

INDUCTIVE TOPOLOGICAL HAUSDORFF DIMENSIONS AND FIBERS OF GENERIC CONTINUOUS FUNCTIONS

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ABSTRACT. In an earlier paper Buczolich, Elekes and the author introduced a new concept of dimension for metric spaces, the so called topological Hausdorff dimension. They proved that it is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic real-valued continuous function (in the sense of Baire category) defined on a compact metric space K .

The goal of this paper is to determine the Hausdorff dimension of the fibers of the generic continuous function from K to \mathbb{R}^n . In order to do so, we define the n th inductive topological Hausdorff dimension, $\dim_{t^n_H} K$. Let $\dim_H K$, $\dim_t K$ and $C_n(K)$ denote the Hausdorff and topological dimension of K and the Banach space of the continuous functions from K to \mathbb{R}^n . We show that $\sup_{y \in \mathbb{R}^n} \dim_H f^{-1}(y) = \dim_{t^n_H} K - n$ for the generic $f \in C_n(K)$, provided that $\dim_t K \geq n$, otherwise every fiber is finite.

In order to prove the above theorem we give some equivalent definitions for the inductive topological Hausdorff dimensions, which can be interesting in their own right. Here we use techniques coming from the theory of topological dimension.

We show that the supremum is actually attained on the left hand side of the above equation.

We characterize those compact metric spaces K for which $\dim_H f^{-1}(y) = \dim_{t^n_H} K - n$ for the generic $f \in C_n(K)$ and the generic $y \in f(K)$. We also generalize a result of Kirchheim by showing that if K is self-similar and $\dim_t K \geq n$ then $\dim_H f^{-1}(y) = \dim_{t^n_H} K - n$ for the generic $f \in C_n(K)$ for every $y \in \text{int } f(K)$.

1. INTRODUCTION

The Hausdorff dimension of a metric space X is denoted by $\dim_H X$, see e.g. [6] or [11]. In this paper we adopt the convention that $\dim_H \emptyset = -1$.

The following theorem is due to Kirchheim [9].

Theorem 1.1 (Kirchheim). *Let $m, n \in \mathbb{N}^+$, $m \geq n$. For the generic continuous function $f: [0, 1]^m \rightarrow \mathbb{R}^n$ (in the sense of Baire category) for all $y \in \text{int } f([0, 1]^m)$*

$$\dim_H f^{-1}(y) = m - n.$$

Buczolich, Elekes and the author introduced in [1] a new dimension for metric spaces, the topological Hausdorff dimension. The main motivation behind this concept was to generalize Kirchheim's theorem for real-valued functions defined

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on arbitrary compact metric spaces. We recall first the definition of the (small inductive) topological dimension.

Definition 1.2. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d - 1 \text{ for every } U \in \mathcal{U}\}.$$

For more information on this concept see [4] or [7]. The topological Hausdorff dimension (introduced in [1]) is defined analogously to the topological dimension. However, it is not inductive, and it can attain non-integer values as well.

Definition 1.3. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d - 1 \text{ for every } U \in \mathcal{U}\}.$$

(Both notions of dimension can attain the value ∞ as well, actually we use the convention $\infty - 1 = \infty$, hence $d = \infty$ is a member of the above set.)

For more information on topological Hausdorff dimension see [1], here we mention only the results concerning level sets of generic continuous functions.

Let K be a compact metric space, n be a positive integer and let $C_n(K)$ denote the space of continuous functions from K to \mathbb{R}^n equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments.

If $\dim_t K = 0$ then the generic $f \in C_1(K)$ is well-known to be one-to-one, so every non-empty level set is a singleton.

Assume $\dim_t K > 0$. The following theorem from [1] shows the connection between the topological Hausdorff dimension and the level sets of the generic $f \in C_1(K)$.

Theorem 1.4 (Balka, Buczolich, Elekes). *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C_1(K)$*

- (i) $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ for every $y \in \mathbb{R}$,
- (ii) for every $d < \dim_{tH} K$ there exists a non-degenerate interval $I_{f,d}$ such that $\dim_H f^{-1}(y) \geq d - 1$ for every $y \in I_{f,d}$.

Corollary 1.5. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C_1(K)$*

$$\sup_{y \in \mathbb{R}} \dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

The following definition is due to Darji and Elekes [3].

Definition 1.6. Let $\dim_{t^n H} \emptyset = -1$ for all $n \in \mathbb{N}$. For a non-empty metric space X set $\dim_{t^0 H} X = \dim_H X$. The *n th inductive topological Hausdorff dimension* is defined inductively as

$$\dim_{t^n H} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ s. t. } \dim_{t^{n-1} H} \partial U \leq d - 1 \text{ for every } U \in \mathcal{U}\}.$$

The main goal of this paper is to generalize Theorem 1.4 to higher dimensions, which can be viewed as an application of the inductive topological Hausdorff dimensions.

If $\dim_t K < n$ then the fibers of the generic map $f \in C_n(K)$ are finite, see Theorem 6.1 below.

Suppose $\dim_t K \geq n$. The main theorem of the paper is the following.

Theorem 6.12 (Main Theorem, simplified version). *Let $n \in \mathbb{N}^+$ and assume that K is a compact metric space with $\dim_t K \geq n$. Then for the generic $f \in C_n(K)$*

- (i) $\dim_H f^{-1}(y) \leq \dim_{t^n_H} K - n$ for all $y \in \mathbb{R}^n$,
- (ii) for every $d < \dim_{t^n_H} K$ there exists a non-empty open ball $U_{f,d} \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \geq d - n$ for every $y \in U_{f,d}$.

Corollary 6.13. *If K is a compact metric space with $\dim_t K \geq n$ then for the generic $f \in C_n(K)$*

$$\sup_{y \in \mathbb{R}^n} \dim_H f^{-1}(y) = \dim_{t^n_H} K - n.$$

If K is also sufficiently homogeneous, for example self-similar, then we can actually say more.

Theorem 7.9 (simplified version). *Let K be a self-similar compact metric space such that $\dim_t K \geq n$. Then for the generic $f \in C_n(K)$ for any $y \in \text{int } f(K)$*

$$\dim_H f^{-1}(y) = \dim_{t^n_H} K - n.$$

In the Preliminaries section we introduce some notation and definitions.

In Section 3 we prove some basic properties of inductive topological Hausdorff dimensions.

In order to prove our Main Theorem in Section 4 we give some equivalent definitions for the inductive topological Hausdorff dimensions. These equivalent definitions are more or less analogous to the corresponding equivalent definitions of the topological dimension. Throughout the section we apply the standard techniques of the theory of topological dimension. Perhaps these results can be interesting in their own right, this is the reason why we work in separable metric spaces instead of compact ones.

In Section 5 we completely describe the possible values of the inductive topological Hausdorff dimensions based on ideas from [1]. This is a supplement for the theory of the dimensions, we will not use this result in the subsequent sections.

In Section 6 we consider two more equivalent definitions for the inductive topological Hausdorff dimensions in compact metric spaces, and we prove the Main Theorem based on Section 4.

In Section 7 we make the Main Theorem more precise. We show that in Corollary 6.13 the supremum is attained. We generalize Kirchheim's theorem for sufficiently homogeneous compact spaces. The proofs of this section rely heavily on the methods developed in [2], where the case of real-valued functions is investigated.

2. PRELIMINARIES

Let (X, d) be a metric space, and let $A, B \subseteq X$ be arbitrary sets. We denote by $\text{cl}(A)$, $\text{int } A$ and ∂A the closure, interior and boundary of A , respectively. The diameter of A is denoted by $\text{diam } A$. We use the convention $\text{diam } \emptyset = 0$. The distance of the sets A and B is defined by $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Let $B(x, r) = \{y \in X : d(x, y) \leq r\}$, $U(x, r) = \{y \in X : d(x, y) < r\}$ and $B(A, r) = \{x \in X : \text{dist}(B, \{x\}) \leq r\}$.

For two metric spaces (X, d_X) and (Y, d_Y) a function $f: X \rightarrow Y$ is *Lipschitz* if there exists a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. A function $f: X \rightarrow Y$ is called *bi-Lipschitz* if f is a bijection and both f and f^{-1} are Lipschitz.

For every $s \geq 0$ the s -Hausdorff content of a metric space X is defined as

$$\mathcal{H}_\infty^s(X) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^s : X \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$

Then the Hausdorff dimension of X is

$$\dim_H X = \inf \{s \geq 0 : \mathcal{H}_\infty^s(X) = 0\}.$$

We adopt the convention that $\dim_H \emptyset = -1$ throughout the paper. It is not difficult to see using the regularity of \mathcal{H}_∞^s that every set is contained in a G_δ set of the same Hausdorff dimension. For more information on these concepts see [6] or [11]. The following facts are easy consequences of the definitions.

Fact 2.1. *If $\mathcal{H}_\infty^s(X) \leq 1$ then $\mathcal{H}_\infty^t(X) \leq \mathcal{H}_\infty^s(X)$ for all $t \geq s$.*

Fact 2.2. *For a metric space X and $s \geq 0$ the following statements are equivalent:*

- (i) $\dim_H X \leq s$;
- (ii) $\mathcal{H}_\infty^{s+\varepsilon}(X) \leq \varepsilon$ for all $\varepsilon > 0$;
- (iii) $\mathcal{H}_\infty^{s+1/i}(X) \leq 1/i$ for all $i \in \mathbb{N}^+$.

Let X be a complete metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is of *second category* if it is not meager. A set is called *co-meager* if its complement is meager. By the Baire Category Theorem a set is co-meager iff it contains a dense G_δ set. We say that the *generic* element $x \in X$ has property \mathcal{P} if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The set $A \subseteq X$ has the *Baire property* if $A = U \Delta M$ where U is open and M is meager. If a set is of second category in every non-empty open set and has the Baire property then it is co-meager.

If X is a metric space and A, B are disjoint subsets of X then we say that $L \subseteq X$ is a *partition between A and B* if there are open sets U, V such that $A \subseteq U, B \subseteq V, U \cap V = \emptyset$ and $L = X \setminus (U \cup V)$. The following lemma is [4, 1.2.11. Lemma].

Lemma 2.3. *Let X be a metric space and let $Z \subseteq X$ be separable with $\dim_t Z = 0$. Then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $L \cap Z = \emptyset$.*

Let us recall the following decomposition theorem for the topological dimension, see [4, 1.5.7. Thm.] and [4, 1.5.8. Thm.].

Theorem 2.4. *For a separable metric space X and $n \in \mathbb{N}$ the following statements are equivalent:*

- (i) $\dim_t X \leq n$;
- (ii) $X = Y \cup Z$ such that $\dim_t Y \leq n - 1$ and $\dim_t Z \leq 0$;
- (iii) $X = Z_1 \cup \dots \cup Z_{n+1}$ such that $\dim_t Z_i \leq 0$ for all $i \in \{1, \dots, n + 1\}$.

Let X be a metric space and let \mathcal{A}, \mathcal{B} be families of subsets of X , where repeated copies of any given member are allowed. Let $\text{mesh } \mathcal{A} = \sup\{\text{diam } A : A \in \mathcal{A}\}$. We say that \mathcal{A} is a *cover* of X if $\bigcup \mathcal{A} = X$, and \mathcal{A} is *locally finite* if every $x \in X$ has a neighborhood that intersects only finitely many $A \in \mathcal{A}$. The family \mathcal{A} is *open* (*closed*) if every $A \in \mathcal{A}$ is open (closed) in X . We say that \mathcal{B} is a *refinement* of the cover \mathcal{A} if \mathcal{B} is a cover of X and for every $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $B \subseteq A$. The following theorem claims that every metric space is paracompact. It is due to Stone, see [5, 4.4.1. Thm.] for a proof.

Theorem 2.5 (Stone). *Every open cover of a metric space has a locally finite open refinement.*

We say that $\mathcal{B} = \{B_i\}_{i \in I}$ is a *shrinking* of the cover $\mathcal{A} = \{A_i\}_{i \in I}$ if \mathcal{B} is a cover of X and $B_i \subseteq A_i$ for all $i \in I$. The following theorem is [4, 1.7.8. Thm.].

Theorem 2.6. *Every finite open cover of a normal space has a closed shrinking.*

3. BASIC PROPERTIES OF INDUCTIVE TOPOLOGICAL HAUSDORFF DIMENSIONS

This section contains some basic properties of inductive topological Hausdorff dimensions that will be useful in the following sections.

