



Generalized Turán results for intersecting cliques

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ABSTRACT

For fixed graphs F and H , the generalized Turán problem asks for the maximum number $\text{ex}(n, H, F)$ of copies of H that an n -vertex F -free graph can have. In this paper, we focus on the case when F is $B_{r,s}$, the graph consisting of two cliques of size r sharing s common vertices. We determine $\text{ex}(n, K_k, B_{r,0})$, $\text{ex}(n, K_k, B_{r,1})$ and $\text{ex}(n, K_{a,b}, B_{3,1})$ for all values of a, b, r, k if n is large enough.

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1. Introduction

A central question in extremal graph theory, the so-called Turán problem asks for the maximum number $\text{ex}(n, F)$ of edges that an n -vertex graph G can have without containing F as a subgraph. Graphs with this property are called F -free. The asymptotics of $\text{ex}(n, F)$ is given by the celebrated Erdős-Stone-Simonovits theorem [10] if the chromatic number of F is at least three. For results and open problems in the case when F is bipartite, see the survey by Füredi and Simonovits [13].

A natural generalization of this problem is to maximize the number of copies of some other graph H while forbidding F as subgraph. This maximum is denoted by $\text{ex}(n, H, F)$. More precisely, we denote by $\mathcal{N}(H, G)$ the number of (unlabeled) copies of H in G , and $\text{ex}(n, H, F) := \max\{\mathcal{N}(H, G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\}$. So the original problem is the $H = K_2$ case and $\text{ex}(n, F) = \text{ex}(n, K_2, F)$. More generally, for a family \mathcal{F} of graphs, we denote by $\text{ex}(n, H, \mathcal{F})$ the maximum number of copies of H in n -vertex graphs that do not contain any member of \mathcal{F} . After some very interesting but sporadic results [4,24,26], these so-called generalized Turán problems were first addressed systematically by Alon and Shikhelman [2].

In this paper we study the case where F consists of two cliques sharing some vertices. Let us denote by $B_{r,s}$ the graph consisting of two r -cliques sharing exactly s vertices. We also call $B_{r,s}$ a *generalized book graph*.

Let us denote by $G_1 + G_2$ the graph consisting of a vertex disjoint pair of copies of G_1 and G_2 and by kG the graph consisting of k vertex-disjoint copies of G . Let $T(m, s)$ denote the *Turán graph*, which is the complete s -partite graph on m vertices with each part having order $\lfloor m/s \rfloor$ or $\lceil m/s \rceil$. For graphs G_1, G_2 , their join $G_1 \vee G_2$ denotes the graph obtained by taking vertex disjoint copies of G_1, G_2 and joining every pair v_1, v_2 of vertices with $v_1 \in V(G_1), v_2 \in V(G_2)$. For a set $U \subset V(G)$, we denote by $G[U]$ the subgraph of G induced by U , i.e., the subgraph we obtain by deleting the vertices not in U .

As observed by Clark, Entringer, McCanna, and Székely [6], the celebrated 6-3 Theorem of Ruzsa and Szemerédi [33] can be reformulated the following way: the largest number of edges in an n -vertex graph where every edge is contained

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in exactly one triangle is $o(n^2)$ but at least $n^{2-o(1)}$. This implies the same bounds on $\text{ex}(n, K_3, B_{3,2})$. Gowers and Janzer [23] (motivated by a rainbow variant of generalized Turán problems [20]) generalized this by showing that for $2 \leq s < r$, we have $n^{s-o(1)} < \text{ex}(n, K_r, \{B_{r,s}, B_{r,s+1}, \dots, B_{r,r-1}\}) = o(n^s)$. Liu and Wang [29] initiated the study of $\text{ex}(n, K_r, B_{r,s})$. They determined its value exactly for $s = 0$ and $s = 1$ in the case n is large enough, and gave bounds in the case of other values of s .

In this paper we extend these investigations to counting other graphs. Let us first discuss other results that fit into this setting. If $s = 0$, we forbid two vertex-disjoint copies of K_r . Moon [32] showed that $\text{ex}(n, \ell K_r) = |E(K_{\ell-1} \vee T(n - \ell + 1, r - 1))|$. Concerning generalized Turán problems, $\text{ex}(n, H, \ell G)$ was studied in [21], in particular the order of magnitude of $\text{ex}(n, K_k, \ell K_r)$ was determined there.

If $s \geq 2$, then $B_{r,s}$ has a color-critical edge, i.e., an edge whose deletion decreases the chromatic number. Simonovits [34] showed that for an r -chromatic graph F with a color-critical edge, $\text{ex}(n, F) = |E(T(n, r - 1))|$, while Ma and Qiu [31] extended this result by showing that if $k < r$, then $\text{ex}(n, K_k, F) = \mathcal{N}(K_k, T(n, r - 1))$. Further results that determine the exact value of $\text{ex}(n, H, B_{r,s})$ for some classes of graphs H if n is large enough can be found in [14, 15, 17, 18, 22].

In the case $s = 1$, we have $\text{ex}(n, B_{r,1}) = |E(T(n, r - 1))| + 1$ and the extremal construction is the Turán graph with an arbitrary additional edge. This was proved in [7] for $r = 3$ and in [5] for larger r . Gerbner and Palmer [22] determined $\text{ex}(n, C_4, B_{3,1})$. The graph $B_{r,1}$ has a color-critical vertex, i.e., a vertex whose deletion decreases the chromatic number (from r to $r - 1$). Gerbner [15] determined $\text{ex}(n, H, F)$ for every r -chromatic graph F with a color-critical vertex if H is a complete balanced $(r - 1)$ -partite graph $K_{a, \dots, a}$ with a large enough. In particular, this determines $\text{ex}(n, K_{a,a}, B_{3,1})$ for every $a > 2$. Further results concerning $\text{ex}(n, H, B_{3,1})$ for bipartite graphs H can be found in [19].

For a family of graphs \mathcal{H} , we denote by $\text{ex}(n, \mathcal{H}, F)$ the largest value of $\sum_{H \in \mathcal{H}} \mathcal{N}(H, G)$ over all n -vertex F -free graphs G . The study of counting multiple graphs in generalized Turán problems was initiated in [16]. Now we are ready to state our results.

Theorem 1.1. *For any r and large enough n , we have the following:*

(i) if $k < r$, then

$$\text{ex}(n, K_k, 2K_r) = \mathcal{N}(K_k, K_1 \vee T(n - 1, r - 1)),$$

(ii) if $r \leq k < 2r$, then

$$\text{ex}(n, \{K_k, K_{k+1}, \dots, K_{2r-1}\}, 2K_r) = \mathcal{N}(K_k, K_{2k-2r+1} \vee T(n - 2k + 2r - 1, 2r - k - 1)).$$

Note that the above theorem determines $\text{ex}(n, K_k, 2K_r)$ for every pair of k and r if n is large enough. The second statement in Theorem 1.1 gives a bit more: if we count the copies of larger cliques in addition to K_k , then we obtain the same bound.

Theorem 1.2. *For any $r \geq 3$, $1 \leq s \leq r - 1$, and $1 \leq t < r - s$, we have*

$$\text{ex}(n, K_{r+t}, B_{r,s}) = \Omega(n^{r-s-t-1}).$$

For any $r \geq 3$, $2s + t + 1 < r$ and n large enough, we have

$$\text{ex}(n, K_{r+t}, B_{r,s}) = \Theta(n^{r-s-t-1}).$$

For any $t \geq 1$, $t + 3 < r$ and n large enough, we have

$$\text{ex}(n, K_{r+t}, B_{r,1}) = \mathcal{N}(K_{r+t}, K_{2t+2} \vee T(n - 2t - 2, r - t - 2)).$$

For any $r \geq 3$, we have

$$\text{ex}(n, K_{2r-2}, B_{r,1}) = \left\lfloor \frac{n}{2r-2} \right\rfloor.$$

For any $r \geq 4$, we have

$$\text{ex}(n, K_{2r-3}, B_{r,1}) = (2r - 2) \left\lfloor \frac{n}{2r-2} \right\rfloor + a,$$

where a is 1 if $n \equiv 2r - 3 \pmod{2r - 2}$ and 0 otherwise.

The above theorem deals with $\text{ex}(n, K_k, B_{r,s})$ in the case $k > r$. As we have mentioned, Liu and Wang [29] studied the case $k = r$. Consider now the case $k < r$. For $s = 0$, the value of $\text{ex}(n, K_k, B_{r,s})$ is determined exactly for sufficiently large n by Theorem 1.1. We have mentioned a result of Ma and Qiu [31] earlier about graphs with a color-critical edge, which determines $\text{ex}(n, K_k, B_{r,s})$ for sufficiently large n if $k < r$ and $s \geq 2$. We now deal with the remaining case $s = 1$. Let $T^+(n, r - 1)$ denote the graph we obtain from $T(n, r - 1)$ by adding an edge to a smallest part.

