



Vector sum-intersection theorems

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ABSTRACT

We introduce the following generalization of set intersection via characteristic vectors: for $n, q, s, t \geq 1$ a family $\mathcal{F} \subseteq \{0, 1, \dots, q\}^n$ of vectors is said to be s -sum t -intersecting if for any distinct $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ there exist at least t coordinates, where the entries of \mathbf{x} and \mathbf{y} sum up to at least s , i.e. $|\{i : x_i + y_i \geq s\}| \geq t$. The original set intersection corresponds to the case $q = 1, s = 2$.

We address analogs of several variants of classical results in this setting: the Erdős–Ko–Rado theorem and the theorem of Bollobás on intersecting set pairs.

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1. Introduction

Many problems in extremal finite set theory ask for the maximum size of a set family that satisfies some intersection property. When members of the family examined are subsets of $[n] := \{1, 2, \dots, n\}$, then there is a one-to-one correspondence between a set F and its 0-1 characteristic vector \mathbf{x}_F of length n , that has a 1-entry in its i th coordinate if and only if $i \in F$ for $i \in [n]$. So one can say that two sets F and G intersect, if the sum of their characteristic vectors (as vectors in \mathbb{Z}^n) contains a 2 in at least one coordinate. The goal of this paper is to introduce a notion of intersection that generalizes set intersection (translated to sum of characteristic vectors) to a type of intersection among q -ary vectors.

For $q, n \geq 1$ we introduce the notation $Q = Q(q) = \{0, 1, \dots, q\}$ and also $Q^n := \{0, 1, \dots, q\}^n$. We denote vectors by boldface letters and the i th coordinate of the vector \mathbf{x} is denoted by x_i .

Intersection problems have been studied for vectors / integer sequences with several possible definitions for the size of the intersection: the *permutation-type* intersection size of $\mathbf{x}, \mathbf{y} \in \{0, 1, \dots, q\}^n$ is $|\mathbf{x} \cap_{\text{perm}} \mathbf{y}| = |\{i : x_i = y_i\}|$; the *multiset-type* intersection size is defined as $|\mathbf{x} \cap_{\text{multi}} \mathbf{y}| = \sum_i \min\{x_i, y_i\}$. Results using the former definition include [9,10], while multiset-type results can be found in e.g. [11,12]. The main definition of our paper is as follows.

Definition 1.1. For integers $n, q, s \geq 1$ and two vectors $\mathbf{x}, \mathbf{y} \in Q^n$, we define the *size of their s -sum intersection* as $|\mathbf{x} \cap_s \mathbf{y}| = |\{i : x_i + y_i \geq s\}|$.

For $t \geq 1$ we say that $\mathbf{x}, \mathbf{y} \in Q^n$ are s -sum t -intersecting, if $|\mathbf{x} \cap_s \mathbf{y}| \geq t$. More generally, $\mathcal{F} \subset Q^n$ is s -sum t -intersecting if any two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ are s -sum t -intersecting.

In case of $t = 1$ we just simply write s -sum intersecting instead of s -sum 1-intersecting.

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Note that the case $q = 1$ and $s = 2$ corresponds to ordinary set intersection.

We will consider analogs of the Erdős–Ko–Rado theorem and theorems about Bollobás’s intersecting set-pair systems. To be able to state our results first we need to define uniformity for families of vectors. One has several options: as in the case of multisets and many other types of problems, we can work with the *weight/rank* $r(\mathbf{x}) := \sum_{i=1}^n x_i$ of $\mathbf{x} \in Q^n$ and say that for an integer $r \geq 0$ a family $\mathcal{F} \subseteq Q^n$ is *r-rank uniform* if $r(\mathbf{x}) = r$ for all $\mathbf{x} \in \mathcal{F}$. Another possibility is to use the size of the support $S_{\mathbf{x}} := \{i : x_i \neq 0\}$ of \mathbf{x} . We say that $\mathcal{F} \subseteq Q^n$ is *r-support uniform* if $|S_{\mathbf{x}}| = r$ for every $\mathbf{x} \in \mathcal{F}$.