Fact 3.1. *If X is a metric space and $n \in \mathbb{N}$ then $\dim_t X \leq \dim_{t^n_H} X$.*

Proof. If $n = 0$ then $\dim_t X \leq \dim_H X$ by [7, Thm. VII 2.]. Let $n \geq 1$ and assume by induction that the inequality holds for $n - 1$. Thus every basis \mathcal{U} of X satisfies $\dim_t \partial U \leq \dim_{t^{n-1}_H} \partial U$ for all $U \in \mathcal{U}$, so the definitions of topological dimension and n th inductive topological Hausdorff dimension imply $\dim_t X \leq \dim_{t^n_H} X$. \square

Fact 3.2. *If X is a metric space and $n \in \mathbb{N}$ then*

$$\dim_t X < n \implies \dim_{t^n_H} X = \dim_t X.$$

Proof. For $n = 0$ the statement is obvious. Let $n \geq 1$ and assume by induction that the statement holds for $n - 1$. Let X be a metric space such that $\dim_t X < n$. As $\dim_t X \leq \dim_{t^n_H} X$ by Fact 3.1, it is enough to show $\dim_{t^n_H} X \leq \dim_t X$. The definition of the topological dimension yields that X has a basis \mathcal{U} such that $\dim_t \partial U \leq \dim_t X - 1 < n - 1$ for all $U \in \mathcal{U}$. Then the inductive hypothesis implies that $\dim_{t^{n-1}_H} \partial U = \dim_t \partial U$ for all $U \in \mathcal{U}$, therefore

$$\dim_{t^n_H} X \leq \sup_{U \in \mathcal{U}} \dim_{t^{n-1}_H} \partial U + 1 = \sup_{U \in \mathcal{U}} \dim_t \partial U + 1 \leq \dim_t X.$$

This concludes the proof. \square

Now we compare the values of different dimensions, for the next theorem see [1].

Theorem 3.3. *For every metric space X*

$$\dim_t X \leq \dim_{t_H} X \leq \dim_H X.$$

Fact 3.4. *If X is a metric space and $n \in \mathbb{N}$ then $\dim_{t^{n+1}_H} X \leq \dim_{t^n_H} X$.*

Proof. If $n = 0$ then this follows from Theorem 3.3. Let $n \geq 1$ and assume by induction that $\dim_{t^n_H} Y \leq \dim_{t^{n-1}_H} Y$ for all metric spaces Y . Hence for every basis \mathcal{U} of X we have $\dim_{t^n_H} \partial U \leq \dim_{t^{n-1}_H} \partial U$ for all $U \in \mathcal{U}$. Thus $\dim_{t^{n+1}_H} X \leq \dim_{t^n_H} X$ easily follows from definition of inductive topological Hausdorff dimensions. \square

Theorem 3.5. *If X is a metric space and $n \in \mathbb{N}$ then*

$$\dim_t X \leq \dim_{t^n_H} X \leq \dim_H X.$$

Proof. If $n = 0$ or $n = 1$ then we are done by Theorem 3.3. Let $n > 1$ and assume by induction that the inequality holds for $n - 1$. Then Fact 3.1, Fact 3.4 and the inductive hypothesis imply $\dim_t X \leq \dim_{t^n_H} X \leq \dim_{t^{n-1}_H} X \leq \dim_H X$. \square

Corollary 3.6 (Extension of the classical dimension). *The n th inductive topological Hausdorff dimension of a countable set equals zero, and for open subspaces of \mathbb{R}^d and for smooth d -dimensional manifolds this dimension equals d .*

Theorem 3.7 (Monotonicity). *If $X \subseteq Y$ are metric spaces then $\dim_{t^n H} X \leq \dim_{t^n H} Y$ for all $n \in \mathbb{N}$.*

Proof. If $n = 0$ then we are done. Let $n \geq 1$ and assume by induction that monotonicity holds for the $(n - 1)$ st inductive topological Hausdorff dimension. If \mathcal{U} is a basis in Y then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is a basis in X such that $\partial_X(U \cap X) \subseteq \partial_Y U$, thus $\dim_{t^{n-1} H} \partial_X(U \cap X) \leq \dim_{t^{n-1} H} \partial_Y U$ holds for all $U \in \mathcal{U}$. Therefore $\dim_{t^n H} X \leq \dim_{t^n H} Y$. \square

Theorem 3.8. *Let X, Y be metric spaces and $n \in \mathbb{N}$. If $f: X \rightarrow Y$ is a Lipschitz homeomorphism then $\dim_{t^n H} Y \leq \dim_{t^n H} X$.*

Proof. If $n = 0$ then we are done. Let $n \geq 1$ and assume by induction that the statement holds for $n - 1$. Since f is a homeomorphism, if \mathcal{U} is a basis in X then $\mathcal{V} = \{f(U) : U \in \mathcal{U}\}$ is a basis in Y , and $\partial f(U) = f(\partial U)$ for all $U \in \mathcal{U}$. As $f|_{\partial U}$ is also a Lipschitz homeomorphism, the inductive hypothesis implies that $\dim_{t^{n-1} H} \partial V = \dim_{t^{n-1} H} \partial f(U) = \dim_{t^{n-1} H} f(\partial U) \leq \dim_{t^{n-1} H} \partial U$ for all $V = f(U) \in \mathcal{V}$. Therefore $\dim_{t^n H} Y \leq \dim_{t^n H} X$. \square

Corollary 3.9 (Bi-Lipschitz invariance). *Let X, Y be metric spaces and $n \in \mathbb{N}$. If $f: X \rightarrow Y$ is bi-Lipschitz then $\dim_{t^n H} X = \dim_{t^n H} Y$.*

Theorem 3.10 (Countable stability for closed sets). *Let X be a separable metric space with $X = \bigcup_{i=0}^{\infty} X_i$, where X_i are closed subsets of X . Then for all $n \in \mathbb{N}$*

$$\dim_{t^n H} X = \sup_{i \in \mathbb{N}} \dim_{t^n H} X_i.$$

Proof. If $n = 0$ then we are done by the countable stability of the Hausdorff dimension. Let $n \geq 1$ and assume by induction that the statement holds for $n - 1$.

Monotonicity clearly implies $\dim_{t^n H} X \geq \sup_{i \in \mathbb{N}} \dim_{t^n H} X_i$. For the other direction we may assume $\sup_{i \in \mathbb{N}} \dim_{t^n H} X_i < \infty$. Let $d > \sup_{i \in \mathbb{N}} \dim_{t^n H} X_i$ be arbitrary. Let \mathcal{U}_i be a countable basis of X_i such that $\dim_{t^{n-1} H} \partial_{X_i} U \leq d - 1$ for all $i \in \mathbb{N}$ and $U \in \mathcal{U}_i$.

Let $Y = \bigcup \{\partial_{X_i} U : i \in \mathbb{N}, U \in \mathcal{U}_i\}$. The countable stability of the $(n - 1)$ st inductive topological Hausdorff dimension for closed sets implies $\dim_{t^{n-1} H} Y \leq d - 1$. The definition of the topological dimension yields $\dim_t(X_i \setminus Y) = 0$ for all $i \in \mathbb{N}$. Then $X_i \setminus Y$ is a closed subspace of the separable metric space $X \setminus Y$, and $X \setminus Y = \bigcup_{i \in \mathbb{N}} (X_i \setminus Y)$. The countable stability of the topological dimension zero for closed sets [4, 1.3.1. Thm.] yields $\dim_t(X \setminus Y) = 0$.

Let us fix an open set $V \subseteq X$ and a point $x \in V$. As $X \setminus Y$ is a separable subspace of X with $\dim_t(X \setminus Y) = 0$, Lemma 2.3 yields that there is a partition L between $\{x\}$ and $X \setminus V$ with $L \subseteq Y$. Thus there exist disjoint open sets $U, U' \subseteq X$ such that $x \in U$, $X \setminus V \subseteq U'$ and $U \cup U' = X \setminus L$. In particular, $x \in U \subseteq V$. Moreover, $\partial_X U \subseteq L \subseteq Y$, thus $\dim_{t^{n-1} H} \partial_X U \leq \dim_{t^{n-1} H} Y \leq d - 1$. Therefore the definition of the n th inductive topological Hausdorff dimension implies $\dim_{t^n H} X \leq d$. As $d > \sup_{i \in \mathbb{N}} \dim_{t^n H} X_i$ was arbitrary, the proof is complete. \square

4. EQUIVALENT DEFINITIONS OF INDUCTIVE TOPOLOGICAL HAUSDORFF DIMENSIONS

The aim of this section is to give some equivalent definitions which play a crucial role in the proof of the Main Theorem.

Let X be a separable metric space and $n \in \mathbb{N}^+$. If $\dim_t X < n$ then Fact 3.2 yields $\dim_{t^n H} X = \dim_t X$, so the n th inductive topological Hausdorff dimension is reduced to the well-known topological dimension. Hence from now on we can restrict our attention to the case $\dim_t X \geq n$.

Notation 4.1. If \mathcal{A} is a family of sets and $m \in \mathbb{N}^+$ then let $T_m(\mathcal{A})$ denote the set of points covered by at least m members of \mathcal{A} .

For a motivation let us repeat the definition of the topological dimension and recall three of its equivalent definitions, for some details consult [4].

Theorem 4.2. *If X is a non-empty separable metric space then*

$$\begin{aligned} \dim_t X &= \min\{n : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq n - 1 \text{ for every } U \in \mathcal{U}\} \\ &= \min\{n : \exists A \subseteq X \text{ such that } \dim_t A \leq 0 \text{ and } \dim_t(X \setminus A) \leq n - 1\} \\ &= \min\{n : \forall (A_1, B_1), \dots, (A_{n+1}, B_{n+1}) \text{ pairs of disjoint closed subsets} \\ &\quad \text{of } X \exists \text{ partitions } L_i \text{ between } A_i \text{ and } B_i \text{ such that } \bigcap_{i=1}^{n+1} L_i = \emptyset\} \\ &= \min\{n : \forall \text{ finite open cover } \mathcal{U} \text{ of } X \exists \text{ a finite open refinement } \mathcal{V} \text{ of } \mathcal{U} \\ &\quad \text{such that } T_{n+2}(\mathcal{V}) = \emptyset\}. \end{aligned}$$

From now on we assume that $n \in \mathbb{N}^+$ and X is a separable metric space with $\dim_t X \geq n$.

Definition 4.3. Let

$$\begin{aligned} P_{t^n H} &= \{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_{t^{n-1} H} \partial U \leq d - 1 \text{ for every } U \in \mathcal{U}\}, \\ P_{d^n H} &= \{d \geq n : \exists A \subseteq X \text{ such that } \dim_H A \leq d - n \text{ and } \dim_t(X \setminus A) \leq n - 1\}, \\ P_{p^n H} &= \{d \geq n : \forall (A_1, B_1), \dots, (A_n, B_n) \text{ pairs of disjoint closed subsets of } X \\ &\quad \exists \text{ partitions } L_i \text{ between } A_i \text{ and } B_i \text{ such that } \dim_H(\bigcap_{i=1}^n L_i) \leq d - n\}, \\ P_{c^n H} &= \{d \geq n : \forall \varepsilon > 0 \forall \text{ finite open cover } \mathcal{U} \text{ of } X \exists \text{ a finite open refinement} \\ &\quad \mathcal{V} \text{ of } \mathcal{U} \text{ such that } \mathcal{H}_\infty^{d-n+\varepsilon}(T_{n+1}(\mathcal{V})) \leq \varepsilon\}. \end{aligned}$$

We assume $\infty \in P_{t^n H}, P_{d^n H}, P_{p^n H}, P_{c^n H}$. In the above notation the letter H refers to Hausdorff, while t, d, p, c come from the first letters of the words topological, decomposition, partition and covering, respectively.

The goal of this section is to prove the following theorem. This implies that the infimum is attained in the definition of inductive topological Hausdorff dimensions and also yields three equivalent definitions similarly to Theorem 4.2.

Theorem 4.4. *If $n \in \mathbb{N}^+$ and X is a separable metric space with $\dim_t X \geq n$ then*

$$\dim_{t^n H} X = \min P_{t^n H} = \min P_{d^n H} = \min P_{p^n H} = \min P_{c^n H}.$$

Theorem 4.4 easily follows from Lemma 4.5 and Theorem 4.6 below.

Lemma 4.5. $\inf P_{d^n H} \in P_{d^n H}$.