Theorem 1.3. *If $k < r$ and n is sufficiently large, then*

$$\text{ex}(n, K_k, B_{r,1}) = \mathcal{N}(K_k, T^+(n, r-1)).$$

In the case $r = 3$, the above theorem implies the result from [7] on $\text{ex}(n, K_2, B_{r,1})$. We can obtain a much more general result.

Theorem 1.4. *For any integers $a \leq b$ and n large enough, $\text{ex}(n, K_{a,b}, B_{3,1}) = \mathcal{N}(K_{a,b}, T)$ for some n -vertex graph T that is obtained from a complete bipartite graph by adding an edge to one of the parts.*

For given a and b , a straightforward optimization shows what T is. In the case $K_{a,b}$ is not a star, i.e., $a, b \geq 2$, the extra edge of T cannot be in any copy of $K_{a,b}$, thus a complete bipartite graph $K_{m,n-m}$ is also an extremal graph in addition to T .

2. Forbidding $2K_r$ and counting cliques

In this section, we prove Theorem 1.1. First we gather some results that we will use in the proof. A family \mathcal{F} of sets is t -intersecting if for any $F, F' \in \mathcal{F}$ we have $|F \cap F'| \geq t$. For a set X we denote by $\binom{X}{k}$ the family of all k -subsets of X . The set $\{1, 2, \dots, n\}$ of the first n positive integers is denoted by $[n]$, and we write $[a, b]$ for the interval $\{s \in \mathbb{N} : a \leq s \leq b\}$.

We will use the following theorem of Frankl.

Theorem 2.1 (Frankl [11]). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be t -intersecting with $|\cap_{F \in \mathcal{F}} F| < t$. If n is large enough, then $|\mathcal{F}| \leq \max\{|\mathcal{F}_1|, |\mathcal{F}_2|\}$, where*

$$\mathcal{F}_1 = \left\{ F \in \binom{[n]}{k} : [t] \subset F, F \cap [t+1, k+1] \neq \emptyset \right\} \cup \binom{[k+1]}{k}$$

and

$$\mathcal{F}_2 = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2]| \geq t+1 \right\}.$$

More precisely, the following simple corollary is enough for us.

Corollary 2.2. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be t -intersecting with $|\cap_{F \in \mathcal{F}} F| < t$. Then $|\mathcal{F}| = O(n^{k-t-1})$.*

Let us remark that later Ahlswede and Khachatrian [1] strengthened this result by determining the maximum size of a t -intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\cap_{F \in \mathcal{F}} F| < t$ for all n, k , and t .

Another tool in the proof will be the following generalization of the Erdős-Simonovits stability theorem [8,34]. We say that two graphs G and G' have *edit distance* at most x if we can obtain G' from G by adding and deleting at most x edges.

Theorem 2.3 (Ma, Qiu [31]). *Let H be a graph with $\chi(H) = r+1 > m \geq 2$. If G is an n -vertex H -free graph with $\mathcal{N}(K_m, G) \geq \mathcal{N}(K_m, T(n, r)) - o(n^m)$, then G and $T(n, r)$ have edit distance $o(n^2)$.*

We will also need the following two well-known results.

Theorem 2.4 (Zykov [35]). *For any $2 \leq k < r \leq n$ we have*

$$\text{ex}(n, K_k, K_r) = \mathcal{N}(K_k, T(n, r-1)).$$

Theorem 2.5 (Removal lemma, Erdős, Frankl, Rödl [9]). *For any graph H on h vertices and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, H)$ such that if a graph $G = (V, E)$ on n vertices contains at most δn^h copies of H , then there exists $E' \subset E$ with $|E'| \leq \varepsilon n^2$ such that the graph $(V, E \setminus E')$ is H -free.*

Proof of Theorem 1.1. Let $G = (V, E)$ be a $2K_r$ -free graph on n vertices.

Suppose first that $k < r$. Let t be the minimum number such that there exists a vertex subset $U \subset V$ of size t such that $G[V \setminus U]$ is K_r -free. Observe that $t \leq r$ as if K is a copy of K_r , then $G[V \setminus V(K)]$ must be K_r -free since G is $2K_r$ -free. By Theorem 2.4, we have $\mathcal{N}(K_k, G[V \setminus U]) \leq \mathcal{N}(K_k, T(n-t, r-1))$. If $t \leq 1$, then this immediately yields the statement of (i).

Assume $2 \leq t \leq r$. Assume that $\mathcal{N}(K_k, G) \geq \mathcal{N}(K_k, T(n-t, r-1))$. Then the number of k -cliques intersecting U is at least $\mathcal{N}(K_k, K_1 \vee T(n-1, r-1)) - \mathcal{N}(K_k, T(n-t, r-1))$. Observe that

$$\begin{aligned} \mathcal{N}(K_k, K_1 \vee T(n-1, r-1)) - \mathcal{N}(K_k, T(n-t, r-1)) \\ = (1 + o(1)) \left((t-1) \mathcal{N}(K_{k-1}, T\left(\frac{r-2}{r-1}n, r-2\right)) + \mathcal{N}(K_{k-1}, T(n-1, r-1)) \right). \end{aligned}$$

If n is large enough, then

$$\frac{\mathcal{N}(K_{k-1}, T(n-1, r-1))}{\mathcal{N}\left(K_{k-1}, T\left(\frac{r-2}{r-1}n, r-2\right)\right)} > c_r > 1$$

for some constant c_r depending only on r . This means that the number of k -cliques meeting U is at least $(t-1 + c_r + o(1))\mathcal{N}(K_{k-1}, T(\frac{r-2}{r-1}n, r-2))$. As the number of k -cliques containing at least two vertices of U is $O(n^{k-2})$, we obtain a contradiction by proving that for any $u \in U$, the number of k -sets $S \subset V$ with $S \cap U = \{u\}$ and $G[S] = K_k$ is at most $(1 + o(1))\mathcal{N}(K_{k-1}, T(\frac{r-2}{r-1}n, r-2))$.

To this end, observe first that for any $u \in U$ the number of $(r-1)$ -subsets A of $V \setminus U$ such that $\{u\} \cup A$ form an r -clique in G is $O(n^{r-2})$. Indeed, as U is minimal, for every $u' \in U$ there exists an r -clique $K_{u'}$ such that $K_{u'} \cap U = \{u'\}$. As $|U| \geq 2$, we can consider $K_{u'}$ for some $u' \neq u$ that does not contain u . Now every $(r-1)$ -subset A that forms an r -clique in G with u must meet $K_{u'}$ as G is $2K_r$ -free, so their number is at most $r \binom{n-2}{r-2}$. By Theorem 2.5, there exists a set E' of $o(n^2)$ edges whose deletion removes all these K_{r-1} s. The number of k -cliques containing u and at least one edge from E' is $o(n^{k-1})$.

As the number of k -cliques in G meeting U is $O(n^{k-1})$, we must have $\mathcal{N}(K_k, G[V \setminus U]) \geq \mathcal{N}(K_k, T(n-t, r-1)) - o(n^k)$. Theorem 2.3 implies that $G[V \setminus U]$ has edit distance $o(n^2)$ from the Turán-graph, i.e., $V \setminus U$ can be partitioned into $r-1$ sets V_1, V_2, \dots, V_{r-1} such that each V_i has order either $\lfloor (n-t)/(r-1) \rfloor$ or $\lceil (n-t)/(r-1) \rceil$, there are $o(n^2)$ edges inside the sets V_i and $o(n^2)$ edges are missing between the sets V_i .

We claim that there exists an i such that $|N_G(u) \cap V_i| = o(n)$. Indeed, otherwise we can pick $\Theta(n^{r-1})$ $(r-1)$ -sets having exactly one element in each $N_G(u) \cap V_i$. Only $o(n^{r-1})$ of these $(r-1)$ -sets contain a pair of vertices v, v' such that vv' is not an edge of G , since only $o(n^2)$ edges between the sets V_i ($i \leq r-1$) are missing from G . Therefore, there are $\Theta(n^{r-1})$ copies of K_{r-1} in the neighborhood of u . However, we have already shown that there are only $O(n^{r-2})$ copies of K_{r-1} that form an r -clique in G with u , a contradiction.

Clearly, there are $o(n^{k-1})$ copies of K_k containing u and a vertex from a V_i that satisfies $|N_G(u) \cap V_i| = o(n)$. This implies that the number of k -cliques containing u is at most $o(n^{k-1}) + \mathcal{N}(K_{k-1}, T(\frac{r-2}{r-1}n, r-2))$ as claimed. This finishes the proof of (i).

