Let  $U = \bigcup_{i=1}^{\infty} B_i \triangle C_i$ , then  $\pi(U) = 0$ . Let  $(\mathcal{Q}'_i)_{i=1}^{\infty}$  be a generating partition sequence of Borel subsets of U, and let  $\mathcal{Q}_i = \mathcal{P}_i|_{J \setminus U} \cup \mathcal{Q}'_i$ . Then  $(\mathcal{Q}_i)_{i=1}^{\infty}$  is a generating partition sequence in  $(J, \mathcal{B})$  such that  $\mathcal{P}_i|_{J \setminus U} = \mathcal{Q}_i|_{J \setminus U}$  for all i.

(iii) $\Rightarrow$ (i): Let  $(\mathcal{Q}_i)_{i=1}^{\infty}$  be a generating sequence of partitions and U, a Borel set with  $\pi(U) = 0$  such that  $\mathcal{P}_i|_{J\setminus U} = \mathcal{Q}_i|_{J\setminus U}$  for all i. Then  $\mathcal{P}_i|_{J\setminus U}$ is a generating partition sequence for the Borel sets in  $J \setminus U$ , and hence for every  $C \in \mathcal{A}$  there is a  $D \in \overline{\mathcal{R}}|_{J\setminus U}$  for which  $C \setminus U = D$ . Then  $D = D_1 \setminus U$ for some  $D_1 \in \overline{\mathcal{R}}$ , and  $\pi(C \triangle D_1) \leq \pi(U) = 0$ .

 $\{(i),(ii),(iii)\} \Rightarrow (iv)$ : By (iii), we may assume that  $\mathcal{R}$  is generating. It suffices to prove that every function  $\mathbb{1}_S (S \in \mathcal{B})$  can be approximated arbitrarily well by finite linear combinations of functions  $\mathbb{1}_A (A \in \mathcal{P}_i)$ , since the functions  $\mathbb{1}_S$  are dense in  $L^1(J, \mathcal{B}, \pi)$ , and  $\mathbb{1}_A \in L^1(J, \mathcal{P}_i, \pi)$ . This follows by (ii).

(iv) $\Rightarrow$ (ii): For every  $S \in \mathcal{B}$  and  $\varepsilon > 0$  there are sets  $A_1, \ldots, A_k \in \mathcal{R}$  and nonzero real numbers  $\alpha_1, \ldots, \alpha_k$  such that

$$\left\|\mathbb{1}_{S}-\alpha_{1}\mathbb{1}_{A_{1}}-\cdots-\alpha_{k}\mathbb{1}_{A_{k}}\right\|_{1}<\varepsilon.$$

Let *i* be the least integer for which  $A_1, \ldots, A_k \in \widehat{\mathcal{P}}_i$ . By splitting an  $A_j$  into partition classes in  $\mathcal{P}_i$  (and adjusting the coefficients as necessary), we may assume that every  $A_j \in P_i$ . Then the  $A_j$  are disjoint. Replacing  $\alpha_j$  by 1 if  $\pi(A_j \cap S) \ge \pi(A_j)/2$ , and by 0 otherwise, we decrease the left hand side. Deleting zero terms, we may assume that every  $\alpha_j = 1$ , and then  $Y = \bigcup_j A_j$ satisfies  $\pi(S \triangle Y) < \varepsilon$ . **Lemma 9.6** Let  $(J, \mathcal{B}, \pi)$  be a Borel probability space and assume that  $\mu$  is a measure on  $J^k$  for some  $k \in \mathbb{N}$  such that its marginal distribution in each coordinate is  $\pi$ . Let  $(\mathcal{P}_i)_{i=1}^{\infty}$  be an exhausting partition sequence with respect to  $\pi$ . Then the partition sequence  $(\mathcal{P}_i^k)_{i=1}^{\infty}$  is exhausting to both  $\pi^k$  and  $\mu$ .

**Proof.** Replacing "exhausting" by "generating", the assertion is easy. For exhausting partition sequences, it follows by Lemma 9.5(iii).

#### 9.4 Unbounded graphons and non-acyclic graphs

We give the details of the arguments for Example 5.10. Recall that  $f: I = [-1, 1] \to \mathbb{R}$  has the following properties:  $f \ge 0$ ; f(-x) = f(x) for all  $x \in I$ ;  $\int_I f(x) dx = 1$ ; f is convex and monotone decreasing for x > 0. This function defines a graphon by

$$W(x,y) = f(x-y) \qquad (x, \in I),$$

where f is extended periodically modulo 2. Clearly W is symmetric and 1-regular. The stationary measure  $\mu$  of the graphon is  $\mu = \lambda/2$ . We claim that as a kernel operator, it is positive semidefinite and compact as  $L^2(\mu) \rightarrow L^2(\mu)$ .