Proof. Let $d = \inf P_{d^n H}$, we may assume $d < \infty$. Set $d_i = d + 1/i$ for all $i \in \mathbb{N}^+$. As $d_i \in P_{d^n H}$, there exist sets $A_i \subseteq X$ such that $\dim_H A_i \leq d_i - n$ and $\dim_t(X \setminus A_i) \leq n - 1$. We may assume that the sets A_i are G_δ , since we can take G_δ hulls with the same Hausdorff dimension. Let $A = \bigcap_{i=1}^\infty A_i$, then clearly $\dim_H A \leq d - n$. As $X \setminus A_i$ are F_σ sets such that $\dim_t(X \setminus A_i) \leq n - 1$ and $X \setminus A \subseteq \bigcup_{i=1}^\infty (X \setminus A_i)$,

monotonicity and countable stability of the topological dimension for F_σ sets [4, 1.5.4. Corollary] yield $\dim_t(X \setminus A) \leq n - 1$. Hence $d \in P_{d^n H}$. \square

Theorem 4.6. *If $n \in \mathbb{N}^+$ and X is a separable metric space with $\dim_t X \geq n$ then*

$$P_{t^n H} = P_{d^n H} = P_{p^n H} = P_{c^n H}.$$

Before proving Theorem 4.6 we need some preparation.

Notation 4.7. If \mathcal{A} is a family of sets and $A \in \mathcal{A}$ then let us define the *star of the set A with respect to the family \mathcal{A}* as

$$\text{St}(A, \mathcal{A}) = \bigcup \{A' \in \mathcal{A} : A \cap A' \neq \emptyset\}.$$

For the next result see [4, 4.1.1. Lemma] and its proof.

Lemma 4.8. *Let X be a normal space and $m \in \mathbb{N}^+$. Assume that \mathcal{V}_i ($i \in \mathbb{N}^+$) are open covers of X such that \mathcal{V}_{i+1} is a refinement of \mathcal{V}_i for every $i \in \mathbb{N}^+$ and $\{\text{St}(V, \mathcal{V}_i) : V \in \mathcal{V}_i, i \in \mathbb{N}^+\}$ is a basis of X . Then every finite open cover \mathcal{U} of X has an open shrinking \mathcal{V} such that $T_m(\mathcal{V}) \subseteq \bigcup_{i=1}^{\infty} T_m(\mathcal{V}_i)$.*

The following lemma helps us to work with $P_{c^n H}$.

Lemma 4.9. *Let X be a separable metric space and let $m \in \mathbb{N}^+$ and $s \geq 0$. Then the following properties are equivalent:*

- (i) *For every $\varepsilon > 0$ and open cover \mathcal{U} of X there is an open refinement \mathcal{V} of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$;*
- (ii) *for every $\varepsilon > 0$ and finite open cover \mathcal{U} of X there is a finite open refinement \mathcal{V} of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$;*
- (iii) *for every $\varepsilon > 0$ and open cover \mathcal{U} of X there is an open shrinking \mathcal{V} of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$;*
- (iv) *for every $\varepsilon > 0$ and finite open cover \mathcal{U} of X there is an open shrinking \mathcal{V} of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$;*
- (v) *for every $\varepsilon > 0$ and m -element open cover $\mathcal{U} = \{U_i\}_{i=1}^m$ of X there is an open shrinking $\mathcal{V} = \{V_i\}_{i=1}^m$ of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(V_1 \cap \dots \cap V_m) \leq \varepsilon$;*
- (vi) *for every open cover \mathcal{U}_0 of X there are locally finite open covers \mathcal{U}_i ($i \in \mathbb{N}^+$) of X such that for all $i \in \mathbb{N}^+$ we have $\text{mesh} \mathcal{U}_i \leq 1/i$, $\mathcal{H}_\infty^{s+1/i}(T_m(\mathcal{U}_i)) \leq 1/i$, and for every $U \in \mathcal{U}_i$ there exists $V \in \mathcal{U}_{i-1}$ such that $\text{cl}(U) \subseteq V$;*
- (vii) *there exist open covers \mathcal{U}_i ($i \in \mathbb{N}^+$) of X such that for all $i \in \mathbb{N}^+$ we have $\text{mesh} \mathcal{U}_i \leq 1/i$, $\mathcal{H}_\infty^{s+1/i}(T_m(\mathcal{U}_i)) \leq 1/i$, and \mathcal{U}_{i+1} is a refinement of \mathcal{U}_i .*

Proof. The proof consists of several implications of different difficulty levels. Proving directions (iv) \Rightarrow (i) and (v) \Rightarrow (iv) need a lot of effort, while the other directions are more or less obvious.

(i) \Rightarrow (iii) and (ii) \Rightarrow (iv): Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let \mathcal{W} be an open refinement of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{W})) \leq \varepsilon$ for some $\varepsilon > 0$. It is enough to find an open shrinking \mathcal{V} of \mathcal{U} with $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$. For every $W \in \mathcal{W}$ let us choose $i(W) \in I$ such that $W \subseteq U_{i(W)}$. Let $V_i = \bigcup \{W \in \mathcal{W} : i(W) = i\}$ for all $i \in I$ and set $\mathcal{V} = \{V_i\}_{i \in I}$. Then \mathcal{V} is an open shrinking of \mathcal{U} with $T_m(\mathcal{V}) \subseteq T_m(\mathcal{W})$, so $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$.

(iii) \Rightarrow (ii): Straightforward.

(iv) \Rightarrow (i): Let \mathcal{U} be an open cover of X and $\varepsilon > 0$, we need to find an open refinement \mathcal{V} of \mathcal{U} with $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$. By Fact 2.1 we may assume that $\varepsilon < 1$.

Suppose that \mathcal{U} is infinite, otherwise we are done. As X is separable, we may assume that \mathcal{U} is countable, let $\mathcal{U} = \{U_j : j \in \mathbb{N}\}$. By Theorem 2.5 we may suppose that \mathcal{U} is locally finite. Let us enumerate the finite subsets of \mathbb{N} as $\mathbb{N}^{<\omega} = \{Q_i : i \in \mathbb{N}^+\}$. For every $i \in \mathbb{N}^+$ let us define a closed set by

$$F_i = \bigcap_{j \in Q_i} \text{cl}(U_j) \cap \bigcap_{j \notin Q_i} (X \setminus U_j).$$

Let $\mathcal{V}_0 = \{V_{0,j} : j \in \mathbb{N}\}$ be the open cover defined as $V_{0,j} = U_j$ for all $j \in \mathbb{N}$. Assume by induction that $i \in \mathbb{N}^+$ and open covers $\mathcal{V}_k = \{V_{k,j} : j \in \mathbb{N}\}$ are already defined for all $k \in \{0, \dots, i-1\}$ such that \mathcal{V}_{k+1} is a shrinking of \mathcal{V}_k for all $k \leq i-2$. Now we define \mathcal{V}_i . As $V_{i-1,j} \subseteq V_{0,j} = U_j$ for all $j \in \mathbb{N}$, the definition of F_i yields that $V_{i-1,j} \subseteq X \setminus F_i$ if $j \notin Q_i$, so $\{X \setminus F_i, V_{i-1,j} : j \in Q_i\}$ is a finite open cover of X . Applying property (iv) for this cover with $\varepsilon 2^{-i} > 0$ and intersecting the open sets with F_i imply that the finite open cover $\{F_i \cap V_{i-1,j} : j \in Q_i\}$ of F_i has an open shrinking $\mathcal{W}_i = \{W_{i,j} : j \in Q_i\}$ such that

$$(4.1) \quad \mathcal{H}_{\infty}^{s+\varepsilon 2^{-i}}(T_m(\mathcal{W}_i)) \leq \varepsilon 2^{-i}.$$

Set $\mathcal{V}_i = \{V_{i,j} : j \in \mathbb{N}\}$, where

$$V_{i,j} = \begin{cases} (V_{i-1,j} \setminus F_i) \cup W_{i,j} & \text{if } j \in Q_i, \\ V_{i-1,j} & \text{if } j \notin Q_i. \end{cases}$$

It is easy to see that $V_{i,j}$ are open sets in X such that

$$(4.2) \quad V_{i-1,j} \setminus V_{i,j} \subseteq F_i \quad (j \in \mathbb{N}).$$

The construction and the inductive hypothesis yield $(\bigcup \mathcal{V}_i) \cap (X \setminus F_i) = (\bigcup \mathcal{V}_{i-1}) \cap (X \setminus F_i) = X \setminus F_i$ and $F_i \subseteq \bigcup \mathcal{V}_i$, thus \mathcal{V}_i covers X . Hence \mathcal{V}_i is an open shrinking of \mathcal{V}_{i-1} .

Let us define $\mathcal{V} = \{V_j : j \in \mathbb{N}\}$ as

$$V_j = \bigcap_{i=0}^{\infty} V_{i,j}.$$

We show that V_j is open for all $j \in \mathbb{N}^+$. Let us fix $j \in \mathbb{N}^+$ and $x \in V_j$. The local finiteness of \mathcal{U} yields that there are $N \in \mathbb{N}$ and an open neighborhood U of x such that $U \cap \text{cl}(U_j) = \emptyset$ for every $j > N$. There exists an $M \in \mathbb{N}$ such that $Q_i \not\subseteq \{0, \dots, N\}$ if $i > M$, thus $U \cap F_i = \emptyset$ for all $i > M$. As \mathcal{V}_M is a cover, there exists $j \in \mathbb{N}$ such that $x \in V_{M,j}$. Then (4.2) yields that $U \cap V_{M,j} \subseteq V_{i,j}$ for all $i > M$, so $U \cap V_{M,j} \subseteq V_j$ is an open neighborhood of x .

We prove that \mathcal{V} is a cover. Let $x \in X$ be arbitrarily fixed, the local finiteness of \mathcal{U} yields that there is an $N \in \mathbb{N}$ such that $x \notin \text{cl}(U_j)$ for every $j > N$. There exists an $M \in \mathbb{N}$ such that $Q_i \not\subseteq \{0, \dots, N\}$ for all $i > M$, thus $x \notin F_i$ for all $i > M$. As \mathcal{V}_M is a cover, there exists $j \in \mathbb{N}$ such that $x \in V_{M,j}$. Then (4.2) implies that $x \in V_{i,j}$ for all $i > M$, thus $x \in V_j$.

Now we show that

$$(4.3) \quad T_m(\mathcal{V}) \subseteq \bigcup_{i=1}^{\infty} T_m(\mathcal{W}_i).$$

Assume $x \in T_m(\mathcal{V})$, then there are distinct indexes $j_1, \dots, j_m \in \mathbb{N}$ such that $x \in V_{j_1} \cap \dots \cap V_{j_m}$. The local finiteness of \mathcal{U} implies that there is a $k \in \mathbb{N}^+$ such

that $x \in F_k$. Then $x \in V_j \subseteq V_{k,j}$ and $x \notin V_{k-1,j} \setminus F_k$ yields $x \in W_{k,j}$ for all $j \in \{j_1, \dots, j_m\}$, thus $x \in T_m(\mathcal{W}_k)$, so (4.3) holds.

Hence \mathcal{V} is an open shrinking (specially a refinement) of $\mathcal{U} = \mathcal{V}_0$. Then (4.3), the subadditivity of $\mathcal{H}_\infty^{s+\varepsilon}$, Fact 2.1, and (4.1) yield

$$\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \sum_{i=1}^{\infty} \mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{W}_i)) \leq \sum_{i=1}^{\infty} \mathcal{H}_\infty^{s+\varepsilon 2^{-i}}(T_m(\mathcal{W}_i)) \leq \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

Thus property (i) holds.

(iv) \Rightarrow (v): Straightforward.

(v) \Rightarrow (iv): Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of X and let $\varepsilon > 0$. We need to prove that there exists an open shrinking \mathcal{V} of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$. We may suppose $k \geq m$, otherwise we are done. By Fact 2.1 we may assume that $\varepsilon < 1$.