Let us start the proof of (ii) with the special case when we only count copies of K_k , i.e., we are interested in $\text{ex}(n, K_k, 2K_r)$. As $r \leq k \leq 2r-1$, any two copies of K_k must meet in at least $2k-2r+1$ vertices, otherwise their union would contain at least $2r$ vertices and thus a copy of $2K_r$. Therefore, the k -uniform hypergraph $H = (V, \mathcal{E})$ with $\mathcal{E} = \{S \in \binom{V}{k} : G[S] = K_k\}$ is $(2k-2r+1)$ -intersecting. By Corollary 2.2, we obtain that either all the k -cliques of G contain a fixed $(2k-2r+1)$ -set, or $\mathcal{N}(K_k, G) = o(n^{k-(2k-2r+1)})$. In the latter case, we are done, since $\mathcal{N}(K_k, K_{2k-2r+1} \vee T(n-2k+2r-1, 2r-k-1)) = \Theta(n^{2r-k-1})$.

Therefore, we can assume that all the k -cliques of G contain a fixed $(2k-2r+1)$ -set K . Then the vertices of K are adjacent to all vertices that are contained in a K_k in G . Let U denote the set of vertices outside K that are contained in at least one copy of K_k in G , so $|U| \leq n - (2k-2r+1)$. If $G[U]$ is K_{2r-k} -free, then

$$\begin{aligned} \mathcal{N}(K_k, G) = \mathcal{N}(K_{2r-k-1}, G[U]) \leq \mathcal{N}(K_{2r-k-1}, T(n-2k+2r-1, 2r-k-1)) \\ = \mathcal{N}(K_k, K_{2k-2r+1} \vee T(n-2k+2r-1, 2r-k-1)). \end{aligned}$$

Finally, if $G[U]$ contains a copy K' of K_{2r-k} , then $G[U \setminus V(K')]$ cannot contain a copy of K_{2r-k-1} as such a copy with K and K' would contain a $2K_r$. Every copy of K_k in G must contain K and intersect U in a copy of K_{2r-k-1} , thus must intersect K' . Therefore, we have $\mathcal{N}(K_k, G) = O(n^{2r-k-2}) = o(\mathcal{N}(K_k, K_{2k-2r+1} \vee T(n-2k+2r-1, 2r-k-1)))$.

Finally, let us consider the general case of (ii). Observe that similarly to the special case above, the $(k+i)$ -uniform hypergraph $H_i = (V, \mathcal{E}_i)$ with $\mathcal{E}_i = \{S \in \binom{V}{k+i} : G[S] = K_{k+i}\}$ is $(2(k+i-r)+1)$ -intersecting. We obtain that the number of $(k+i)$ -cliques is $O(n^{2r-k-i-1})$ and thus the number of cliques larger than k is $O(n^{2r-k-2})$. This means that

- in order to contain $\Theta(n^{2r-k-1})$ cliques of size at least k , all the k -cliques of G must contain the same $2k-r+1$ vertices just as in the special case,
- as any vertex contained in an $(k+i)$ -clique is also contained in a k -clique, every clique of size at least k is contained in U ,
- if $G[U]$ is K_{2r-k} -free, then there are no cliques of size larger than k in G , so the proof finishes as in the special case,
- if $G[U]$ does contain a $(2r-k)$ -clique, then, just like in the special case, there are $O(n^{2r-k-2})$ copies of K_k in G . As the number of cliques larger than k is $O(n^{2r-k-2})$, we obtain the same conclusion as above. \square

3. Forbidding $B_{r,s}$ and counting large cliques

In this section we prove Theorem 1.2. Again, we begin by collecting the tools we will use. The following is a simple corollary of Theorem 2.3.

Proposition 3.1. *For any $r \geq 3$ and large enough n , we have $\text{ex}(n, K_{r-1}, K_r + K_{r-1}) = \mathcal{N}(K_{r-1}, T(n, r-1))$. Furthermore, if G is an n -vertex $(K_r + K_{r-1})$ -free graph with $\mathcal{N}(K_{r-1}, G) = \mathcal{N}(K_{r-1}, T(n, r-1)) - o(n^{r-1})$, then G has edit distance $o(n^2)$ from $T(n, r-1)$.*

Proof. If an n -vertex graph G contains a copy K of K_r , then all the copies of K_{r-1} must meet K , so their number is $O(n^{r-2}) = o(\mathcal{N}(K_{r-1}, T(n, r-1)))$. If G is K_r -free, then by Theorem 2.4, we have $\mathcal{N}(K_{r-1}, G) \leq \mathcal{N}(K_{r-1}, T(n, r-1))$ and the furthermore part follows from Theorem 2.3. \square

If L is a set of non-negative integers, we say that a family \mathcal{F} of sets is L -intersecting if for any distinct $F, F' \in \mathcal{F}$, we have $|F \cap F'| \in L$.

Theorem 3.2 (Frankl, Füredi [12]). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a $\{0, 1, \dots, \ell-1, k-\ell', k-\ell'+1, \dots, k-1\}$ -intersecting family. Then the following statements hold.*

- (i) *There exists a constant d_k such that $|\mathcal{F}| \leq d_k n^{\max\{\ell, \ell'\}}$.*
- (ii) *If $\ell' > \ell$ and $n \geq n_0(k)$, then $|\mathcal{F}| \leq \binom{n-k+\ell'}{\ell'}$ and equality holds if and only if there exists a $(k-\ell')$ -subset X of $[n]$ such that $\mathcal{F} = \{F \in \binom{[n]}{k} : X \subset F\}$.*
- (iii) *If $\ell \geq \ell'$ and $k-\ell$ has a prime power divisor q with $q > \ell'$, then*

$$|\mathcal{F}| \leq (1 + o(1)) \binom{n}{\ell} \frac{\binom{k+\ell'}{\ell'}}{\binom{k+\ell'}{\ell}}.$$

Theorem 3.2 itself will not be sufficient for us and we will also need some consequences of the proof. In the following lemma, we gather the parts of the Frankl-Füredi proof that we will use. To state the lemma we need to define the i -shadow of a family \mathcal{F} of sets as $\Delta_i(\mathcal{F}) := \{G : |G| = i, \exists F \in \mathcal{F} \text{ such that } G \subset F\}$.

Lemma 3.3 (Lemma 6.1 and several propositions in [12]). *If $\ell < \ell'$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a $\{0, 1, \dots, \ell-1, k-\ell', k-\ell'+1, \dots, k-1\}$ -intersecting family, then \mathcal{F} can be partitioned into $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_h \cup \mathcal{F}_{h+1}$ such that the followings hold.*

- (i) $|\mathcal{F}_{h+1}| = O(n^{\ell'-1})$.
- (ii) *For every $1 \leq j \leq h$ there exists a $(k-\ell')$ -set A_j such that $\mathcal{F}_j \subseteq \{G \in \binom{[n]}{k} : A_j \subset G\}$.*
- (iii) *Let $\mathcal{H}_j = \{F \setminus A_j : F \in \mathcal{F}_j\}$. Then the ℓ -shadows of the families \mathcal{H}_j are pairwise disjoint, i.e., for every $1 \leq i < j \leq h$ we have that $\Delta_\ell(\mathcal{H}_i) \cap \Delta_\ell(\mathcal{H}_j) = \emptyset$.*

We will also use the Lovász version [30] of the Kruskal-Katona shadow theorem [28,27].

Theorem 3.4 (Lovász, [30]). *If a family \mathcal{H} of k -subsets has size $\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$ for some real x , then for any $i \leq k$ we have $|\Delta_i(\mathcal{H})| \geq \binom{x}{i}$.*

We will also use the following theorem.

Theorem 3.5 (Andrásfai, Erdős, and Sós [3]). *Every n -vertex K_r -free graph with chromatic number at least r contains a vertex of degree at most $\left(1 - \frac{1}{(r-1)-4/3}\right)n$.*

For integers n, r, t with $r > t+2$ and $n > 2t+2$, let us define the function

$$f(n) = f_{r,t}(n) = \mathcal{N}(K_{r+t}, K_{2t+2} \vee T(n-2t-2, r-t-2)) = \prod_{i=0}^{r-t-3} \left\lfloor \frac{n-2t-2+i}{r-t-2} \right\rfloor.$$

Observe that for fixed r, t , the function $f(n)$ is a polynomial of n of degree $r-t-2$. We will need the following simple properties of $f(n)$.