We use standard notation. For any set X , we denote by $\binom{X}{r}$ the family of all r -subsets of X and 2^X denotes the power set of X . For a set $F \subset [n]$ we denote its complement $[n] \setminus F$ by \bar{F} and for \mathcal{F} a family of subsets of $[n]$ we introduce the notation $\bar{\mathcal{F}} := \{\bar{F} : F \in \mathcal{F}\}$.

As a vector analog for any $\mathbf{x} \in Q^n$ we define its ‘complement’ $\bar{\mathbf{x}}$ by letting $\bar{x}_i := q - x_i$ for all $i \in [n]$ and for a family \mathcal{F} of vectors in Q^n we write $\bar{\mathcal{F}} := \{\bar{\mathbf{x}} : \mathbf{x} \in \mathcal{F}\}$.

The structure of the paper is as follows. In Subsection 1.1 we state various results about s -sum intersecting families of vectors, while in Subsection 1.2 we list our results about intersecting vector pairs. In Section 2 and Section 3 we prove our results about intersecting vectors and intersecting vector pairs, respectively. In Section 4—as concluding results—we give a new intersection definition to provide analogs of some results that would not work with s -sum intersection.

1.1. Results on intersecting families of vectors

Let us start with stating the seminal result of Erdős, Ko and Rado [6].

Theorem 1.2 (Erdős, Ko, Rado [6]). For $n, r \geq 1$ with $2r \leq n$ if $\mathcal{F} \subseteq \binom{[n]}{r}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Moreover, if $2r < n$ and $|\mathcal{F}| = \binom{n-1}{r-1}$, then $\mathcal{F} = \mathcal{F}_i := \{F : i \in F \in \binom{[n]}{r}\}$ holds for some $i \in [n]$.

Furthermore, for any $1 \leq t < r$ there exists $n_0 = n_0(r, t)$ such that if $\mathcal{F} \subseteq \binom{[n]}{r}$ is t -intersecting, then $|\mathcal{F}| \leq \binom{n-t}{r-t}$, and equality holds if and only if $\mathcal{F} = \{F : T \subset F \in \binom{[n]}{r}\}$ for some $T \in \binom{[n]}{t}$.

The exact value of the smallest possible $n_0(r, t)$ was obtained by Frankl [8] and Wilson [22]. The largest possible size of an r -uniform t -intersecting family for all values of $n, t, r \geq 1$ was determined by Ahlswede and Khachatrian [2].

Our first result is a generalization of the Erdős–Ko–Rado (EKR) theorem for r -support uniform families. If s is even, then the vector family corresponding to \mathcal{F}_i of Theorem 1.2 is $\{\mathbf{x} \in Q^n : x_i \geq \frac{s}{2}, |S_{\mathbf{x}}| = r\}$. If s is odd, then to the family $\{\mathbf{x} \in Q^n : x_i \geq \lceil \frac{s}{2} \rceil, |S_{\mathbf{x}}| = r\}$ one can add vectors \mathbf{y} with $y_i = \lfloor \frac{s}{2} \rfloor$ that pairwise s -sum intersect on some other coordinate.

Observe that if $s \leq q$ holds, then any vector having at least one entry at least s can be added to any s -sum intersecting family, so it is enough to consider the case $q < s$.

Theorem 1.3. For any $2 \leq q < s$ and integer $r \geq 1$, if $\mathcal{F} \subseteq Q^n$ is r -support uniform s -sum intersecting with $n \geq qr^2$, then

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)q^{r-1} \binom{n-1}{r-1} & \text{if } s \text{ is even,} \\ 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^r \binom{n-i}{r-i} q^{r-i} & \text{if } s \text{ is odd,} \end{cases} \quad (1)$$

and these bounds are best possible.