First we prove that there is an open shrinking $\mathcal{W} = \{W_i\}_{i=1}^k$ of \mathcal{U} such that $\mathcal{H}_\infty^{s+\varepsilon}(W_1 \cap \dots \cap W_m) \leq \delta$, where $\delta = \varepsilon / \binom{k}{m}$. Let us define $\mathcal{U}' = \{U'_i\}_{i=1}^m$ such that $U'_i = U_i$ if $i \in \{1, \dots, m-1\}$ and $U'_m = \bigcup_{i=m}^k U_i$. Then (v) yields that there is an open shrinking $\mathcal{W}' = \{W'_i\}_{i=1}^m$ of \mathcal{U}' such that

$$(4.4) \quad \mathcal{H}_\infty^{s+\delta}(W'_1 \cap \dots \cap W'_m) \leq \delta.$$

Let us define $\mathcal{W} = \{W_i\}_{i=1}^k$ such that $W_i = W'_i$ if $i \in \{1, \dots, m-1\}$ and $W_i = W'_m \cap U_i$ if $i \in \{m, \dots, k\}$. Then $W'_m \subseteq U'_m$ yields $\bigcup_{i=1}^k W_i = \bigcup_{i=1}^m W'_i = X$, so \mathcal{W} is an open shrinking of \mathcal{U} . Fact 2.1 with $\delta < \varepsilon < 1$, the definition of \mathcal{W} and (4.4) imply

$$\begin{aligned} \mathcal{H}_\infty^{s+\varepsilon}(W_1 \cap \dots \cap W_m) &\leq \mathcal{H}_\infty^{s+\delta}(W_1 \cap \dots \cap W_m) \\ &\leq \mathcal{H}_\infty^{s+\delta}(W'_1 \cap \dots \cap W'_m) \leq \delta. \end{aligned}$$

Now the iteration of the above statement yields the required \mathcal{V} . More precisely, let \mathcal{N} be the collection of m -element subsets of $\{1, \dots, k\}$ and let $n = \binom{k}{m}$. Consider a bijection $\phi: \{1, \dots, n\} \rightarrow \mathcal{N}$. For $j \in \{1, \dots, n\}$ and $l \in \{1, \dots, m\}$ let $\phi(j, l)$ be the l th element of $\phi(j)$ corresponding to the natural ordering. Let $\mathcal{U}_0 = \mathcal{U}$ and denote $\mathcal{U}_0 = \{U_{0,i}\}_{i=1}^k$, where $U_{0,i} = U_i$. Assume by induction that the open cover $\mathcal{U}_{j-1} = \{U_{j-1,i}\}_{i=1}^k$ of X is already defined for some $j \in \{1, \dots, n\}$. Applying the above statement for a rearranged copy of \mathcal{U}_{j-1} we obtain that there is an open shrinking $\mathcal{U}_j = \{U_{j,i}\}_{i=1}^k$ of \mathcal{U}_{j-1} such that

$$(4.5) \quad \mathcal{H}_\infty^{s+\varepsilon} \left(\bigcap_{l=1}^m U_{j,\phi(j,l)} \right) \leq \delta.$$

Now k -element open covers $\mathcal{U}_0, \dots, \mathcal{U}_n$ of X are defined such that \mathcal{U}_n is a shrinking of \mathcal{U}_j for all $j \in \{0, \dots, n\}$. Therefore the subadditivity of $\mathcal{H}_\infty^{s+\varepsilon}$ and (4.5) imply

$$\begin{aligned} \mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{U}_n)) &\leq \sum_{j=1}^n \mathcal{H}_\infty^{s+\varepsilon} \left(\bigcap_{l=1}^m U_{n,\phi(j,l)} \right) \\ &\leq \sum_{j=1}^n \mathcal{H}_\infty^{s+\varepsilon} \left(\bigcap_{l=1}^m U_{j,\phi(j,l)} \right) \leq n\delta = \varepsilon. \end{aligned}$$

Hence $\mathcal{V} = \mathcal{U}_n$ is an open shrinking of \mathcal{U} satisfying (iv).

(iii) \Rightarrow (vi): Let \mathcal{U}_0 be an open cover of X , we need to define open covers \mathcal{U}_i ($i \in \mathbb{N}^+$) satisfying (vi). Assume by induction that $i \in \mathbb{N}^+$ and \mathcal{U}_{i-1} is already defined. Let \mathcal{V}_i be an open refinement of \mathcal{U}_{i-1} such that $\text{mesh } \mathcal{V}_i \leq 1/i$ and for every $V \in \mathcal{V}_i$ there exists $U \in \mathcal{U}_{i-1}$ such that $\text{cl}(V) \subseteq U$. By Theorem 2.5 we may assume that \mathcal{V}_i is locally finite. Applying (iii) for \mathcal{V}_i and $\varepsilon = 1/i$ yields that there is an open shrinking \mathcal{U}_i of \mathcal{V}_i such that $\mathcal{H}_\infty^{s+1/i}(T_m(\mathcal{U}_i)) \leq 1/i$. Then the defined covers \mathcal{U}_i clearly satisfy (vi).

(vi) \Rightarrow (vii): Straightforward.

(vii) \Rightarrow (iv): Let \mathcal{U} be a finite open cover of X and let $\varepsilon > 0$. We need to find an open shrinking \mathcal{V} of \mathcal{U} with $\mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) \leq \varepsilon$. By Fact 2.1 we may assume that $\varepsilon < 1$. Assume that open covers \mathcal{U}_i of X are given according to (vii). Fix $k \in \mathbb{N}^+$ such that $k \geq 1/\varepsilon$ and let $\mathcal{V}_i = \mathcal{U}_{k2^i}$ for all $i \in \mathbb{N}^+$. Since $\text{mesh } \mathcal{V}_i \rightarrow 0$ as $i \rightarrow \infty$, the family $\{\text{St}(V, \mathcal{V}_i) : V \in \mathcal{V}_i, i \in \mathbb{N}^+\}$ is a basis of X . Then Lemma 4.8 yields that there is an open shrinking \mathcal{V} of \mathcal{U} such that $T_m(\mathcal{V}) \subseteq \bigcup_{i=1}^\infty T_m(\mathcal{V}_i)$. Therefore the subadditivity of $\mathcal{H}_\infty^{s+\varepsilon}$, Fact 2.1 for $\frac{1}{k2^i} \leq \varepsilon < 1$ and property (vii) imply

$$\begin{aligned} \mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V})) &\leq \sum_{i=1}^\infty \mathcal{H}_\infty^{s+\varepsilon}(T_m(\mathcal{V}_i)) \\ &\leq \sum_{i=1}^\infty \mathcal{H}_\infty^{s+1/(k2^i)}(T_m(\mathcal{U}_{k2^i})) \\ &\leq \sum_{i=1}^\infty \frac{1}{k2^i} = \frac{1}{k} \leq \varepsilon, \end{aligned}$$

thus property (iv) holds. \square

The following four lemmas conclude the proof of Theorem 4.6.

Lemma 4.10. $P_{t^n H} \subseteq P_{d^n H}$.

Proof. Suppose $d \in P_{t^n H}$ and $d < \infty$, we need to prove that $d \in P_{d^n H}$. The proof is induction on n . We prove the base case $n = 1$ and the inductive step for $n > 1$ simultaneously, in the latter case we assume by induction that $P_{t^{n-1} H}(Y) \subseteq P_{d^{n-1} H}(Y)$ for all separable metric spaces Y . Fact 3.1 implies $\dim_{t^n H} K \geq \dim_t K \geq n$, thus $d \geq n$. There exists a countable basis \mathcal{U} of X such that $\dim_{t^{n-1} H} \partial U \leq d-1$ for all $U \in \mathcal{U}$. Let $F = \bigcup_{U \in \mathcal{U}} \partial U$, then clearly $\dim_t(X \setminus F) \leq 0$. The countable stability of the $(n-1)$ st inductive topological Hausdorff dimension for closed sets yields $\dim_{t^{n-1} H} F \leq d-1$. If $n = 1$ then $\dim_H F \leq d-1$ and $\dim_t(X \setminus F) \leq 0$ implies $d \in P_{d^1 H}$. If $n > 1$ then the inductive hypothesis yields $d-1 \in P_{t^{n-1} H}(F) \subseteq P_{d^{n-1} H}(F)$, so there is a set $A \subseteq F$ such that $\dim_H A \leq (d-1) - (n-1) = d-n$ and $\dim_t(F \setminus A) \leq n-2$. Since $(X \setminus A) = (F \setminus A) \cup (X \setminus F)$, Theorem 2.4 implies $\dim_t(X \setminus A) \leq n-1$. Thus A witnesses $d \in P_{d^n H}$. \square

Lemma 4.11. $P_{d^n H} \subseteq P_{p^n H}$.

Proof. Assume $d \in P_{d^n H}$ and $d < \infty$. Then there exists $Y \subseteq X$ such that $\dim_H Y \leq d-n$ and $\dim_t(X \setminus Y) \leq n-1$. Therefore Theorem 2.4 yields that $X \setminus Y = \bigcup_{i=1}^n Z_i$ such that $\dim_t Z_i \leq 0$ for all $i \in \{1, \dots, n\}$. Let $(A_1, B_1), \dots, (A_n, B_n)$ be pairs of disjoint closed subsets of X . Applying Lemma 2.3 for A_i, B_i and Z_i yields that there exist partitions L_i between A_i and B_i such that $L_i \cap Z_i = \emptyset$ for all $i \in \{1, \dots, n\}$.

Then $\bigcap_{i=1}^n L_i \subseteq X \setminus (\bigcup_{i=1}^n Z_i) = Y$, so $\dim_H(\bigcap_{i=1}^n L_i) \leq \dim_H Y \leq d - n$. Thus $d \in P_{p^n H}$. \square

Lemma 4.12. $P_{p^n H} \subseteq P_{c^n H}$.

Proof. Assume $d \in P_{p^n H}$ and $d < \infty$, we need to prove $d \in P_{c^n H}$. Let $\varepsilon > 0$ and $\mathcal{U} = \{U_i\}_{i=1}^{n+1}$ be an $(n+1)$ -element open cover of X . By Lemma 4.9 it is enough to find an open shrinking $\mathcal{V} = \{V_i\}_{i=1}^{n+1}$ of \mathcal{U} such that $\mathcal{H}_\infty^{d-n+\varepsilon}(V_1 \cap \dots \cap V_{n+1}) \leq \varepsilon$. By Theorem 2.6 the finite open cover \mathcal{U} has a closed shrinking $\mathcal{A} = \{A_i\}_{i=1}^{n+1}$. For all $i \in \{1, \dots, n\}$ let us define $B_i = X \setminus U_i$. The sequence $(A_1, B_1), \dots, (A_n, B_n)$ consists of n pairs of disjoint closed subsets of X . Thus $d \in P_{p^n H}$ yields that there exist partitions L_i between A_i and B_i such that $\dim_H(\bigcap_{i=1}^n L_i) \leq d - n$. For all $i \in \{1, \dots, n\}$ consider open sets $V_i, W_i \subseteq X$ such that

$$(4.6) \quad A_i \subseteq V_i, \quad B_i \subseteq W_i, \quad V_i \cap W_i = \emptyset, \quad \text{and} \quad X \setminus L_i = V_i \cup W_i.$$

Let $L = \bigcap_{i=1}^n L_i$. Then $\dim_H L \leq d - n$ yields $\mathcal{H}_\infty^{d-n+\varepsilon}(L) = 0$, so we can choose an open set $U \subseteq X$ such that $L \subseteq U$ and $\mathcal{H}_\infty^{d-n+\varepsilon}(U) \leq \varepsilon$. Let us consider $\mathcal{V} = \{V_i\}_{i=1}^{n+1}$, where

$$(4.7) \quad V_{n+1} = U_{n+1} \cap \left(U \cup \bigcup_{i=1}^n W_i \right).$$

Now we show that \mathcal{V} is an open shrinking of \mathcal{U} . Since $V_{n+1} \subseteq U_{n+1}$ and (4.6) yields $V_i \subseteq X \setminus B_i = U_i$ for all $i \in \{1, \dots, n\}$, we need to prove that \mathcal{V} covers X . Equation (4.6) and $A_{n+1} \subseteq U_{n+1}$ imply

$$(4.8) \quad \left(\bigcup_{i=1}^n V_i \right) \cup U_{n+1} \supseteq \bigcup_{i=1}^{n+1} A_i = X.$$

Equation (4.6) and $L \subseteq U$ yield

$$(4.9) \quad \begin{aligned} \left(\bigcup_{i=1}^n V_i \right) \cup \left(U \cup \bigcup_{i=1}^n W_i \right) &= \bigcup_{i=1}^n (V_i \cup W_i) \cup U \\ &= \bigcup_{i=1}^n (X \setminus L_i) \cup U \\ &= (X \setminus L) \cup U = X. \end{aligned}$$

Now (4.7), (4.8) and (4.9) imply $\bigcup_{i=1}^{n+1} V_i = X$, so \mathcal{V} is an open shrinking of \mathcal{U} .