Proposition 3.6. (i) *Let n_1 and n_2 be positive integers that are larger than all the roots of f . Then we have $f(n_1) + f(n_2) \leq f(n_1 + n_2)$.*
(ii) *If $r-t-2 \geq 2$, $y = o(x)$ and $x = o(n)$, then $f(n-x) + f(x+y) < f(n) - \Omega(xn^{r-t-3})$.*

Proof. For $i = 1, 2$, let V_i be a vertex set of order n_i and let $A_i \subset V_i$ with $|A_i| = 2t + 2$. Let G_i denote the graph with vertex set V_i where the vertices of A_i have degree $n_i - 1$ and there is a $T(n_i - 2t - 2, r - t - 2)$ on the other vertices. Then G_i is isomorphic to $K_{2t+2} \vee T(n_i - 2t - 2, r - t - 2)$ and thus contains $f(n_i)$ copies of K_{r+t} . The vertex-disjoint union of G_1 and G_2 contains $f(n_1) + f(n_2)$ copies of K_{r+t} . If we delete A_2 and connect each other vertex of V_2 to the vertices of A_1 , then each deleted copy of K_{r+t} is replaced by a new copy of K_{r+t} , thus the resulting graph G_3 contains $f(n_1) + f(n_2)$ copies of K_{r+t} . Clearly G_3 is a subgraph of $K_{2t+2} \vee T(n_1 + n_2 - 2t - 2, r - t - 2)$, completing the proof of the first statement.

To see (ii), we write $f(z) = \sum_{i=0}^{r-t-2} a_i z^i$. If we set $z = x + y$, then the term of largest order of magnitude is x^{r-t-2} . Now consider $f(n) - f(n-x) - f(x+y)$. The terms of the largest order of magnitude are $a_{r-t-2}n^{r-t-2}$ and $-a_{r-t-2}n^{r-t-2}$. The term of the second largest order of magnitude is $n^{r-t-3}x$ with coefficient $-a_{r-t-2}$, which is negative, completing the proof. \square

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Each of the lower bounds is obtained from the following construction: let $m = s + 2t + 1$ and G be the join $K_m \vee T(n - m, r - s - t - 1)$. As $m + r - s - t - 1 = r + t$, we have $\mathcal{N}(K_{r+t}, G) = \mathcal{N}(K_{r-s-t-1}, T(n - m, r - s - t - 1)) = (1 + o(1))\left(\frac{n-m}{r-s-t-1}\right)^{r-s-t-1}$. To see that G is $B_{r,s}$ -free, observe that out of the $2r - s$ vertices of a copy of $B_{r,s}$, at least $2r - s - m = 2r - 2s - 2t - 1$ vertices belong to $T(n - m, r - s - t - 1)$. Therefore at least $r - s - t$ vertices belong to the same K_r of $B_{r,s}$ in $T(n - m, r - s - t - 1)$. As there is no clique of $r - s - t$ vertices in $T(n - m, r - s - t - 1)$, the graph G is indeed $B_{r,s}$ -free.

For the general upper bound, let G be a $B_{r,s}$ -free graph on n vertices. Define the $(r+t)$ -uniform family $\mathcal{F}_G := \{K \subseteq \binom{V(G)}{r+t} : G[K] = K_{r+t}\}$. Observe that \mathcal{F}_G is $\{0, 1, \dots, s-1, 2t+s+1, 2t+s+1, \dots, r+t-1\}$ -intersecting. Indeed, assume that two cliques K, K' each of size $r+t$ intersect in at least s , but less than $(r+t) - (r-t-s-1) = 2t+s+1$ vertices. Then the union of K and K' contains at least $2(r+t) - (2t+s) = 2r-s$ vertices and their intersection contains at least s vertices, and thus G contains a copy of $B_{r,s}$, a contradiction. We can apply Theorem 3.2 (i) to show that $\mathcal{N}(K_{r+t}, G) = O(n^{r-s-t-1})$.

Finally, we consider the case $s = 1$. Let $k = r + t$, and $\ell' = r - t - 2$. Suppose G is an n -vertex $B_{r,1}$ -free graph such that $\mathcal{N}(K_{r+t}, G) \geq f(n)$. Assume first $t + 3 < r$, and thus the condition of Proposition 3.6 (ii) is satisfied and also we have $1 < \ell'$. Therefore we can apply Lemma 3.3 to \mathcal{F}_G with ℓ' and $\ell = s = 1$. We obtain a partition $\mathcal{F}_G = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_h \cup \mathcal{F}_{h+1}$ and m -sets A_1, A_2, \dots, A_h with the properties ensured by Lemma 3.3, where $m = k - \ell' = 2t + 2$. We introduce positive reals x_1, \dots, x_h such that $|\mathcal{F}_i| = |\mathcal{H}_i| = \binom{x_i}{\ell'}$. Without loss of generality, $x_1 \geq x_2 \geq \dots \geq x_h$. Let $M_j = \cup_{F \in \mathcal{F}_j} F$ and clearly, we have $|M_j| \geq x_j$. We gather some facts that we will use several times throughout our proof: by Theorem 3.4, we know that $|\Delta_1(\mathcal{H}_i)| \geq x_i$. By (iii) of Lemma 3.3, the 1-shadows are pairwise disjoint, thus

$$\sum_{i=1}^h x_i \leq n.$$

As $x_i \geq x_j$ implies $\frac{\binom{x_i}{\ell'}}{x_i} \geq \frac{\binom{x_j}{\ell'}}{x_j}$, we have

$$\sum_{j=1}^h |\mathcal{F}_j| = \sum_{j=1}^h |\mathcal{H}_j| = \sum_{j=1}^h x_j \frac{\binom{x_j}{\ell'}}{x_j} \leq \frac{\binom{x_1}{\ell'}}{x_1} \sum_{j=1}^h x_j \leq \frac{\binom{x_1}{\ell'}}{x_1} n = O((x_1/n)^{\ell'-1} n^{\ell'}). \quad (1)$$

Finally, by the definition of the hypergraph \mathcal{F}_G , for every $j \leq h$, every vertex of M_j is adjacent to every vertex of A_j . This implies that

$$G[M_j \setminus A_j] \text{ is } (K_{r-t-1} + K_{r-t-2})\text{-free} \quad (\text{or equivalently } (K_{\ell'+1} + K_{\ell'})\text{-free}), \quad (2)$$

and thus, by Proposition 3.1,

$$|\mathcal{F}_j| = \mathcal{N}(K_{r+t}, G[M_j]) \leq f(|M_j|). \quad (3)$$

Claim 3.7. *There exists an integer n_0 and a constant C such that if $n \geq n_0$, then $\Delta_1(\mathcal{F}_1) = |M_1| \geq n - C$.*

Proof. First we show that $|M_1| = \Omega(n)$. Assume not, and thus $x_1 = o(n)$. Then (1) with $i = 1$ implies $\sum_{j=1}^h |\mathcal{F}_j| = o(n^{\ell'})$ and so $|\mathcal{F}| = o(n^{\ell'}) + |\mathcal{F}_{h+1}| = o(n^{\ell'})$. This is a contradiction as $\mathcal{N}(K_{r+t}, G) \geq f(n) = \Omega(n^{\ell'})$. So $|M_1| = \Omega(n)$ as claimed.

Next we will show that $|M_1| \geq n - n^{2/3} \log n$. By (iii) of Lemma 3.3, we have that the sets $M_j \setminus A_j$ are pairwise disjoint and thus $|M_j \setminus \cup_{j'=1}^{j-1} M_{j'}| \geq |M_j| - (j-1)m$. Let j_1 be the largest index j with $|M_j| \geq (m+1)n^{2/3}$. We claim that $j_1 \leq n^{1/3}$. Indeed, if not, then for $j \leq n^{1/3}$, we have $|M_j \setminus \cup_{j'=1}^{j-1} M_{j'}| \geq |M_j| - jm > n^{2/3}$ and $n \geq \sum_{j=1}^{n^{1/3}} |M_j \setminus \cup_{j'=1}^{j-1} M_{j'}| > n^{1/3} n^{2/3}$, a contradiction. The fact that the 1-shadows $M_j \setminus A_j$ are pairwise disjoint also implies that

$$\sum_{j=1}^{j_1} |M_j| = \sum_{j=1}^{j_1} |M_j \setminus A_j| + \sum_{j=1}^{j_1} |A_j| \leq n + j_1 m = n + O(n^{1/3}). \quad (4)$$

Observe that (1) with $i = j_1 + 1$ and $j_1 \leq n^{1/3}$ imply that $\sum_{j=j_1+1}^h |\mathcal{F}_j| = O(n^{\ell'-1/3})$, and (i) in Lemma 3.3 gives $|\mathcal{F}_{h+1}| = O(n^{\ell'-1})$. Using (3), (4) and Proposition 3.6 (i) we obtain

$$\begin{aligned} \sum_{j=1}^{h+1} |\mathcal{F}_j| &= |\mathcal{F}_1| + \sum_{j=2}^{j_1} |\mathcal{F}_j| + \sum_{j=j_1+1}^h |\mathcal{F}_j| + |\mathcal{F}_{h+1}| \\ &\leq f(|M_1|) + f(n + j_1 m - |M_1|) + O(n^{\ell'-1/3}) + O(n^{\ell'-1}). \end{aligned} \quad (5)$$

Assume that $|M_1| < n - n^{2/3} \log n$. Let $x = n^{2/3} \log n - j_1 m$ and $y = j_1 m$. Then Proposition 3.6 (ii) yields that $f(|M_1|) + f(n + j_1 m - |M_1|) < f(n) - \Omega(xn^{\ell'-1})$. Therefore, the right hand side of (5) is less than $f(n) - \Omega(xn^{\ell'-1}) + O(n^{\ell'-1/3}) + O(n^{\ell'-1}) < f(n) \leq |\mathcal{F}_G|$, a contradiction.