The statement and proof of Theorem 1.3 can be adjusted for the r -rank uniform case, too. Instead, we provide a different proof in the special case $s = q + 1$ that works for all meaningful values of n . Before stating our theorem, observe that if both $\mathbf{x}, \mathbf{y} \in Q^n$ have rank less than $\frac{q+1}{2}$, then they cannot $(q+1)$ -sum intersect, while if both of them have rank greater than $\frac{qn}{2}$, then they always $(q+1)$ -sum intersect. We denote by $Q(n, r)$ the set of all vectors in Q^n of rank r . Just as in Theorem 1.3, if $q+1$ is even, then extremal families are stars (all vectors with entry at least $\frac{q+1}{2}$ in one fixed coordinate), while if $q+1$ is odd, then one can add further vectors to the star.

Theorem 1.4. Let $n, q, r \geq 1$ and $\mathcal{F} \subseteq Q^n$ be an r -rank uniform $(q+1)$ -sum intersecting family with $\frac{q+1}{2} \leq r \leq \frac{qn}{2}$. Then

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=\frac{q+1}{2}}^q |Q(n-1, r-j)| & \text{if } q+1 \text{ is even,} \\ 1 + \sum_{j=\lceil \frac{q+1}{2} \rceil}^q \sum_{i=\lfloor \frac{2(r-1)}{q} \rfloor}^{\lfloor \frac{2(r-1)}{q} \rfloor} |Q(n-i, r-j - \frac{(i-1)q}{2})| & \text{if } q+1 \text{ is odd,} \end{cases} \quad (2)$$

and these bounds are best possible.

Now we continue with s -sum t -intersecting families with $t \geq 2$. If s is even, then again one can consider a t -subset $T \subset [n]$ and the corresponding family

$$\mathcal{F}_{n,q,s,r,T} := \left\{ \mathbf{x} \in Q^n : x_i \geq \frac{s}{2} \text{ for all } i \in T, |S_{\mathbf{x}}| = r \right\}.$$

In Section 2, we will prove that for large enough n these families contain the largest number of vectors among all s -sum t -intersecting families. We will also determine the extremal families if s is odd, but as in that case their definition is more technical, we postpone their introduction to Section 2.

1.2. Results on intersecting pairs of vectors

We continue with stating Bollobás's classical theorem on intersecting set-pair systems for which we prove sum-intersecting analogs. To do so, we recall that if $\mathcal{S} = \{(A_i, B_i) : i = 1, 2, \dots, n\}$ with $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq n$, then

- \mathcal{S} is called a *strong ISP-system* (shorthand for intersecting set-pair system) if $A_i \cap B_j \neq \emptyset$ for all $1 \leq i \neq j \leq n$;
- \mathcal{S} is called a *weak ISP-system* if at least one of $A_i \cap B_j \neq \emptyset$ and $B_i \cap A_j \neq \emptyset$ holds for all $1 \leq i \neq j \leq n$.

If also $a = \max_{1 \leq i \leq n} |A_i|$ and $b = \max_{1 \leq i \leq n} |B_i|$, then \mathcal{S} is a strong or weak (a, b) -system.

Theorem 1.5 (Bollobás [3]). *If $\mathcal{S} = \{(A_j, B_j) : j = 1, 2, \dots, m\}$ is a strong ISP-system, then the inequality*

$$\sum_{j=1}^m \frac{1}{\binom{|A_j|+|B_j|}{|A_j|}} \leq 1$$

holds. In particular, if \mathcal{S} is a strong (a, b) -system, then $m \leq \binom{a+b}{a}$.

The following general inequality is valid for weak ISP-systems.