Finally, applying (4.7) and $V_i \cap W_i = \emptyset$ for $i \in \{1, \dots, n\}$ yield

$$\begin{aligned} \bigcap_{i=1}^{n+1} V_i &\subseteq \bigcap_{i=1}^n V_i \cap \left(U \cup \bigcup_{i=1}^n W_i \right) \\ &\subseteq \left(\bigcap_{i=1}^n V_i \cap U \right) \cup \left(\bigcap_{i=1}^n V_i \cap \bigcup_{i=1}^n W_i \right) \\ &\subseteq U \cup \emptyset = U. \end{aligned}$$

Therefore

$$\mathcal{H}_\infty^{d-n+\varepsilon} \left(\bigcap_{i=1}^{n+1} V_i \right) \leq \mathcal{H}_\infty^{d-n+\varepsilon}(U) \leq \varepsilon,$$

and the proof is complete. \square

Lemma 4.13. $P_{c^n H} \subseteq P_{t^n H}$.

Proof. Suppose $d \in P_{c^n H}$ and $d < \infty$, we need to prove that $d \in P_{t^n H}$. The proof is induction on n . We prove the base case $n = 1$ and the inductive step for $n > 1$ simultaneously, in the latter case we assume by induction that $P_{c^{n-1} H}(Y) \subseteq P_{t^{n-1} H}(Y)$ for all separable metric spaces Y .

Let $x \in X$ and let V_0 be an open set such that $x \in V_0$. For $d \in P_{t^n H}$ it is enough to construct an open set $U \subseteq V_0$ such that $x \in U$ and $\dim_{t^{n-1} H} \partial U \leq d - 1$. Let $A_0 = \{x\}$ and $B_0 = X \setminus V_0$. Let \mathcal{U}_0 be an open cover of X such that if $U \in \mathcal{U}_0$ and $\text{cl}(U) \cap A_0 \neq \emptyset$ then $\text{cl}(U) \cap B_0 = \emptyset$. Applying Lemma 4.9 for $s = d - n$ and $m = n + 1$ yields that there exist locally finite open covers \mathcal{U}_i ($i \in \mathbb{N}^+$) of X such that for all $i \in \mathbb{N}^+$ we have $\text{mesh} \mathcal{U}_i \leq 1/i$ and $\mathcal{H}_\infty^{d-n+1/i}(T_{n+1}(\mathcal{U}_i)) \leq 1/i$, and for every $U \in \mathcal{U}_i$ there exists $V \in \mathcal{U}_{i-1}$ such that $\text{cl}(U) \subseteq V$.

Assume by induction that $i \in \mathbb{N}^+$ and disjoint closed sets A_{i-1} and B_{i-1} are already defined. Consider $A_i = X \setminus G_i$ and $B_i = X \setminus H_i$, where

$$G_i = \bigcup \{U \in \mathcal{U}_i : \text{cl}(U) \cap A_{i-1} = \emptyset\},$$

$$H_i = \bigcup \{U \in \mathcal{U}_i : \text{cl}(U) \cap A_{i-1} \neq \emptyset\}.$$

As G_i and H_i are open sets such that $G_i \cup H_i = X$, we obtain that A_i and B_i are disjoint closed sets.

We prove that for all $i \in \mathbb{N}^+$

$$(4.10) \quad \text{if } U \in \mathcal{U}_i \text{ and } \text{cl}(U) \cap A_{i-1} \neq \emptyset \text{ then } \text{cl}(U) \cap B_{i-1} = \emptyset.$$

If $i = 1$ then the definition of \mathcal{U}_0 and the fact that \mathcal{U}_1 is a refinement of \mathcal{U}_0 imply (4.10). If $i > 1$ and $\text{cl}(U) \cap A_{i-1} \neq \emptyset$ then there is a $V \in \mathcal{U}_{i-1}$ such that $\text{cl}(U) \subseteq V$. Then clearly $V \cap A_{i-1} \neq \emptyset$, thus $V \not\subseteq G_{i-1}$. Therefore $V \subseteq H_{i-1}$, thus $V \cap B_{i-1} = \emptyset$. Hence $\text{cl}(U) \cap B_{i-1} = \emptyset$, so (4.10) holds.

The local finiteness of \mathcal{U}_i implies that

$$\text{cl}(G_i) = \bigcup \{\text{cl}(U) : U \in \mathcal{U}_i \text{ and } \text{cl}(U) \cap A_{i-1} = \emptyset\},$$

$$\text{cl}(H_i) = \bigcup \{\text{cl}(U) : U \in \mathcal{U}_i \text{ and } \text{cl}(U) \cap A_{i-1} \neq \emptyset\}.$$

Therefore $\text{cl}(G_i) \cap A_{i-1} = \emptyset$ and (4.10) implies $\text{cl}(H_i) \cap B_{i-1} = \emptyset$. Thus we obtain $A_{i-1} \subseteq X \setminus \text{cl}(G_i) = \text{int } A_i$ and $B_{i-1} \subseteq X \setminus \text{cl}(H_i) = \text{int } B_i$. Therefore $U_A = \bigcup_{i=0}^\infty A_i$ and $U_B = \bigcup_{i=0}^\infty B_i$ are disjoint open sets containing A_0 and B_0 , respectively.

Let us define $L = X \setminus (U_A \cup U_B) = \bigcap_{i=1}^\infty (G_i \cap H_i)$, then L is a partition between A_0 and B_0 . It is enough to prove $\dim_{t^{n-1} H} L \leq d - 1$, because then $x \in U_A \subseteq V_0$, and $\partial U_A \subseteq L$ implies $\dim_{t^{n-1} H} \partial U_A \leq \dim_{t^{n-1} H} L \leq d - 1$. For all $i \in \mathbb{N}^+$ consider

$$\mathcal{W}_i = \{U \cap L : U \in \mathcal{U}_i \text{ and } \text{cl}(U) \cap A_{i-1} \neq \emptyset\}.$$

Let us fix $i \in \mathbb{N}^+$. Since $L \subseteq H_i$, we obtain that \mathcal{W}_i is an open cover of L , and $\text{mesh} \mathcal{U}_i \leq 1/i$ yields $\text{mesh} \mathcal{W}_i \leq 1/i$.

As $L \subseteq G_i$, for every $x \in L$ there exists an $U \in \mathcal{U}_i$ with $\text{cl}(U) \cap A_{i-1} = \emptyset$, thus $T_n(\mathcal{W}_i) \subseteq T_{n+1}(\mathcal{U}_i)$. Hence $\mathcal{H}_\infty^{d-n+1/i}(T_n(\mathcal{W}_i)) \leq 1/i$.

We show that \mathcal{W}_{i+1} is a refinement of \mathcal{W}_i . Let $U \cap L \in \mathcal{W}_{i+1}$, where $U \in \mathcal{U}_{i+1}$ and $\text{cl}(U) \cap A_i \neq \emptyset$. Then there exists $V \in \mathcal{U}_i$ such that $\text{cl}(U) \subseteq V$. Then clearly $V \cap A_i \neq \emptyset$, thus $V \not\subseteq G_i$, so $\text{cl}(V) \cap A_{i-1} \neq \emptyset$. Hence $V \cap L \in \mathcal{W}_i$ and it contains $U \cap L$.

If $n = 1$ then $\mathcal{H}_\infty^{d-1+1/i}(L) \leq \mathcal{H}_\infty^{d-1+1/i}(\bigcup \mathcal{W}_i) \leq 1/i$ for all $i \in \mathbb{N}^+$, therefore Fact 2.2 yields $\dim_{t^0 H} L = \dim_H L \leq d - 1$, and we are done. If $n > 1$ then applying Lemma 4.9 with $s = d - n$ and $m = n$ for the open covers \mathcal{W}_i implies $d - 1 \in P_{c^{n-1} H}(L)$. Then the inductive hypothesis yields $d - 1 \in P_{t^{n-1} H}(L)$, thus $\dim_{t^{n-1} H} L \leq d - 1$. The proof is complete. \square

5. THE POSSIBLE VALUES OF INDUCTIVE TOPOLOGICAL HAUSDORFF DIMENSIONS

In this section we provide a complete description of the possible values of the inductive topological Hausdorff dimensions. We prove that all values satisfying the conditions of Facts 3.1, 3.2, and 3.4 can be realized even by compact metric spaces. This implies that these dimensions are new and independent in the following sense: The n th inductive topological Hausdorff dimension is not the function of the topological dimension and the k th inductive topological Hausdorff dimensions, where k runs over $\mathbb{N} \setminus \{n\}$. This generalizes a theorem in [1] concerning the possible values of the topological Hausdorff dimension. The material developed here will be not used in the subsequent sections.

First we need some preparation. By product of two metric spaces (X, d_X) and (Y, d_Y) we mean the l^2 -product, that is,

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

Now we recall a well-known statement, see [6, Chapters 3] and [6, Product formula 7.3] for the definition of the upper box dimension and the proof, respectively. In fact, [6] works in Euclidean spaces only, but the proof goes through verbatim to general metric spaces.

Lemma 5.1. *Let X, Y be non-empty metric spaces and let us denote by $\overline{\dim}_B$ the upper box dimension. Then*

$$\dim_H(X \times Y) \leq \dim_H X + \overline{\dim}_B Y.$$

For the sake of notational simplicity we adopt the convention that $[0, 1]^0 = \{0\}$. Let us recall that $\dim_{t^0 H} X = \dim_H X$.

Lemma 5.2. *Let X be a non-empty separable metric space and let $n \in \mathbb{N}$. Then*

$$\dim_{t^n H}(X \times [0, 1]^n) = \dim_H(X \times [0, 1]^n) = \dim_H X + n.$$

Proof. From Theorem 3.5 it follows that $\dim_{t^n H}(X \times [0, 1]^n) \leq \dim_H(X \times [0, 1]^n)$.

Applying Lemma 5.1 for $Y = [0, 1]^n$ we deduce that

$$\dim_H(X \times [0, 1]^n) \leq \dim_H X + \overline{\dim}_B [0, 1]^n = \dim_H X + n.$$

Finally, we prove that $\dim_H X + n \leq \dim_{t^n H}(X \times [0, 1]^n)$. Let us define

$$\text{pr}_X: X \times [0, 1]^n \rightarrow X, \quad \text{pr}_X(x, y) = x.$$

Let $Z = X \times [0, 1]^n$. As $\dim_t Z \geq n$, Theorem 4.4 (moreover, the easy Lemma 4.10) yields that there is a set $A \subseteq Z$ such that $\dim_H A \leq \dim_{t^n H} Z - n$ and $\dim_t(Z \setminus A) \leq n - 1$. Then $\dim_t(Z \setminus A) \leq n - 1 < n = \dim_t [0, 1]^n$ implies that A intersects $\{x\} \times [0, 1]^n$ for all $x \in X$, thus $\text{pr}_X(A) = X$. Projections do not increase the Hausdorff dimension, thus

$$\dim_{t^n H} Z - n \geq \dim_H A \geq \dim_H \text{pr}_X(A) = \dim_H X.$$

Hence $\dim_{t^n H}(X \times [0, 1]^n) \geq \dim_H X + n$, and the proof is complete. \square

Applying the above lemma for $X \times [0, 1]^{k-n}$ in place of X yields the following.

Corollary 5.3. *Let X be a non-empty separable metric space and let $n, k \in \mathbb{N}$ with $n \leq k$. Then*

$$\dim_{t^n H} (X \times [0, 1]^k) = \dim_H X + k.$$

Let X be a non-empty metric space. If $\dim_t X = \infty$ then Fact 3.1 implies that $\dim_{t^k H} X = \infty$ for all $k \in \mathbb{N}$, thus we may assume that $\dim_t X = n$ for some $n \in \mathbb{N}$. Fact 3.2 yields $\dim_{t^k H} X = n$ for all $k > n$, therefore it is enough to describe the possible $(n+1)$ -tuples $(\dim_{t^n H} X, \dots, \dim_{t^0 H} X)$. Facts 3.1 and 3.4 imply that

$$n \leq \dim_{t^n H} X \leq \dots \leq \dim_{t^0 H} X.$$

The following theorem claims that the above inequality is the only constraint.