We have obtained that $r(n) := n - |M_1| = o(n)$. Suppose towards a contradiction that $r(n)$ tends to infinity. Let us write $p(n) := m(r(n))^{2/3}$ and let j^* be the largest index with $|M_{j^*}| \geq p(n)$. We claim that $j^* \leq (r(n))^{1/3}$. We can argue similarly as for the upper bound on j_1 earlier: if $(r(n))^{1/3} \leq j^*$, then for $j \leq (r(n))^{1/3}$ we have $|M_j \setminus \bigcup_{i=1}^{j-1} M_i| \geq |M_j| - jm \geq 2(r(n))^{2/3}$ and $r(n) \geq \sum_{j=2}^{(r(n))^{1/3}} |M_j \setminus \bigcup_{j'=1}^{j-1} M_{j'}| > 2(r(n))^{1/3} (r(n))^{2/3}$, a contradiction. We also obtained $\sum_{j=2}^{j^*} |M_j| = \sum_{j=2}^{j^*} |M_j \setminus A_j| + j^* m \leq r(n) + p(n)$. By (1), we obtain $\sum_{j=j^*+1}^{h+1} |\mathcal{F}_j| = O(n(p(n))^{\ell'-1}) = O(n^{\ell'-1} p(n))$. Applying Proposition 3.6 (i) and (ii) with $x = r(n)$ and $y = p(n)$, we obtain

$$\begin{aligned} \mathcal{N}(K_{r+t}, G) &= \sum_{j=1}^{h+1} |\mathcal{F}_j| \leq f(|M_1|) + \sum_{j=2}^{j^*} f(M_j) + O(n^{\ell'-1} p(n)) \\ &\leq f(n - r(n)) + f(r(n) + p(n)) + O(n^{\ell'-1} p(n)) < f(n), \end{aligned}$$

a contradiction. \square

By Claim 3.7, we may assume that $|M_1| \geq n - C$ for some constant C . Therefore $x_2 \leq C + m$ and thus (1) implies $\sum_{j=2}^h |\mathcal{F}_j| = O(n)$. Taking \mathcal{F}_{h+1} into consideration, this means $\mathcal{N}(K_{r+t}, G) \leq f(|M_1|) + O(n^{\ell'-1})$. As $\mathcal{N}(K_{r+t}, G) \geq f(n)$, we must have $\mathcal{N}(K_{\ell'}, G[M_1 \setminus A_1]) \geq f(|M_1|) - D \cdot n^{\ell'-1}$ for some constant D . Now (2), (3) and the moreover part of Proposition 3.1 imply that G can be made ℓ' -partite by deleting $o(n^2)$ edges. Let us delete those edges and let $U_1, U_2, \dots, U_{\ell'}$ be the corresponding partition. We say that a vertex $v \in U_i$ is *problematic* if there exists $j \neq i$ such that there are at least $\frac{|U_j|}{(\ell')^2}$ vertices in U_j not adjacent to v . A set of vertices $W \subset M_1 \setminus A_1$ is *good* if it does not contain any problematic vertices.

Claim 3.8. *There exists a set $X \subset M_1 \setminus A_1$ with $|X| = O(1)$ such that $M_1 \setminus (A_1 \cup X)$ is good.*

Proof. First we find a set X_1 of vertices with $|X_1| = O(1)$ such that the removal of X_1 from $G[M_1 \setminus A_1]$ makes the remaining graph ℓ' -partite. Then we show that there are $O(1)$ problematic vertices in the remaining ℓ' -partite graph. By (2), $G[M_1 \setminus A_1]$ is $(K_{\ell'+1} + K_{\ell'})$ -free. So if it contains a copy K of $K_{\ell'+1}$, then $G[M_1 \setminus (A_1 \cup K)]$ is $K_{\ell'+1}$ -free. We write $A'_1 = A_1 \cup K$ if K exists and $A'_1 = A_1$ otherwise.

Suppose first that $\chi(G[M_1 \setminus A'_1]) \geq \ell' + 1$. Then by Theorem 3.5, there exists a vertex v with degree in $G[M_1 \setminus A'_1]$ at most $(1 - \frac{1}{\ell'-4/3} + o(1))|M_1 \setminus A'_1|$. As $G[M_1 \setminus A'_1]$ is $K_{\ell'+1}$ -free, $G[N_G(v) \cap (M_1 \setminus A'_1)]$ is $K_{\ell'}$ -free and the number of copies of K_{r+t} in $G[M_1]$ containing v is at most

$$\mathcal{N}(K_{\ell'-1}, G[N_G(v) \cap (M_1 \setminus A'_1)]) \leq \mathcal{N}\left(K_{\ell'-1}, T\left(\left(1 - \frac{1}{\ell'-4/3} + o(1)\right)|M_1 \setminus A'_1|, \ell' - 1\right)\right) =: x.$$

Now observe that the difference between the number of copies of $K_{\ell'}$ in $T(|M_1 \setminus A'_1|, \ell')$ containing a fixed vertex u and x is at least $\alpha n^{\ell'-1}$ for some constant α . So we remove v and add it to X_1 . If the remaining graph is ℓ' -partite, then we are done with the first step, otherwise we use Theorem 3.5 to find another vertex of low degree, and so on. Observe that if $|X_1| \alpha$ is larger than D , then $\mathcal{N}(K_{r+t}, G) \leq f(n)$, so indeed we can guarantee that the size of X_1 is bounded by a constant.

From now on, we can assume that the remaining graph $G[(M_1 \setminus A'_1) \setminus X_1]$ is ℓ' -chromatic with partition $U_1, \dots, U_{\ell'}$. If a vertex $u \in U_i$ is problematic, then the number of copies of K_{r+t} in $G[M_1]$ containing v is at most $(1 - \frac{1}{(\ell')^2}) \prod_{j \neq i} |U_j|$, so again some $\beta n^{\ell'-1}$ smaller than in the appropriate Turán graph. Remove problematic vertices one by one and let X_2 be the set of vertices removed this way. As in the above paragraph, if $|X_2| \beta$ is larger than D , then $\mathcal{N}(K_{r+t}, G) \leq f(n)$, so indeed we can guarantee that the size of X_2 is bounded by a constant. \square

We claim that if an $(r+t)$ -clique W contains a vertex from $V \setminus M_1$, then W and $M_1 \setminus X$ are disjoint. Indeed, assume to the contrary that for a clique $W \not\subseteq M_1$ we have $|W \cap (M_1 \setminus X)| \geq 1$ and let y be an element of $W \cap A_1$ if such an element exists and $y \in W \cap M_1 \setminus (A_1 \cup X)$ otherwise.

Let us go through the indices i with $y \notin U_i$ in an arbitrary order. For each i , we pick a vertex $v_i \in U_i \setminus W$ that is adjacent to y and to every vertex already picked. As the number of vertices picked this way is at most $r-t-3$, at most $(r-t-3)|U_i|/(r-t-2)^2 + o(n)$ vertices of U_i are forbidden, thus we can pick the desired vertex. Then we can add the vertices of $A_1 \cup \{y\}$ to obtain a clique W' of size $m + \ell' = r+t$. Because y is in both W and W' , we have $|W \cap W'| \geq 1$. We claim that $|W \cup W'| \geq m + 2\ell' + 1 = 2r - 1$. Indeed, as W contains a vertex from $V \setminus M_1$, we cannot have $A_1 \subset W$ and thus by construction, we have $|W \cap W'| < m$. Observe that $1 \leq |W \cap W'|$ and $|W \cup W'| \geq 2r - 1$ imply that $W \cup W'$ contains a copy of $B_{r,1}$. This contradiction shows that W is indeed disjoint with $M_1 \setminus X$.

The number of $(r+t)$ -cliques disjoint with $M_1 \setminus X$ is at most

$$\binom{|(V \setminus M_1) \cup X|}{r+t} = O(1) = O(n^{\ell'-2}).$$

As a consequence we obtain that the number of $(r+t)$ -cliques of G meeting $V \setminus M_1$ is $o(n^{\ell'-1})$, while $f(n) - f(n-C) = \Omega(n^{\ell'-1})$ as long as C is positive by (ii) of Proposition 3.6. Therefore, we must have that $|M_1| = n$. But then $G = G[M_1]$ and $\mathcal{N}(K_{r+t}, G[M_1]) = \mathcal{N}(K_{r+t}, G) \leq f(n)$ by (3). This proves the third statement of the theorem.