Theorem 1.6 (Tuza [18]). *Let $0 < p < 1$ be any real number and $q = 1 - p$. If $\{(A_j, B_j) : j = 1, 2, \dots, m\}$ is a weak ISP-system, then the inequality*

$$\sum_{j=1}^m p^{|A_j|} q^{|B_j|} \leq 1$$

holds. Moreover, for every $a, b \in \mathbb{N}$ there exists a weak (a, b) -system for which equality holds for all $0 < p < 1$ and $q = 1 - p$.

For a general overview on ISP-systems and their applications in extremal combinatorics we refer to the two-part survey [19,20]. Theorem 1.6 implies the upper bound $m \leq \frac{(a+b)^{a+b}}{a^a b^b}$ for weak (a, b) -systems. The best lower bounds on the maximum size of weak (a, b) -systems are due to Király, Nagy, Pálvölgyi and Visontai [16], and Wagner [21].

Now we would like to generalize these notions to vector pairs in the s -sum intersecting setting. Note that there is no assumption on the size of the ground set of ISP-systems. Let us denote by $Q^{<\mathbb{N}} (\subset \mathbb{Z}^{<\mathbb{N}})$ the set of all $\mathbf{x} \in \mathbb{Q}^{\mathbb{N}}$ with finite support $S_{\mathbf{x}} = \{i : x_i > 0\}$.

Assume that for $\mathcal{F} = \{(\mathbf{x}^j, \mathbf{y}^j) \in Q^{<\mathbb{N}} \times Q^{<\mathbb{N}} : j = 1, 2, \dots, m\}$ we have $|\mathbf{x}^j \cap_s \mathbf{y}^j| = 0$ for all $j = 1, 2, \dots, m$. We say that \mathcal{F} is a *strong s -sum IVP-system* in $Q^{<\mathbb{N}}$, if $|\mathbf{x}^i \cap_s \mathbf{y}^j| \neq 0$ for all $1 \leq i \neq j \leq m$ and \mathcal{F} is a *weak s -sum IVP-system* in $Q^{<\mathbb{N}}$, if for all $1 \leq i \neq j \leq m$ at least one pair of $\mathbf{x}^i, \mathbf{y}^j$ or $\mathbf{x}^j, \mathbf{y}^i$ is s -sum intersecting. If the supports of all \mathbf{x}^j have size at most a , and the supports of all \mathbf{y}^j have size at most b , then we will talk about *strong and weak s -sum (a, b) -systems*.

The next observation shows that it is enough to deal with $(q+1)$ -sum IVP-systems in $Q^{<\mathbb{N}}$.

Observation 1.7.

- If \mathcal{F} is a strong/weak s -sum (a, b) -system, then for all $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ and all $i \leq m$ we have $x_i, y_i < s$.
- If $\mathcal{F} \subset Q^{<\mathbb{N}} \times Q^{<\mathbb{N}}$ is a strong/weak $(q+t)$ -sum (a, b) -system with $t > 1$, then there exists a $(q-t+2)$ -sum strong/weak (a, b) -system $\mathcal{F}' \subset (\{0, 1, \dots, q-t+1\}^{<\mathbb{N}})^2$ with $|\mathcal{F}| = |\mathcal{F}'|$.

Proof. If $x_i \geq s$ or $y_j \geq s$ for some $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$, then $|\mathbf{x} \cap_s \mathbf{y}| > 0$. This implies (i).

To see (ii), for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ introduce $(\mathbf{x}', \mathbf{y}')$ with $x'_i = \max\{x_i - t + 1, 0\}$, $y'_i = \max\{y_i - t + 1, 0\}$ for all indices i . Clearly, for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ and index j , we have $x'_j + y'_j < q + t - 2(t-1) = q - t + 2$. Furthermore, if $|\mathbf{x}^{h_1} \cap_{q+t} \mathbf{y}^{h_2}| > 0$, then there exists an index j with $q + t \leq x_j^{h_1} + y_j^{h_2}$. So $x_j^{h_1'} + y_j^{h_2'} \geq q + t - 2(t-1) = q - t + 2$, and thus the system $\mathcal{F}' = \{(\mathbf{x}', \mathbf{y}') : (\mathbf{x}, \mathbf{y}) \in \mathcal{F}\} \subset (\{0, 1, \dots, q-t+1\}^{<\mathbb{N}})^2$ is a $(q-t+2)$ -sum strong/weak (a, b) -system. \square