Theorem 5.4. *Let $n \in \mathbb{N}$ and let $d_0, \dots, d_n \in [n, \infty]$ such that $d_n \leq \dots \leq d_0$. Then there exists a compact metric space K such that $\dim_t K = n$ and $\dim_{t^k H} K = d_k$ for all $k \in \{0, \dots, n\}$.*

Proof. For all $i \in \{0, \dots, n\}$ let K_i be a compact metric space with $\dim_t K_i = 0$ and $\dim_H K_i = d_i - i$. It is well-known that there exist such Cantor spaces: If $d_i < \infty$ then K_i can be constructed in a Euclidean space, if $d_i = \infty$ then let $K_i = \prod_{m=1}^{\infty} \frac{C}{m}$ endowed with the l_2 -metric, where C is the classical ‘middle-third’ Cantor set. Let

$$K = \bigcup_{i=0}^n (K_i \times [0, 1]^i),$$

where the union is understood as the disjoint sum of metric spaces.

Let $i \in \{0, \dots, n\}$. Then the Cartesian product theorem [4, 1.5.16.] implies $\dim_t (K_i \times [0, 1]^i) \leq \dim_t K_i + \dim_t [0, 1]^i = i$. By monotonicity we obtain that $\dim_t (K_i \times [0, 1]^i) \geq \dim_t [0, 1]^i = i$, thus $\dim_t (K_i \times [0, 1]^i) = i$. The countable stability of the topological dimension for closed sets [4, 1.5.3.] yields that

$$\dim_t K = \max_{i \in \{0, \dots, n\}} \dim_t (K_i \times [0, 1]^i) = \max_{i \in \{0, \dots, n\}} i = n.$$

Let $i, k \in \{0, \dots, n\}$. If $i < k$, then $\dim_t (K_i \times [0, 1]^i) = i < k$, therefore Fact 3.2 yields that $\dim_{t^k H} (K_i \times [0, 1]^i) = i$. If $i \geq k$ then Corollary 5.3 implies $\dim_{t^k H} (K_i \times [0, 1]^i) = \dim_H K_i + i = d_i$. Therefore the stability of the k th inductive topological Hausdorff dimension for closed sets yields that

$$\dim_{t^k H} K = \max_{i \in \{0, \dots, n\}} \dim_{t^k H} (K_i \times [0, 1]^i) = \max\{0, \dots, k-1, d_k, \dots, d_n\} = d_k.$$

This completes the proof. \square

Let $n \in \mathbb{N}$ be arbitrary. The above theorem yields that there exist compact metric spaces X, Y such that $\dim_t X = \dim_t Y$ and $\dim_{t^k H} X = \dim_{t^k H} Y$ for all $k \in \mathbb{N} \setminus \{n\}$ but $\dim_{t^n H} X \neq \dim_{t^n H} Y$. This implies the following corollary.

Corollary 5.5. *Assume that $n \in \mathbb{N}$. Then $\dim_{t^n H} X$ cannot be calculated from $\dim_t X$ and $\dim_{t^k H} X$ ($k \in \mathbb{N} \setminus \{n\}$), even for compact metric spaces.*

6. THE PROOF OF THE MAIN THEOREM

The goal of this section is to prove Theorem 6.12 based on Section 4. In order to do so we need two new equivalent definitions for the n th inductive topological Hausdorff dimension in compact metric spaces.

For a compact metric space K and $n \in \mathbb{N}^+$ let us denote by $C_n(K)$ the space of continuous functions from K to \mathbb{R}^n equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments.

Let us first note that the case $\dim_t K < n$ is completed by the following theorem of Hurewicz, see [10, p. 124.].

Theorem 6.1 (Hurewicz). *If K is a compact metric space with $\dim_t K < n$ then $\#f^{-1}(y) \leq n$ for the generic $f \in C_n(K)$ for all $y \in \mathbb{R}^n$.*

Corollary 6.2. *If K is a compact metric space with $\dim_t K < n$ then every non-empty fiber of the generic $f \in C_n(K)$ is of Hausdorff dimension 0.*

Hence from now on we assume that $n \in \mathbb{N}^+$ and a compact metric space K are given such that $\dim_t K \geq n$.

Definition 6.3. Let

$$P_l^n = \{d \geq n : \exists G \subseteq K \text{ such that } \dim_H G \leq d - n \text{ and we have} \\ \#(f^{-1}(y) \setminus G) \leq n \text{ for the generic } f \in C_n(K) \text{ for all } y \in \mathbb{R}^n\}.$$

Definition 6.4. We say that $f \in C_n(K)$ is *d-level narrow*, if there exists a dense set $S_f \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \leq d - n$ for every $y \in S_f$. Let $\mathcal{N}_n(d)$ be the set of d -level narrow functions. Define

$$P_w^n = \{d \geq n : \mathcal{N}_n(d) \text{ is somewhere dense in } C_n(K)\}.$$

We assume $\infty \in P_l^n, P_w^n$. The characters l and w come from the first and last letters of the words level set and narrow, respectively.

For the definitions of $P_{t^n H} = P_{t^n H}(K)$, $P_{d^n H} = P_{d^n H}(K)$ and $P_{p^n H} = P_{p^n H}(K)$ see Definition 4.3 again. Now we show the following theorem.

Theorem 6.5. *If K is a compact metric space with $\dim_t K \geq n$ then*

$$P_{t^n H} = P_l^n = P_w^n.$$

Theorem 6.5 and Theorem 4.4 immediately yield two new equivalent definitions for the n th inductive topological Hausdorff dimension.

Theorem 6.6. *If K is a compact metric space with $\dim_t K \geq n$ then*

$$\dim_{t^n H} K = \min P_l^n = \min P_w^n$$

Before proving Theorem 6.5 we need the following well-known lemma.

Lemma 6.7. *Let $K_1 \subseteq K_2$ be compact metric spaces and*

$$R: C_n(K_2) \rightarrow C_n(K_1), \quad R(f) = f|_{K_1}.$$

If $\mathcal{F} \subseteq C_n(K_1)$ is co-meager then so is $R^{-1}(\mathcal{F}) \subseteq C_n(K_2)$.

Proof. The map R is clearly continuous. Using the Tietze Extension Theorem it is not difficult to see that it is also open. We may assume that \mathcal{F} is a dense G_δ set in $C_n(K_1)$. The continuity of R implies that $R^{-1}(\mathcal{F})$ is also G_δ , thus it is enough

to prove that $R^{-1}(\mathcal{F})$ is dense in $C_n(K_2)$. Let $\mathcal{U} \subseteq C_n(K_2)$ be non-empty open, then $R(\mathcal{U}) \subseteq C_n(K_1)$ is also non-empty open, hence $R(\mathcal{U}) \cap \mathcal{F} \neq \emptyset$, and therefore $\mathcal{U} \cap R^{-1}(\mathcal{F}) \neq \emptyset$. \square

The following four lemmas clearly conclude the proof of Theorem 6.5.

Lemma 6.8. $P_{p^n H} \subseteq P_{t^n H} \subseteq P_{d^n H}$.

Proof. Theorem 4.6 yields the statement. \square

Lemma 6.9. $P_{d^n H} \subseteq P_{l^n}$.

Proof. Assume $d \in P_{d^n H}$ and $d < \infty$. There is a set $G \subseteq K$ such that $\dim_H G \leq d - n$ and $\dim_t(K \setminus G) \leq n - 1$. By taking a G_δ hull of the same Hausdorff dimension we can assume that G is G_δ . As $K \setminus G$ is F_σ , we can choose compact sets K_i such that $K \setminus G = \bigcup_{i=1}^\infty K_i$ and $K_i \subseteq K_{i+1}$ for all $i \in \mathbb{N}^+$. For all $i \in \mathbb{N}^+$ let

$$\mathcal{F}_i = \{f \in C_n(K_i) : \forall y \in \mathbb{R}^n, \#f^{-1}(y) \leq n\},$$

and let $R_i: C_n(K) \rightarrow C_n(K_i)$ defined as $R_i(f) = f|_{K_i}$. As $\dim_t K_i \leq \dim_t(K \setminus G) \leq n - 1$, Theorem 6.1 implies that the sets $\mathcal{F}_i \subseteq C_n(K_i)$ are co-meager. Lemma 6.7 yields that $R_i^{-1}(\mathcal{F}_i) \subseteq C_n(K)$ are co-meager, too. As a countable intersection of co-meager sets $\mathcal{F} = \bigcap_{i=1}^\infty R_i^{-1}(\mathcal{F}_i) \subseteq C_n(K)$ is also co-meager. Clearly, every $f \in \mathcal{F}$ satisfies $\#(f^{-1}(y) \cap K_i) \leq n$ for all $y \in \mathbb{R}^n$ and $i \in \mathbb{N}^+$, so $\bigcup_{i=1}^\infty K_i = K \setminus G$ and $K_i \subseteq K_{i+1}$ ($i \in \mathbb{N}^+$) yield $\#(f^{-1}(y) \setminus G) \leq n$ for all $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$. Hence $d \in P_{l^n}$. \square

Lemma 6.10. $P_{l^n} \subseteq P_{w^n}$.

Proof. Assume $d \in P_{l^n}$ and $d < \infty$. The definition of P_{l^n} yields that there exists $G \subseteq K$ such that $\dim_H G \leq d - n$ and $\#(f^{-1}(y) \setminus G) \leq n$ for the generic $f \in C_n(K)$ for all $y \in \mathbb{R}^n$. Then $\dim_H G \leq d - n$ and $d \geq n$ yield $\dim_H f^{-1}(y) \leq d - n$, so $\mathcal{N}_n(d)$ is co-meager, thus (everywhere) dense. Hence $d \in P_{w^n}$. \square

Lemma 6.11. $P_{w^n} \subseteq P_{p^n H}$.

Proof. Assume $d \in P_{w^n}$ and $d < \infty$. Then we can fix $f \in C_n(K)$ and $\varepsilon > 0$ such that $\mathcal{N}_n(d)$ is dense in $B(f, \varepsilon)$. The uniform continuity of f implies that there is a $\delta > 0$ such that if $A \subseteq K$ with $\text{diam } A \leq \delta$ then $\text{diam } f(A) < \varepsilon/n$.

Now we prove that there is a compact set $C \subseteq K$ such that $\text{diam } C \leq \delta$ and $P_{p^n H}(C) = P_{p^n H}(K)$. Let us write K as a union of finitely many compact sets with diameter at most δ . The countable stability of the n th inductive topological Hausdorff dimension for closed sets implies that this union has a compact member $C \subseteq K$ such that $\dim_{t^n H} C = \dim_{t^n H} K$, and $\text{diam } C \leq \delta$ by definition. Theorem 4.4 yields that $\dim_{t^n H} C = \min P_{p^n H}(C)$ and $\dim_{t^n H} K = \min P_{p^n H}(K)$, so $\min P_{p^n H}(C) = \min P_{p^n H}(K)$. Hence $P_{p^n H}(C) = P_{p^n H}(K)$, because both sets are of the form $[r, \infty]$.

Finally, it is enough to show $d \in P_{p^n H}(C)$. Let $(A_1, B_1), \dots, (A_n, B_n)$ be arbitrary pairs of disjoint closed subsets of C . We need to show that for all $i \in \{1, \dots, n\}$ there are partitions $L_i \subseteq C$ between A_i and B_i such that $\dim_H(\bigcap_{i=1}^n L_i) \leq d - n$. Let $f_1, \dots, f_n \in C_1(K)$ be such that $f = (f_1, \dots, f_n)$ and observe that we may construct for all $i \in \{1, \dots, n\}$ functions $g_i \in C_1(K)$ such that

- (i) $\max g_i(A_i) < \min g_i(B_i)$;
- (ii) $g_i \in B(f_i, \varepsilon/n)$;

(iii) The function $g = (g_1, \dots, g_n) \in C_n(K)$ satisfies $g \in \mathcal{N}_n(d)$.

Indeed, as $\text{diam } f_i(C) \leq \text{diam } f(C) < \varepsilon/n$, for every $i \in \{1, \dots, n\}$ we can define g_i first on $A_i \cup B_i$ and then we can extend it to K by the Tietze Extension Theorem such that (i) and (ii) hold. Property (ii) implies that $g = (g_1, \dots, g_n) \in B(f, \varepsilon)$. As $g \in B(f, \varepsilon)$ and $\mathcal{N}_n(d)$ is dense in $B(f, \varepsilon)$, we may assume that $g \in \mathcal{N}_n(d)$, so (iii) holds.