Finally, we deal with the cases $t = r - 2$ and $t = r - 3$. The lower bounds are given by $\lfloor n/(2r-2) \rfloor$ vertex-disjoint copies of K_{2r-2} and if $n \equiv 2r-3 \pmod{2r-2}$, then an additional copy of K_{2r-3} disjoint from the copies of K_{2r-2} . In the case $t = r - 2$, observe that the $B_{r,1}$ -free property of G implies that copies of K_{2r-2} must be vertex disjoint. This yields $\mathcal{N}(K_{2r-2}, G) \leq \lfloor \frac{n}{2r-2} \rfloor$.

In the case $t = r - 3$, let $K_1, K_2, \dots, K_\alpha$ be the vertex sets of copies of K_{2r-3} in G . The $B_{r,1}$ -free property implies that $|K_i \cap K_j|$ is either 0 or $2r-4$. If all the K_i 's are pairwise disjoint, then $\alpha \leq \frac{n}{2r-3}$ and we are done. So suppose $|K_1 \cap K_2| = 2r-4$. Then as G is $B_{r,1}$ -free, for any other K_i we either have that $K_i \cap (K_1 \cup K_2) = \emptyset$ or $K_i \subset K_1 \cup K_2$ or $K_i \cap K_1 = K_i \cap K_2 = K_1 \cap K_2$. This means that connected components of the $(2r-3)$ -uniform hypergraph of cliques of G are either $(2r-3)$ -subsets of a $(2r-2)$ -set or form a Δ -system with kernel size $2r-4$, i.e., the cliques in the components share the same $2r-4$ vertices.

Let y be the number of vertices contained in at least one K_{2r-3} , so $y \leq n$, and let y' be the number of vertices in the above mentioned Δ -systems. Then $y - y'$ is a multiple of $2r-2$ and $\mathcal{N}(K_{2r-3}, G) \leq (y - y') + (y' - (2r-4))$. Simple case analysis shows that unless $n \equiv 2r-3 \pmod{2r-2}$, this expression is maximized when $y' = 0$, while if $n \equiv 2r-3 \pmod{2r-2}$, then there can be exactly one Δ -system. \square

4. Forbidding $B_{r,1}$ and counting small cliques

In this section we prove Theorem 1.3. Recall that it states that for $k < r$ and n large enough, $\text{ex}(n, K_k, B_{r,1}) = \mathcal{N}(K_k, T^+(n, r-1))$. We will use the asymptotic result $\text{ex}(n, K_k, B_{r,1}) = (1+o(1))\mathcal{N}(K_k, T^+(n, r-1)) = (1+o(1))\mathcal{N}(K_k, T(n, r-1))$. This follows from a theorem of Alon and Shikhelman [2] that states that if $\chi(F) = r > k$, then $\text{ex}(n, K_k, F) = (1+o(1))\mathcal{N}(K_k, T(n, r-1))$.

Proof of Theorem 1.3. Let G be a $B_{r,1}$ -free graph on n vertices. By Theorem 2.3, if G has at least $\mathcal{N}(K_k, T(n, r-1)) - o(n^k)$ copies of K_k , then G can be obtained from $T(n, r-1)$ by adding and removing $o(n^2)$ edges. We consider an $(r-1)$ -partite subgraph G' of G with the maximum number of edges. Let V_1, \dots, V_{r-1} be the parts of G' , then there are $o(n^2)$ edges inside the parts V_i . Moreover, each vertex is adjacent to at most as many vertices in its part as in any other part. Also, every V_i has order $(1+o(1))\frac{n}{r-1}$, as otherwise there are at most $\mathcal{N}(K_k, T(n, r-1)) - \Theta(n^k)$ copies of K_k in G' and also in G .

Let us pick $\alpha < (r-2)/(r-1)$.

CASE I: Every vertex has degree at least αn .

We partition V_i to V'_i and V''_i with V'_i containing those vertices of V_i that are adjacent to all but $o(n)$ vertices outside V_i . By the assumption on the degrees, all $v \in V''_i$ are incident to $\Omega(n)$ edges inside V_i . This implies $|V''_i| = o(n)$. We will use that for any i , for any set $U \subset V'_1 \cup \dots \cup V'_i$ with $|U| = O(1)$, the common neighborhood of the vertices of U contains all but $o(n)$ vertices from $V'_{i+1} \cup \dots \cup V'_{r-1}$.

Without loss of generality the number m of edges inside V_1 is not smaller than the number of edges inside any V_i .

CASE IA: $m > 1$.

We claim that there cannot be two edges within V'_1 . Indeed, if u, v, w is a path, then in the common neighborhood of u, v, w outside V_1 , we can find two disjoint cliques each of size $r-2$, so their union with $\{u, v, w\}$ spans a copy of $B_{r,1}$. Similarly, if uv and wz are edges in V'_1 , then in the common neighborhood of u, v, w, z outside V_1 one can find a copy B of $B_{r-2,1}$, and B together with u, v, w, z form a copy of $B_{r,1}$. These contradictions prove our claim.

Since there is only one edge inside V'_1 and there are at least two edges inside V_1 , we obtain that V'_1 is non-empty. Let $u \in V'_1$ and thus u has $n' = \Omega(n)$ neighbors in V_1 in G . We also know that for any $i > 1$, u has at least n' neighbors in V_i by the choice of G' , thus u has at least $n' - o(n) \geq n'/2$ neighbors in V'_i . Let U_i be an arbitrary set of $\lceil n'/2 \rceil$ neighbors of u in V'_i , and put $U = U_2 \cup \dots \cup U_{r-1}$. We claim that U is $2K_{r-2}$ -free in G . Indeed, if K, K' were two cliques of each of size $r-2$ in U , then the common neighborhood of the vertices of $K \cup K'$ would contain all but $o(n)$ vertices in V_1 (since these vertices are chosen from V'_i for $i \neq 1$), in particular the common neighborhood would contain all but $o(n)$ neighbors of u in V_1 . So K, K', u and two such neighbors would form a $B_{r,1}$ in G . This contradiction proves our claim.

We can now count the copies of K_k . There are $o(n^k)$ copies of K_k containing an edge inside a V_i . Let G'' denote the complete $(r-1)$ -partite graph with parts V_1, \dots, V_{r-1} . Now compare the number of copies of K_k inside G' to the number of copies of K_k in G'' . We consider only those copies that contain a vertex of V_1 and a K_{k-1} inside U . There are at most $|V_1| \cdot \text{ex}(|U|, K_{k-1}, 2K_{r-2}) = (1 + o(1)) |V_1| \cdot \mathcal{N}(K_{k-1}, T(|U|, r-3))$ such copies of K_k in G' and $|V_1| \cdot \mathcal{N}(K_{k-1}, T(|U|, r-2))$ such copies of K_k in G'' . The difference between these two quantities is $\Theta(n^k)$. Since G has $\mathcal{N}(K_k, G') + o(n^k)$ copies of K_k , we obtain that G has fewer copies of K_k than the complete $(r-1)$ -partite graph G'' , which has at most as many copies of K_k as the Turán graph $T(n, r-1)$, completing the proof in this case.

CASE IB: $m = 1$, i.e., there is at most one edge inside each V_i .

If there are at least two such edges, say $uv \in V_1$ and $xy \in V_2$, then we pick $U = \{u_1, u_3, \dots, u_{r-1}\}$ and $U' = \{v_2, v_3, v_4, \dots, v_{r-1}\}$ with $u_i, v_i \in V_i$. If both $U \cup \{x, y\}$ and $U' \cup \{u, v\}$ induce cliques, then we find $B_{r,1}$, a contradiction. Thus there is a missing edge between parts V_i among these vertices, thus there are $\Omega(n)$ missing edges between parts altogether. We can again compare the number of copies of K_k inside G' to the number of copies of K_k in the complete $(r-1)$ -partite graph with parts V_1, \dots, V_{r-1} . The additional at most $r-1$ edges inside parts create $O(n^{k-2})$ copies of K_k , while the missing edges $\Omega(n)$ between parts are in $\Omega(n^{k-1})$ copies of K_k .

If there is an edge uv inside only one part, say V_1 , then we have to show that the order of the parts is as balanced as possible. If we have $|V_i| \geq |V_j| + 2$, then we move a vertex w from V_i to V_j . The number of edges between V_i and V_j increases. Therefore, the number of copies of K_k not containing uv increases by $\Omega(n^{k-2})$. Indeed, the number copies that intersect both V_i and V_j can be counted by picking an edge, and then $k-2$ vertices from other parts, $\Theta(n^{k-2})$ ways. The number of copies that intersect at most one of V_i and V_j does not change.