To obtain bounds on the size of $(q+1)$ -sum IVP-systems, we write $m(q, k)$ and $m'(q, k)$ for the maximum number of vector pairs in a strong/weak $(q+1)$ -sum (k, k) -system. In particular, for $q = 2$ and $s = 3$ let $m(k) := m(2, k)$.

To estimate $m(k)$, we let

$$f(k) := \max \frac{(x+y+z)!}{x!y!z!},$$

where the maximum is taken over all nonnegative integers x, y, z such that $x+z \leq k$ and $y+z \leq k$. The following inequalities provide an almost tight bound on $m(k)$, with only a linear multiplicative error in k , while the function is exponential.

Theorem 1.8. *For every $k \geq 1$ we have*

$$f(k) \leq m(k) \leq k \cdot f(k).$$

Finally, we determine the order of magnitude of the maximum size of strong and weak $(q+1)$ -sum IVP systems in $Q^{<\mathbb{N}}$ up to a polynomial factor.

Theorem 1.9. *For any $q \geq 1$, $\lim_{k \rightarrow \infty} \sqrt[k]{m(q, k)} = \lim_{k \rightarrow \infty} \sqrt[k]{m'(q, k)} = (\sqrt{q} + 1)^2$.*

A standard calculation shows that the maximum in the definition of $f(k)$ is attained when $z = (1 - \frac{1}{\sqrt{2}})k + O(1)$ and $x = y = k - z$. Plugging in these values, we obtain that $f(k) = (c + o(1)) \frac{1}{k} (3 + 2\sqrt{2})^k$ for some real $c < 1$. The upper bound of Theorem 1.8 on strong 3-sum (k, k) -systems is a constant factor smaller than the upper bound of Theorem 1.9 on weak 3-sum (k, k) -systems.

2. Sum-intersecting families of vectors

This section contains the proofs of Theorem 1.3, Theorem 1.4 and Theorem 2.11, but we consider first non-uniform $(q+1)$ -sum intersecting vector families.

Proposition 2.1. *For $n, q \geq 1$ if $\mathcal{F} \subseteq Q^n$ is $(q+1)$ -sum intersecting, then $|\mathcal{F}| \leq \lceil \frac{(q+1)^n}{2} \rceil$ and this bound is best possible.*

Proof. Note that we cannot have \mathbf{x} and $\bar{\mathbf{x}}$ both belong to \mathcal{F} . Moreover, there exists one vector \mathbf{x} with $\bar{\mathbf{x}} = \mathbf{x}$ if and only if q is even. This proves the upper bound. For the lower bound consider the family of all vectors with rank larger than $\frac{qn}{2}$ together with one vector from each pair of (the not necessarily different vectors) $\mathbf{x}, \bar{\mathbf{x}}$ of rank $\frac{qn}{2}$ (if such pairs exist). \square

Corollary 2.2. *For $n, q, s \geq 1$ with $q \geq s$ if $\mathcal{F} \subseteq Q^n$ is s -sum intersecting, then $|\mathcal{F}| \leq (q+1)^n - s^n + \lceil \frac{s^n}{2} \rceil$ and this bound is best possible.*

Proof. If a vector contains an entry at least s , then it s -sum intersects every other vector. The number of such vectors is $(q+1)^n - s^n$, and then we apply Proposition 2.1 to the set of all other vectors. \square

Now we turn our attention to (rank- or support-) uniform families of vectors. We start with the proof of Theorem 1.4, but we need several definitions and some results from the literature.