As $g \in \mathcal{N}_n(d)$, there is a dense set $S_g \subset \mathbb{R}^n$ such that $\dim_H g^{-1}(s) \leq d - n$ for all $s \in S_g$, see Definition 6.4. We can choose $s = (s_1, \dots, s_n) \in S_g$ such that for every $i \in \{1, \dots, n\}$ its i th coordinate s_i satisfies $\max g_i(A_i) < s_i < \min g_i(B_i)$. Let us define for all $i \in \{1, \dots, n\}$

$$S_i = \{(y_1, \dots, y_n) \in g(K) : y_i = s_i\},$$

then (i) implies that S_i is a partition between $g(A_i)$ and $g(B_i)$ in $g(K)$ for every $i \in \{1, \dots, n\}$. For all $i \in \{1, \dots, n\}$ let us define $L_i = (g|_C)^{-1}(S_i)$. Then L_i is a partition between A_i and B_i in C such that

$$\bigcap_{i=1}^n L_i = \bigcap_{i=1}^n (g|_C)^{-1}(S_i) = (g|_C)^{-1} \left(\bigcap_{i=1}^n S_i \right) = (g|_C)^{-1}(s) \subseteq g^{-1}(s).$$

Therefore $s \in S_g$ implies

$$\dim_H \left(\bigcap_{i=1}^n L_i \right) \leq \dim_H g^{-1}(s) \leq d - n,$$

thus $d \in P_{p^n H}(C)$. The proof is complete. \square

Now we are ready to prove the Main Theorem.

Theorem 6.12 (Main Theorem). *Let $n \in \mathbb{N}^+$ and assume that K is a compact metric space with $\dim_t K \geq n$. Then there exists a G_δ set $G \subseteq K$ with $\dim_H G = \dim_{t^n H} K - n$ such that for the generic $f \in C_n(K)$*

- (i) $\#(f^{-1}(y) \setminus G) \leq n$ for all $y \in \mathbb{R}^n$, thus $\dim_H f^{-1}(y) \leq \dim_{t^n H} K - n$ for all $y \in \mathbb{R}^n$,
- (ii) for every $d < \dim_{t^n H} K$ there exists a non-empty open ball $U_{f,d} \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \geq d - n$ for every $y \in U_{f,d}$.

Proof. Theorem 6.6 implies that $\dim_{t^n H} K = \min P_{t^n}$, so there exists a set $G \subseteq K$ with $\dim_H G = \dim_{t^n H} K - n$ such that $\#(f^{-1}(y) \setminus G) \leq n$ for the generic $f \in C_n(K)$ for all $y \in \mathbb{R}^n$. By taking a G_δ hull of the same Hausdorff dimension we can assume that G is G_δ . Then $\dim_{t^n H} K = \min P_{t^n} \geq n$ yields $\dim_H G = \dim_{t^n H} K - n \geq 0$, thus $\dim_H f^{-1}(y) \leq \dim_H G = \dim_{t^n H} K - n$ for the generic $f \in C_n(K)$ for all $y \in \mathbb{R}^n$. Hence (i) holds.

Let us now prove (ii). Choose a sequence $d_k \nearrow \dim_{t^n H} K$. Theorem 6.6 implies that $d_k < \dim_{t^n H} K = \min P_{w^n}$ for every $k \in \mathbb{N}^+$, so $\mathcal{N}_n(d_k)$ is nowhere dense by the definition of P_{w^n} . It follows from the definition of $\mathcal{N}_n(d)$ that for every $f \in C_n(K) \setminus \mathcal{N}_n(d_k)$ there exists a non-empty open ball $U_{f,d_k} \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \geq d_k - n$ for every $y \in U_{f,d_k}$. But then (ii) holds for every $f \in C_n(K) \setminus (\bigcup_{k=1}^\infty \mathcal{N}_n(d_k))$, and this latter set is clearly co-meager, which concludes the proof of the theorem. \square

Corollary 6.13. *If K is a compact metric space with $\dim_t K \geq n$ then for the generic $f \in C_n(K)$*

$$\sup_{y \in \mathbb{R}^n} \dim_H f^{-1}(y) = \dim_{t^n} K - n.$$

7. STRENGTHENING OF THE MAIN THEOREM

The proofs of this section are more or less analogous to the proofs of the one-dimensional results in [2], so we only describe the necessary modifications.

Let us fix a compact metric space K and $n \in \mathbb{N}^+$. If $\dim_t K < n$ then the fibers of the generic map $f \in C_n(K)$ are finite, see Theorem 6.1.

Thus we assume $\dim_t K \geq n$ in the sequel.

7.1. Fibers of maximal dimension. Corollary 6.13 states that if $\dim_t K \geq n$ then $\sup_{y \in \mathbb{R}^n} \dim_H f^{-1}(y) = \dim_{t^n} K - n$ for the generic $f \in C_n(K)$. We show that in this statement the supremum is attained.

Theorem 7.1. *Let K be a compact metric space with $\dim_t K \geq n$. Then for the generic $f \in C_n(K)$*

$$\max_{y \in \mathbb{R}^n} \dim_H f^{-1}(y) = \dim_{t^n} K - n.$$

Proof. Buczolicz, Elekes and the author proved this theorem for $n = 1$, see [2, Thm. 4.1.]. The proof goes through with the obvious changes. The only significant modification is that we need to apply Lemma 7.2 instead of its one-dimensional special case [2, Lemma 2.14.]. \square

Lemma 7.2. *Let K be a compact metric space with a fixed $x_0 \in K$. Let $K_i \subseteq K$, $i \in \mathbb{N}$ be compact sets such that*

- (i) $\dim_t K_i \geq n$ for all $i \in \mathbb{N}$ and
- (ii) $\text{diam}(K_i \cup \{x_0\}) \rightarrow 0$ if $i \rightarrow \infty$.

Then for the generic $f \in C_n(K)$ we have $f(x_0) \in f(K_i)$ for infinitely many $i \in \mathbb{N}$.

Proof. Clearly it is enough to show that the sets

$$\mathcal{F}_k = \{f \in C_n(K) : f(x_0) \notin f(K_i) \text{ for all } i \geq k\}$$

are nowhere dense in $C_n(K)$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$, $f_0 \in C_n(K)$ and $r > 0$ be arbitrarily fixed, it is enough to find a ball in $B(f_0, 3r) \setminus \mathcal{F}_k$. We may assume that $x_0 \notin K_i$ for all $i \geq k$, otherwise $\mathcal{F}_k = \emptyset$ and the statement is obvious. By the continuity of f_0 and (ii) we can fix $m \geq k$ such that $f_0(K_m) \subseteq B(f_0(x_0), r)$. As $\dim_t K_m \geq n$, by [7, Thm. VI 2.] there is a continuous map $g_0: K_m \rightarrow \mathbb{R}^n$ with a stable value, that is, there exist $y \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $y \in g(K_m)$ for all $g \in B(g_0, 2\varepsilon)$. Therefore $B(y, \varepsilon) \subseteq g(K_m)$ for all $g \in B(g_0, \varepsilon)$. By applying an affine transformation we may assume that $y = f_0(x_0)$ and $g_0(K_m) \subseteq B(f_0(x_0), r)$, so $\varepsilon \leq r$. Then $x_0 \notin K_m$, $f_0(K_m) \cup g_0(K_m) \subseteq B(f_0(x_0), r)$ and the Tietze Extension Theorem imply that there is an $f_1 \in B(f_0, 2r)$ such that $f_1(x_0) = f_0(x_0)$ and $f_1|_{K_m} = g_0$. For all $f \in B(f_1, \varepsilon)$ we have $f(x_0) \in B(f_1(x_0), \varepsilon) = B(y, \varepsilon)$ and $f|_{K_m} \in B(g_0, \varepsilon)$ implies $B(y, \varepsilon) \subseteq f(K_m)$. Hence $f(x_0) \in f(K_m)$, so $f \notin \mathcal{F}_k$. Thus $B(f_1, \varepsilon) \cap \mathcal{F}_k = \emptyset$, so $f_1 \in B(f_0, 2r)$ and $\varepsilon \leq r$ imply $B(f_1, \varepsilon) \subseteq B(f_0, 3r) \setminus \mathcal{F}_k$. \square

Remark 7.3. In [2] a compact set $K \subseteq \mathbb{R}^2$ is constructed such that the generic $f \in C_1(K)$ has a unique level set of maximal Hausdorff dimension. Therefore we cannot strengthen Theorem 7.1 in general.

7.2. Fibers on fractals. If K is sufficiently homogenous then we can improve the Main Theorem.

Definition 7.4. If K is a compact metric space then let

$$\text{supp}_n K = \{x \in K : \forall r > 0, \dim_{t^n H} B(x, r) = \dim_{t^n H} K\}.$$

We say that K is *homogeneous for the n th inductive topological Hausdorff dimension* if $\text{supp}_n K = K$.

Remark 7.5. The stability of the n th inductive topological Hausdorff dimension for closed sets clearly yields $\text{supp}_n K \neq \emptyset$. Corollary 3.9 implies that if K is self-similar then it is also homogeneous for the n th inductive topological Hausdorff dimension.

Theorem 7.6. *Let K be a compact metric space with $\dim_t K \geq n$. The following statements are equivalent:*

- (i) $\dim_H f^{-1}(y) = \dim_{t^n H} K - n$ for the generic $f \in C_n(K)$ for generic $y \in f(K)$;
- (ii) K is homogeneous for the n th inductive topological Hausdorff dimension.

Proof. The proof of [2, Thm. 3.3.] works with the obvious modifications. Let us note that the half of this proof is actually in [1]. \square

Definition 7.7. Let K be a compact metric space. We say that K is *weakly self-similar* if for all $x \in K$ and $r > 0$ there exist a compact set $K_{x,r} \subseteq B(x, r)$ and a bi-Lipschitz map $\phi_{x,r}: K_{x,r} \rightarrow K$.

Remark 7.8. If K is self-similar then it is also weakly self-similar. If K is weakly self-similar then Corollary 3.9 yields that it is homogeneous for the n th inductive topological Hausdorff dimension.

The following theorem is the main result of the section, it generalizes Kirchheim's theorem for weakly self-similar compact metric spaces.

Theorem 7.9. *Let K be a weakly self-similar compact metric space such that $\dim_t K \geq n$. Then for the generic $f \in C_n(K)$ for any $y \in \text{int } f(K)$*

$$\dim_H f^{-1}(y) = \dim_{t^n H} K - n.$$

Before proving Theorem 7.9 we need a definition and a lemma. Basically we follow the proof of [2, Thm. 3.6.], but Lemma 7.11 is not similar to [2, Lemma 2.12.] and their applications are also different.

Definition 7.10. Let K be a compact metric space and $n \in \mathbb{N}^+$. For all $m \in \mathbb{N}^+$ consider

$$\mathcal{D}_m = \{f \in C_n(K) : \exists \varepsilon > 0 \text{ such that for all } g \in B(f, \varepsilon) \text{ and for all } y \in g(K) \setminus B(\partial g(K), 1/m) \text{ we have } y \in f(K)\}.$$

If $f \in \mathcal{D}_m$ then one can fix a witness $\varepsilon(f, m) > 0$ corresponding to the definition.

Lemma 7.11. *Let K be a compact metric space such that $B(x, r)$ is uncountable for all $x \in K$ and $r > 0$. If $m, n \in \mathbb{N}^+$ then $\mathcal{D}_m = \mathcal{D}_m(K, n)$ is dense in $C_n(K)$.*

Proof. Let $f_0 \in C_n(K)$ and $r > 0$ be given, we need to show that $\mathcal{D}_m \cap B(f_0, 3r) \neq \emptyset$. Since K is compact and f_0 is uniformly continuous, there are finitely many distinct $x_1, \dots, x_k \in K$ and $\delta > 0$ such that

$$(7.1) \quad K = \bigcup_{i=1}^k B(x_i, \delta)$$

and for all $i \in \{1, \dots, k\}$

$$(7.2) \quad f_0(B(x_i, \delta)) \subseteq B(f_0(x_i), r/n).$$

Choose $0 < \delta' < \delta$ such that the balls $B(x_i, \delta')$ are disjoint. As the balls $B(x_i, \delta'/2)$ are uncountable, for all $i \in \{1, \dots, k\}$ there are sets $C_i \subseteq B(x_i, \delta'/2)$ homeomorphic to the triadic Cantor set, see [8, 6.5. Corollary].