The number of copies of K_k containing uv decreases by $O(n^{k-3})$, since the deleted copies each contain u, v and w . Therefore, making the parts more balanced increases the number of copies of K_k , completing the proof in this case.

CASE II: There exist vertices of degree less than αn .

We remove such vertices one by one until we arrive to a graph G_0 with no such vertices. Assume that we removed ℓ vertices. If $\ell < n/2$, then G_0 has sufficiently many vertices, thus at most $\mathcal{N}(K_k, T^+(n-\ell, r-1))$ copies of K_k by the above part of the proof. We removed at most $\ell \cdot \text{ex}(\alpha n, K_{k-1}, 2K_{r-1}) = (1 + o(1)) \ell \cdot \mathcal{N}(K_{k-1}, T(\alpha n, r-2))$ copies of K_k , using Theorem 1.1. If we add ℓ vertices to $T^+(n-\ell, r-1)$ to form the $T^+(n, r-1)$ instead, then we add $(1 + o(1)) \ell \cdot \mathcal{N}(K_{k-1}, T(\frac{r-2}{r-1}n, r-2))$ copies of K_k , thus we obtain more copies than in G , completing the proof.

If $\ell > n/2$, then we cannot apply the previous argument, since $|V(G_0)|$ may not be large enough. However, we can use the known asymptotic bound mentioned at the beginning of this section to show G_0 contains at most $(1 + o(1)) \mathcal{N}(K_k, T^+(n-\ell, r-1))$ copies of K_k . We removed at most

$$\begin{aligned} \ell \cdot \text{ex}(\alpha n, K_{k-1}, 2K_{r-1}) &= (1 + o(1)) \ell \cdot \mathcal{N}(K_{k-1}, T(\alpha n, r-2)) \\ &= \ell \cdot \mathcal{N}\left(K_{k-1}, T\left(\frac{r-2}{r-1}n, r-2\right)\right) - \Theta(n^k) \end{aligned}$$

copies of K_k , since $\alpha < \frac{r-2}{r-1}$. If we change G_0 to $T^+(n-\ell, r-1)$, we lose $o(n^k)$ copies of K_k . If we add ℓ vertices to $T^+(n-\ell, r-1)$ to form the $T^+(n, r-1)$, we gain $\Theta(n^k)$ more copies compared to G , thus $\mathcal{N}(K_k, G) \leq \mathcal{N}(K_k, T(n, r-1)) - \Theta(n^k)$, a contradiction completing the proof. \square

5. Forbidding $B_{3,1}$ and counting complete bipartite graphs

Let us begin this section by describing the symmetrization method due to Zykov [35], who used it to show that $\text{ex}(n, K_k, K_r) = \mathcal{N}(K_k, T(n, r-1))$. We say that we *symmetrize* a vertex u to another vertex v in a graph G when we delete all the edges incident to u and for each edge vw , we add the edge uw . In other words, we replace the neighborhood of u by the neighborhood of v . We apply this operation to non-adjacent vertices. One can show that if G is K_r -free, then the graph G' we obtain by symmetrizing u to v is also K_r -free.

Let $d(H, v)$ denote the number of copies of H containing vertex v . Extending Zykov's idea, Györi, Pach and Simonovits [25] showed that if H is a complete multipartite graph and $d(H, u) \leq d(H, v)$, then this symmetrization does not decrease the total number of copies of H . Thus, for any pair of non-adjacent vertices (u, v) we can symmetrize one to the other such that the total number of copies of H does not decrease. We apply such symmetrization steps as long as we can find two non-adjacent vertices with different neighborhoods. At the end of the symmetrization process we obtain a K_r -free

complete multipartite graph with at least $\mathcal{N}(H, G)$ copies of H , which implies that $\text{ex}(n, H, K_r)$ is attained by a complete $(r - 1)$ -partite graph (one also needs to show that this process terminates).

In a sense, this is the most general application of Zykov symmetrization for generalized Turán problems: if F is any graph that is not a clique, then symmetrization may ruin the F -free property. If H is any graph that is not complete multipartite, then it is possible that both symmetrizing u to v and symmetrizing v to u decreases the total number of copies of H . However, Liu and Wang [29] introduced a restricted version of symmetrization that avoids the first of these problems. Here we state the general version of the basic idea.

Recall that $B_{r,s}$ consists of two r -cliques sharing exactly s vertices. We call the vertices shared by the two r -cliques *rootlet* vertices, and the other vertices of the book graph are *page* vertices. The page vertices are partitioned into two pages, according to which of the two r -cliques they belong.

Proposition 5.1. *Let G be a $B_{r,s}$ -free graph, u and v be non-adjacent vertices of G , and assume that v is not a rootlet vertex of any $B_{r,s+1}$ in G . Let G' be the graph obtained from G by symmetrizing u to v . Then G' is $B_{r,s}$ -free.*

Proof. Assume that there is a copy B of $B_{r,s}$ in G' . Then B has to contain u , otherwise B would be contained in G . If B does not contain v , then we can replace u with v to obtain a copy of $B_{r,s}$ that is also present in G , a contradiction. If B contains v , then, as u and v are not adjacent in G' , they are both page vertices of B on different pages. But they have the same neighborhood, thus v is adjacent to every vertex of B but u . Then the s rootlet vertices of B with v form the rootlet vertices of a copy B' of $B_{r,s+1}$ in G , where the pages of B' are the pages of B without u and v . Thus v is a rootlet vertex of a $B_{r,s+1}$ in G , contradicting our assumption. \square

Let us repeatedly apply symmetrization on the vertices that are not rootlet vertices of any $B_{r,s+1}$. Suppose that the process terminates and let G_0 be the resulting graph (in the proof of Theorem 1.4 we will show that the process terminates). Let G_1 be the subgraph of G_0 induced by vertices that are not rootlet vertices of any $B_{r,s+1}$ in G_0 . Then the proposition above implies that G_1 is a complete multipartite graph. Moreover, vertices in the same partite set of G_1 have the exact same neighborhood among the other vertices of G_0 .

Now we are ready to prove Theorem 1.4 which we restate for convenience.

Theorem. *For any integers $a \leq b$ and n large enough, $\text{ex}(n, K_{a,b}, B_{3,1}) = \mathcal{N}(K_{a,b}, T)$ for some n -vertex graph T that is obtained from a complete bipartite graph by adding an edge to one of the parts.*

Proof. Let G be an n -vertex $B_{3,1}$ -free graph with $\text{ex}(n, K_{a,b}, B_{3,1})$ copies of $K_{a,b}$. Let Q denote the set of vertices in G that are not rootlet vertices of a $B_{3,2}$. If there are two non-adjacent vertices u and v in Q with $d(K_{a,b}, u) < d(K_{a,b}, v)$, then we symmetrize u to v and obtain a $B_{3,1}$ -free graph with more copies of H , a contradiction. Thus we can assume that for non-adjacent vertices u, v in Q , we have $d(K_{a,b}, u) = d(K_{a,b}, v)$ (and also later, after any symmetrization). This means that for non-adjacent vertices u and v , we can choose whether we symmetrize u to v or v to u and the total number of copies of H will not change.

Recall that $B_{3,2}$ consists of two triangles sharing a common edge. We call the graph consisting of $k \geq 2$ triangles sharing an edge a *book graph with k pages*. Observe that for any vertex $v \in Q$ there exists at most one inclusion-wise maximal book B_v of which v is a page vertex. Indeed, assume indirectly that there are two books containing v as a page vertex, then the two books have at least 3 rootlet vertices. If the two books have four rootlet vertices, we find a $B_{3,1}$. Thus the two books have three rootlet vertices x, y, z , each adjacent to v , such that xy and yz are in $E(G)$. But then v and y are rootlet vertices of a $B_{3,2}$ with x, z being the page vertices contradicting $v \in Q$.

Let u and v be page vertices from different copies of $B_{3,2}$ that are not rootlet vertices of any $B_{3,2}$. If B_u has more pages than B_v , then we symmetrize v to u . If they have the same number of pages, then we symmetrize arbitrarily. If u is the page vertex of a $B_{3,2}$ but v is not, then we symmetrize v to u . For any i , we consider i largest maximal books, i.e., i maximal books with the most pages. Let $g(i)$ denote the sum of the number of pages of the i largest maximal books.

We claim that after such a symmetrization, $g(i)$ does not decrease for any i , and for some i , $g(i)$ increases. Indeed, observe that if B_v is among the largest i maximal books, then so is B_u . We deleted at most one page containing v , but added a page when we connected v to the neighbors of u . This proves the first part of the claim. Assume that there are $i - 1$ maximal books with more pages than B_u . Those are unaffected by the symmetrization, and B_u has one more page, proving the second part of our claim.