Definition 2.3. The *shadow* $\Delta(F)$ of a set F is $\{G \subset F : |G| = |F| - 1\}$ and the *shadow* $\Delta(\mathcal{F})$ of a family \mathcal{F} of sets is $\cup_{F \in \mathcal{F}} \Delta(F)$. If \mathcal{F} is r -uniform and $0 \leq \ell < r$, then $\Delta_\ell(\mathcal{F}) := \{G : |G| = \ell \text{ and } \exists F \in \mathcal{F} \text{ s.t. } G \subset F\}$.

We introduce the notation $<_{\text{colex}}$ for the *colex ordering* of all finite subsets of the positive integers. In this ordering for two finite sets A and B we have $A <_{\text{colex}} B$ if and only if the largest element of the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B belongs to B .

Kruskal and Katona independently proved the following fundamental theorem.

Theorem 2.4 (Kruskal [17], Katona [15]). *Let $n, r, m \geq 1$ and $L_{r,m}$ be the initial segment of $\binom{[n]}{r}$ of size m with respect to the colex ordering. For any $\mathcal{F} \subseteq \binom{[n]}{r}$ of size m , we have $|\Delta(\mathcal{F})| \geq |\Delta(L_{r,m})|$.*

We can introduce the notion of shadow for vectors, too.

Definition 2.5. The *shadow* $\Delta(\mathbf{x})$ of a vector $\mathbf{x} \in Q^n$ is $\{\mathbf{y} < \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x}) - 1\}$, where $<$ denotes the coordinate-wise ordering, i.e., for two vectors \mathbf{x} and \mathbf{y} we have $\mathbf{y} < \mathbf{x}$ if and only if $y_i \leq x_i$ for all $1 \leq i \leq n$ and $y_i < x_i$ for at least one i . Then for $\mathcal{F} \subseteq Q^n$ we define the *shadow* $\Delta(\mathcal{F})$ of \mathcal{F} as $\cup_{\mathbf{x} \in \mathcal{F}} \Delta(\mathbf{x})$ and for r -rank uniform \mathcal{F} and $\ell < r$ we let $\Delta_\ell(\mathcal{F}) = \{\mathbf{y} : r(\mathbf{y}) = \ell \text{ and } \exists \mathbf{x} \in \mathcal{F} \text{ such that } \mathbf{y} < \mathbf{x}\}$. We will write $\mathbf{x} \leq \mathbf{y}$ for $\mathbf{x} < \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

Analogously to the set case, we can introduce the *colex ordering* of Q^n , i.e., for $\mathbf{x}, \mathbf{y} \in Q^n$ we have $\mathbf{x} <_{\text{colex}} \mathbf{y}$ if and only if $x_i < y_i$ where i is the largest coordinate in which \mathbf{x} and \mathbf{y} differ.

Clements and Lindström proved a generalization of the Kruskal-Katona theorem for the shadows of vectors introduced in Definition 2.5.

Theorem 2.6 (Clements, Lindström [4]). Let $q, r, m, n \geq 1$, and let $L_{q,r,m}$ be the initial segment of $Q(n, r)$ of size m with respect to the colex ordering. For any $\mathcal{F} \subseteq Q(n, r)$ of size m , we have $|\Delta(\mathcal{F})| \geq |\Delta(L_{q,r,m})|$.

One can easily check the following properties of the colex ordering of sets and vectors, so we omit their proof.

Proposition 2.7. Suppose $n \geq r \geq 1$.

(i) Both in $\binom{[n]}{r}$ and in $Q(n, r)$, the shadow of an initial segment is an initial segment, so one can iterate Theorems 2.4 and 2.6 to obtain that initial segments minimize the size of shadows of any lower rank.

(ii) If \mathcal{F} is the family of the largest m sets of $\binom{[n]}{r}$ with respect to the colex ordering, then $\overline{\mathcal{F}} = L_{n-r,m}$.