The eigenfunctions of W are  $\sin(k\pi x)$  and  $\cos(k\pi x)$ , and hence the eigenvalues can be obtained as the Fourier coefficients of f(x). By the symmetry of f, eigenvalues associated with the eigenfunction  $\sin(k\pi x)$  are zero. The other eigenvalues can be expressed for even  $k \ge 0$  as

$$\lambda_{k} = \int_{-1}^{1} f(x) \cos(k\pi x) \, dx = 2 \int_{0}^{1} f(x) \cos(k\pi x) \, dx$$
$$= \frac{2}{k} \int_{0}^{k} f\left(\frac{y}{k}\right) \cos(\pi y) \, dy = \frac{2}{k} \sum_{j=0}^{k/2-1} \int_{0}^{2} f\left(\frac{y+2j}{k}\right) \cos(\pi y) \, dy.$$
(74)

To see that this is nonnegative, notice that  $\cos(\pi y) = -\cos(\pi(1-y)) = -\cos(\pi(1+y)) = \cos(\pi(2-y))$ , and so we can write (74) as

$$\lambda_{k} = \frac{2}{k} \sum_{j=0}^{k/2-1} \int_{0}^{1/2} \left[ f\left(\frac{2j+y}{k}\right) - f\left(\frac{2j+1-y}{k}\right) - f\left(\frac{2j+1+y}{k}\right) \right] + f\left(\frac{2j+2-y}{k}\right) \cos(\pi y) \, dy.$$
(75)

Here each integrand is nonnegative by the convexity of f. For odd k, we get an extra term

$$\frac{2}{k}\int_{k-1}^{k}f\left(\frac{y}{k}\right)\cos(\pi y)\,dy \ge \frac{2}{k}\int_{k-1}^{k}f\left(\frac{y}{k}\right)dy\int_{k-1}^{k}\cos(\pi y)\,dy = 0,$$

where we used Chebyshev's sum inequality on the monotone decreasing functions f(y/k) and  $\cos(\pi y)$ . This proves that W is positive semidefinite.

By the Riemann–Lebesgue Lemma,  $\lambda_k \to 0$ . This implies that W defines a compact operator  $L^2(\mu) \to L^2(\mu)$ .

As a useful special case, we consider the function defined by

$$f(x) = \frac{1}{x(2 - \ln(x))^2} = \left(\frac{1}{2 - \ln(x)}\right)'$$

for x > 0, and f(x) = f(-x) for x < 0. (For x = 0 we can define f(x) = 0.) We have

$$\int_{-1}^{1} f(x) \, dx = 2 \left[ \frac{1}{2 - \ln(x)} \right]_{0}^{1} = 1.$$

The conditions that f is monotone decreasing and convex for x > 0 are easy to check. To determine the order of magnitude of  $\lambda_k$ , note that the first term in (75) is

$$a_k = \frac{1}{k} \int_{0}^{1/2} \left( f\left(\frac{y}{k}\right) - f\left(\frac{1-y}{k}\right) - f\left(\frac{1+y}{k}\right) + f\left(\frac{2-y}{k}\right) \right) \cos(\pi y) \, dy$$
$$\geq \frac{1}{k} \int_{0}^{1/4} \left( f\left(\frac{y}{k}\right) - f\left(\frac{1-y}{k}\right) - f\left(\frac{1+y}{k}\right) + f\left(\frac{2-y}{k}\right) \right) \cos(\pi y) \, dy$$

Using the inequality  $f(3x) \leq f(x)/2$  and  $f(5x) \leq f(x)/4$  valid for x < 1/(4k) if k is large enough, we can estimate the expression in the large parenthesis as

$$f\left(\frac{y}{k}\right) - f\left(\frac{1-y}{k}\right) - f\left(\frac{1+y}{k}\right) + f\left(\frac{2-y}{k}\right)$$
$$\geq \left(1 - \frac{1}{2} - \frac{1}{4}\right) f\left(\frac{y}{k}\right) > \frac{1}{4} f\left(\frac{y}{k}\right).$$

Hence

$$\lambda_k \ge a_k \ge \frac{1}{4k} \int_0^{1/4} f\left(\frac{y}{k}\right) \cos\frac{\pi}{4} \, dy = \frac{1}{4\sqrt{2}} \int_0^{1/(4k)} f(x) \, dx = \frac{1}{4\sqrt{2}(2+\ln(4k))}.$$

It follows that no operator power of W has finite trace, so  $t(C_n, W) = \infty$  for all n. Since f is bounded away from 0, it follows that  $t(G, W) = \infty$  for every graph G containing a cycle.