Let us apply such symmetrization steps as long as we can. We claim that after finitely many steps this process terminates. Indeed, for any i the total number of pages in the largest i books can increase at most $i \cdot n$ times.

Let G_0 be the resulting graph, G_1 be the subgraph induced on the vertices that are not rootlet vertices of any $B_{3,2}$ and G_2 be the subgraph induced by the other vertices. Note that the vertex set of G_1 might be different from Q , as symmetrization may destroy or create copies of $B_{3,2}$. Non-adjacent vertices of G_1 have the same neighborhood, as otherwise we would symmetrize one vertex to another. This means that G_1 is a complete m -partite graph for some m with parts A_1, \dots, A_m . Observe that $m \leq 3$ because there are no rootlet vertices in G_1 .

Observe that a page vertex of a $B_{3,2}$ can be a rootlet vertex of a $B_{3,2}$ only if these two books are actually contained in a K_4 , as otherwise one can easily find a $B_{3,1}$. Furthermore, every vertex outside the K_4 can have at most one neighbor inside

that K_4 . This implies that for every $i \leq m$, the vertices of A_i have at most one neighbor in a K_4 . Therefore, if u and v are the rootlet vertices of a $B_{3,2}$ that is not a K_4 , then its page vertices are in G_1 , and belong to the same part set A_i . It is easy to see that if $u'v'$ is an edge in G_2 and $\{u', v'\} \neq \{u, v\}$, then we cannot have that u' and v' are both adjacent to a vertex $w \in A_i$. Indeed, if u, v, u', v' are four distinct vertices, then they form $B_{3,1}$ with w . If, say, $u = u'$, then u, v', w and u, v, w' for any page vertex w' from B_w form $B_{3,1}$. In other words, the only copies of $B_{3,2}$ with page vertices in A_i are those with rootlet vertices u and v .

Assume first that $m = 3$. Then G_1 is a triangle because there are no rootlet vertices in G_1 . This implies that the page vertices of a $B_{3,2}$ cannot belong to the same part A_i , thus each $B_{3,2}$ is contained in a K_4 . Then G_0 is the vertex-disjoint union of a K_3 and copies of K_4 , with additional edges between these subgraphs. However, every vertex v is adjacent to at most one vertex in every copy of K_4 and K_3 (except the one containing v). This means that the degree of v is at most $d := 2 + (n+1)/4$. Then we can count the copies of $K_{a,b}$ the following way. Assume that $a \neq b$; if $a = b$, then each of our bounds are divided by 2 because of symmetry. Either we pick a vertex v , a of its neighbors and $b-1$ of their common neighbors other than v , or pick b neighbors of v and $a-1$ of their common neighbors other than v . This way we count every copy of $K_{a,b}$ exactly $a+b$ times, since we can pick any vertex as the first vertex. Thus we obtain the upper bound $n \left(\binom{d}{a} \binom{d}{b-1} + \binom{d}{b} \binom{d}{a-1} \right) / (a+b)$. We can count the copies of $K_{a,b}$ in $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ the same way and we get $n \left(\binom{d'}{a} \binom{d'}{b-1} + \binom{d'}{b} \binom{d'}{a-1} \right) / (a+b)$ copies, where $d' = \lfloor n/2 \rfloor$. Clearly this is a larger quantity if n is large enough, a contradiction to our assumption that G is extremal.

Assume now that $m \leq 2$. We will handle together the cases $m = 1$ and $m = 2$; in what follows A_2 may be empty. Recall that vertices of G_1 have the same neighborhood even inside G_2 . Therefore, two edges in such a neighborhood could be extended to a $B_{3,1}$. In other words, G_2 contains at most two edges uv and $u'v'$ such that u, v are adjacent to the vertices in A_1 and u', v' are adjacent to the vertices of A_2 . Note that G_2 also may contain several copies of K_4 , with additional edges between these subgraphs. However, for every i and each such copy of K_4 , at most one vertex of the 4-clique can be adjacent to vertices in A_i . Also, such a vertex can be adjacent to vertices of at most one of A_1 and A_2 otherwise G_0 would contain a K_4 and a K_3 meeting in exactly one vertex, thus a $B_{3,1}$.

Let us assume that there are $p \geq 1$ copies of K_4 in G_0 and let U be the set of their vertices. Recall that G_2 has at most 4 other vertices: u, v adjacent to the vertices in A_1 and u', v' are adjacent to the vertices of A_2 .

We will define a new graph G'_0 on $V(G_0)$ the following way. Let us start by defining vertex sets A'_1 and A'_2 . For each vertex $w \in U$, if w is adjacent to the vertices of A_1 , we add w to A'_1 , and if w is adjacent to the vertices of A_2 , we add w to A'_2 . Then we add the remaining vertices of U to A'_1 and A'_2 such that for each K_4 , two of its vertices are in A'_1 and two are in A'_2 . Finally, we add A_1 to A'_1 and A_2 to A'_2 . Now we define the edge set of G'_0 . We keep the edges of G_0 outside U , and connect the vertices of A'_1 to the vertices of A'_2 and to u and v , and similarly we connect the vertices of A'_2 to u' and v' . Clearly G'_0 is $B_{3,1}$ -free.

Claim 5.2. G'_0 contains more copies of $K_{a,b}$ than G_0 .

Proof. Assume first that $p \geq 4$. Then the degree of any vertex w has not decreased. If $w \in A_1 \cup A_2$, then the edges incident to w in G_0 are also in G'_0 . Otherwise, w was adjacent to at most $p+2$ vertices of U in G_0 and is adjacent to $2p$ vertices of U in G'_0 . Furthermore, if w was adjacent to vertices in G_1 , those edges remain intact. Finally, w was adjacent to at most 3 vertices out of u, v, u', v' .

Consider a copy H_0 of $K_{a,b}$ in G_0 that intersects G_1 in a non-empty set X of vertices. If X intersects both parts of G_1 , then H_0 is also in G'_0 , as the edges of G_0 that are incident to vertices of G_1 are present in G'_0 . If X is a subset of A_1 , then the vertices of X belong to the same part of H_0 and the other part of H_0 belongs to the set of vertices adjacent to A_1 in G_0 . Thus we have to pick the remaining vertices of H_0 from the common neighbors of these vertices; they have more common neighbors in G'_0 , thus we can find more copies of $K_{a,b}$ this way in G'_0 . This shows that there are more copies of $K_{a,b}$ in G'_0 intersecting $A_1 \cup A_2$ than in G_0 .

Finally, if a copy of $K_{a,b}$ does not intersect G_1 , then we can pick it the following way. We pick a vertex w , then we pick either b of its (at most $p+2$) neighbors in G_2 and $a-1$ of the at most $(p+2)$ common neighbors of those in G_2 , or we pick a of its neighbors and $b-1$ of their common neighbors. We can pick more copies of $K_{a,b}$ in G'_0 the same way, as there are $2p$ neighbors of w from $V(G_2)$ in G'_0 and those have at least $2p$ common neighbors from $V(G_2)$ in G'_0 .

Assume now that $1 \leq p \leq 3$. Observe that if $|A_1| = o(n)$ or $|A_2| = o(n)$ then we have $o(n^{a+b})$ copies of $K_{a,b}$ in G_0 , less than in $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$, a contradiction. In a K_4 , at most two vertices are adjacent to vertices in G_1 , and the other at least two vertices have degree at most $4+p \leq 7$ in G_0 . By deleting the edges incident to those two vertices, we removed $O(n^{b-1})$ copies of $K_{a,b}$. By adding those two vertices to A'_1 or A'_2 , we added $\Omega(n^{a+b-1})$ copies of $K_{a,b}$ to G'_0 , thus the number of copies of $K_{a,b}$ increases. \square

The above claim finishes the proof if $p \geq 1$. Finally, if $p = 0$, then G_0 without uv and $u'v'$ is a complete bipartite graph, thus we are done if $m = 1$ or if $u'v'$ does not exist. Observe that a triangle on the vertices u, v, u', v' would create a $B_{3,1}$ with a vertex of A_1 or A_2 . Thus, we can assume without loss of generality that uu' and vv' are not edges of G_0 . Let G''_0 be the graph obtained by deleting from G_0 the edge $u'v'$ and adding uu' and vv' . We have removed some copies of $K_{a,b}$ only

if $a = b = 2$ or if $a = 1$. In the first case, we removed only one copy, and we created more copies of $K_{2,2}$. In the second case, the degree of every vertex remained the same or (for two vertices) increased. Thus in both cases, the number of copies of $K_{a,b}$ increased, a contradiction. \square

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Data availability

No data was used for the research described in the article.

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