Let $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ be the standard basis of \mathbb{R}^n . For $y \in \mathbb{R}^n$ and $d > 0$ let us denote by $Q(y, d)$ the n -cube with center y and edge length d homothetic to $[0, 1]^n$. For all $i \in \{1, \dots, k\}$ choose $d_i \in [2r/n, 3r/n]$ such that the $2nk$ many hyperplanes determined by the faces of the cubes $Q_i = Q(f_0(x_i), d_i)$ are distinct. For all $j \in \{1, \dots, n\}$ let us denote by \mathcal{S}_j the collection of those hyperplanes according to these cubes that are orthogonal to e_j . Set

$$\theta = \min \{ \text{dist}(S, S') : S, S' \in \mathcal{S}_j, S \neq S', 1 \leq j \leq n \} > 0.$$

Now we construct $f \in B(f_0, 3r)$ such that $f(B(x_i, \delta)) = Q_i$ for all $i \in \{1, \dots, k\}$. As C_i are homeomorphic to the triadic Cantor set, by [8, 4.18. Thm.] there are continuous onto maps $g_i: C_i \rightarrow Q_i$. Then $d_i \geq 2r/n$ and (7.2) imply $f_0(B(x_i, \delta)) \subseteq Q_i$, so applying the Tietze Extension Theorem for the coordinate functions yields that for all $i \in \{1, \dots, k\}$ there are functions $\hat{g}_i: B(x_i, \delta) \rightarrow Q_i$ such that $\hat{g}_i = g_i$ on C_i and $\hat{g}_i = f_0$ on $B(x_i, \delta) \setminus U(x_i, \delta')$. Let $f(x) = \hat{g}_i(x)$ for all $x \in B(x_i, \delta)$ and $i \in \{1, \dots, k\}$, then (7.1) implies that f is defined for all $x \in K$. The construction easily yields that $f \in C_n(K)$ is well-defined and $f(B(x_i, \delta)) = Q_i$ for all $i \in \{1, \dots, k\}$, so $f(K) = \bigcup_{i=1}^k Q_i$. Since $f_0(B(x_i, \delta)) \cup f(B(x_i, \delta)) \subseteq Q_i$ and $\text{diam } Q_i = \sqrt{n}d_i \leq \sqrt{n}3r/n \leq 3r$, we obtain that $f \in B(f_0, 3r)$.

Finally, we prove that $\varepsilon = \min\{\theta/4, 1/(2mn)\} > 0$ witnesses $f \in \mathcal{D}_m$. Let $g \in B(f, \varepsilon)$ and $y_0 \in g(K) \setminus B(\partial g(K), 1/m)$, and assume to the contrary that $y_0 \notin f(K)$. We construct points $y_1, \dots, y_n \notin f(K)$ near y_0 such that some properties of y_n lead to a contradiction. Assume by induction that $y_{j-1} \notin f(K)$ is already defined for some $j \in \{1, \dots, n\}$. The definition of θ with $2\varepsilon \leq \theta/2$ and $y_{j-1} \notin f(K) = \bigcup_{i=1}^k Q_i$ imply that there exists $c_j \in [-2\varepsilon, 2\varepsilon]$ such that $y_{j-1} + c_j e_j \notin f(K)$ and

$$(7.3) \quad \min_{S \in \mathcal{S}_j} \text{dist}(\{y_{j-1} + c_j e_j\}, S) \geq 2\varepsilon.$$

Let $y_j = y_{j-1} + c_j e_j$, then y_1, \dots, y_n are defined. The construction yields that $y_n - y_j$ are parallel to e_j , so $\text{dist}(\{y_n\}, S) = \text{dist}(\{y_j\}, S)$ for all $j \in \{1, \dots, n\}$ and $S \in \mathcal{S}_j$. Therefore $y_n \notin f(K) = \bigcup_{i=1}^k Q_i$ and (7.3) imply

$$(7.4) \quad \begin{aligned} \text{dist}(\{y_n\}, f(K)) &\geq \min_{1 \leq j \leq n} \min_{S \in \mathcal{S}_j} \text{dist}(\{y_n\}, S) \\ &= \min_{1 \leq j \leq n} \min_{S \in \mathcal{S}_j} \text{dist}(\{y_j\}, S) \geq 2\varepsilon. \end{aligned}$$

Then $y_0 \in g(K) \setminus B(\partial g(K), 1/m)$ implies that $B(y_0, 1/m) \subseteq g(K)$, and

$$|y_n - y_0| = \sqrt{\sum_{j=1}^n c_j^2} \leq 2\varepsilon\sqrt{n} \leq 1/m.$$

Hence $y_n \in g(K)$. Choose $x \in K$ such that $g(x) = y_n$, then $g \in B(f, \varepsilon)$ yields $|f(x) - y_n| = |f(x) - g(x)| \leq \varepsilon$, thus $\text{dist}(\{y_n\}, f(K)) \leq \varepsilon$, but this contradicts (7.4). Therefore $\mathcal{D}_m \cap B(f_0, 3r) \neq \emptyset$, and the proof is complete. \square

Proof of Theorem 7.9. The Main Theorem yields $\dim_H f^{-1}(y) \leq \dim_{t^n H} K - n$ for the generic $f \in C_n(K)$ for all $y \in \mathbb{R}^n$, thus we only need to verify the opposite inequality.

Fact 3.1 implies $\dim_{t^n H} K \geq \dim_t K \geq n$, therefore we can choose a sequence $n-1 < d_m \nearrow \dim_{t^n H} K$. Let us fix $m \in \mathbb{N}^+$. The Main Theorem implies that for the generic $f \in C_n(K)$ there exists a non-empty open set $U_{f, d_m} \subseteq \mathbb{R}^n$ such that $\dim_H f^{-1}(y) \geq d_m - n$ for all $y \in U_{f, d_m}$.

By the Baire Category Theorem there are $0 < r_1 < r_2$ and $y_0 \in \mathbb{R}^n$ such that

$$\mathcal{H}_m = \{f \in C_n(K) : f(K) \subseteq B(y_0, r_2) \text{ and we have} \\ \dim_H f^{-1}(y) \geq d_m - n \text{ for all } y \in B(y_0, r_1)\}$$

is of second category. Note that $d_m > n-1$ implies that for every $f \in \mathcal{H}_m$ we have $B(y_0, r_1) \subseteq f(K)$. Let us define

$$\mathcal{G}_m = \{f \in C_n(K) : \dim_H f^{-1}(y) \geq d_m - n \\ \text{for all } y \in f(K) \setminus B(\partial f(K), 1/m)\}.$$

Following the proof of [2, Lemma 3.7.] we obtain that \mathcal{H}_m and \mathcal{G}_m have the Baire property.

It is sufficient to verify that \mathcal{G}_m is co-meager, since by taking the intersection of the sets \mathcal{G}_m for all $m \in \mathbb{N}^+$ we obtain the desired co-meager set in $C_n(K)$. In order to prove this we show that \mathcal{G}_m contains ‘certain copies’ of \mathcal{H}_m . Since \mathcal{G}_m has the Baire property, it is enough to prove that \mathcal{G}_m is of second category in every non-empty open subset of $C_n(K)$. As $\dim_t K \geq n$, we obtain that K is uncountable, and the weak self-similarity of K yields that $B(x, r)$ is uncountable for all $x \in K$ and $r > 0$. Hence we can apply Lemma 7.11. Let us fix an arbitrary $f_0 \in \mathcal{D}_m$ and a witness $\varepsilon = \varepsilon(f_0, m) > 0$ corresponding to Definition 7.10. As \mathcal{D}_m is dense in $C_n(K)$ by Lemma 7.11, it is enough to show that $\mathcal{G}_m \cap B(f_0, \varepsilon)$ is of second category.

Since K is compact and f_0 is uniformly continuous, there are finitely many distinct $x_1, \dots, x_k \in K$ and $\delta > 0$ such that

$$(7.5) \quad K = \bigcup_{i=1}^k B(x_i, \delta)$$

and for each $i \in \{1, \dots, k\}$ the oscillation of f_0 on $B(x_i, \delta)$ is less than

$$(7.6) \quad \omega = \frac{\varepsilon r_1}{2r_2} < \frac{\varepsilon}{2}.$$

Choose $0 < \delta' < \delta$ such that the balls $B(x_i, \delta')$ are disjoint. Using the weak self-similarity property we can choose for every $i \in \{1, \dots, k\}$ a set $K_i \subseteq B(x_i, \delta')$

and a bi-Lipschitz map $\phi_i: K_i \rightarrow K$. Let us fix $i \in \{1, \dots, k\}$. We define the affine function $\psi_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$(7.7) \quad \psi_i(B(y_0, r_1)) = B(f_0(x_i), \omega).$$

Let $G_i: C_n(K) \rightarrow C_n(K_i)$ defined by $G_i(f) = \psi_i \circ f \circ \phi_i$. The maps $\phi_i: K_i \rightarrow K$ and $\psi_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are homeomorphisms, hence G_i is a homeomorphism, too. Therefore, since \mathcal{H}_m is of second category in $C_n(K)$, we obtain that

$$\widehat{\mathcal{F}}_i = \{\psi_i \circ f \circ \phi_i : f \in \mathcal{H}_m\} = G_i(\mathcal{H}_m)$$

is of second category in $C_n(K_i)$.

Now we prove that $\widehat{\mathcal{F}}_i \subseteq B(f_0|_{K_i}, \varepsilon)$. Let $f \in \mathcal{H}_m$, then the form of ψ_i , (7.7) and (7.6) imply

$$\begin{aligned} \text{diam}(\psi_i \circ f \circ \phi_i)(K_i) &= \text{diam} \psi_i(f(K)) \leq \text{diam} \psi_i(B(y_0, r_2)) \\ &= \frac{r_2}{r_1} \text{diam} \psi_i(B(y_0, r_1)) = \frac{r_2}{r_1} 2\omega = \varepsilon. \end{aligned}$$

Then $f_0(K_i) \subseteq B(f_0(x_i), \omega) \subseteq (\psi_i \circ f \circ \phi_i)(K_i)$, so $\psi_i \circ f \circ \phi_i \in B(f_0|_{K_i}, \varepsilon)$. Set

$$\mathcal{F}_i = \{f \in B(f_0, \varepsilon) : f|_{K_i} \in \widehat{\mathcal{F}}_i\} \quad \text{and} \quad \mathcal{F} = \bigcap_{i=1}^k \mathcal{F}_i.$$

Clearly $\mathcal{F} \subseteq B(f_0, \varepsilon)$, and repeating the proof of [2, Lemma 3.8.] verbatim yields that \mathcal{F} is of second category in $B(f_0, \varepsilon)$.

Therefore it is enough to prove $\mathcal{F} \subseteq \mathcal{G}_m$, which implies that \mathcal{G}_m is of second category in $B(f_0, \varepsilon)$. Assume that $g \in \mathcal{F}$ and $y \in g(K) \setminus B(\partial g(K), 1/m)$. The definition of $\varepsilon = \varepsilon(f_0, m)$ and $g \in B(f_0, \varepsilon)$ yield $y \in f_0(K)$. Hence the definition of ω and (7.5) imply that there is an $i \in \{1, \dots, k\}$ such that $y \in B(f_0(x_i), \omega)$. The definition of \mathcal{F} yields that there exists an $f \in \mathcal{H}_m$ such that $g|_{K_i} = \psi_i \circ f \circ \phi_i$. Then (7.7) implies $\psi_i^{-1}(y) \in B(y_0, r_1)$, and $f \in \mathcal{H}_m$ yields $\dim_H f^{-1}(\psi_i^{-1}(y)) \geq d_m - n$. By the bi-Lipschitz property of ϕ_i we infer

$$\begin{aligned} \dim_H g^{-1}(y) &\geq \dim_H (g|_{K_i})^{-1}(y) \\ &= \dim_H \phi_i^{-1}(f^{-1}(\psi_i^{-1}(y))) \\ &= \dim_H f^{-1}(\psi_i^{-1}(y)) \\ &\geq d_m - n. \end{aligned}$$

Therefore $g \in \mathcal{G}_m$, so $\mathcal{F} \subseteq \mathcal{G}_m$. This completes the proof. \square

We can analogously define inductive dimensions by replacing Hausdorff dimension with packing, or lower box, or upper box dimension, respectively. However, one can show that these definitions and some natural modifications of them do not satisfy the analogous version of the Main Theorem. The reason why these concepts behave differently is that box dimensions are not even countable stable, and packing dimension does not allow to take G_δ hulls: It is easy to see that every G_δ hull of \mathbb{Q} has packing dimension 1. Therefore the following question is quite natural, the author cannot answer it even in the special case $n = 1$. For other problems concerning the topological Hausdorff dimension see [1].

Question 7.12. *What is the right notion to describe the packing, or lower box, or upper box dimension of the fibers of the generic continuous function $f \in C_n(K)$?*

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