(iii) If \mathcal{F} is the family of the largest m vectors of $Q(n, r)$ with respect to the colex ordering, then $\overline{\mathcal{F}} = L_{q,qn-r,m}$.

Before the proof of Theorem 1.4, let us briefly recall the proof of the upper bound in Theorem 1.2 that uses the Kruskal-Katona shadow theorem (Theorem 2.4) and was obtained by Daykin [5] as we would like to mimic it.

Suppose contrary to the statement of Theorem 1.2 that there exists an intersecting family $\mathcal{F} \subseteq \binom{[n]}{r}$ of size larger than $\binom{n-1}{r-1}$. Consider the family $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$ and observe that as \mathcal{F} is intersecting and $n \geq 2r$, we must have $\mathcal{F} \cap \Delta_r(\overline{\mathcal{F}}) = \emptyset$. Clearly, $|\overline{\mathcal{F}}| = |\mathcal{F}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$. Applying Theorem 2.4, any $(n-r)$ -uniform family of size larger than $\binom{y}{n-r}$ has r -shadow larger than $\binom{y}{r}$. So $\binom{n}{r} = |\binom{[n]}{r}| \geq |\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. This contradiction proves the upper bound in Theorem 1.2.

This proof seems to be very lucky that it includes miraculous equalities $\binom{n-1}{r-1} = \binom{n-1}{n-r}$ and $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$, so let us recite it without any calculation. Consider greedily the largest sets of $\binom{[n]}{r}$ with respect to the colex order as long as they form an intersecting family. Let \mathcal{F}_0 be the family when we need to stop. If $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = \binom{[n]}{r}$, then \mathcal{F}_0 is a largest possible intersecting family. Indeed, if $|\mathcal{F}| > |\mathcal{F}_0|$, then as $\overline{\mathcal{F}_0}$ is an initial segment, by Proposition 2.7 (i) and (ii), we have $|\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| > |\mathcal{F}_0| + |\Delta_r(\overline{\mathcal{F}_0})| = \binom{n}{r}$, so \mathcal{F} cannot be intersecting. To obtain the results of Theorem 1.2 about intersecting families, all we need to observe is that $\mathcal{F}_0 = \{F \in \binom{[n]}{r} : n \in F\}$ and $\Delta_r(\overline{\mathcal{F}_0}) = \{F \in \binom{[n]}{r} : n \notin F\}$.

Before the proof of Theorem 1.4 let us restate it.

Theorem 1.4. Let $n, q, r \geq 1$ and $\mathcal{F} \subseteq Q^n$ be an r -rank uniform $(q+1)$ -sum intersecting family with $\frac{q+1}{2} \leq r \leq \frac{qn}{2}$. Then

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=\frac{q+1}{2}}^q |Q(n-1, r-j)| & \text{if } q+1 \text{ is even,} \\ 1 + \sum_{j=\lceil \frac{q+1}{2} \rceil}^q \sum_{i=1}^{\lfloor \frac{2(r-1)}{q} \rfloor} |Q(n-i, r-j-\frac{(i-1)q}{2})| & \text{if } q+1 \text{ is odd,} \end{cases} \quad (3)$$

and these bounds are best possible.

Proof. Clearly $\mathbf{x} \in Q^n$ does not $(q+1)$ -sum intersect a vector $\mathbf{y} \in Q^n$ if and only if $\mathbf{y} \leq \bar{\mathbf{x}}$. Also, $\mathcal{F} \subseteq Q(n, r)$ is a $(q+1)$ -sum intersecting family if and only if $\mathcal{F} \cap \Delta_r(\overline{\mathcal{F}})$ contains at most one vector as \mathcal{F} may contain one vector \mathbf{x} that does not $(q+1)$ -sum intersect itself. Indeed, if $\mathbf{x} \neq \mathbf{y}$ and $|\mathbf{x} \cap_{q+1} \mathbf{y}| = 0$, then $\mathbf{x}, \mathbf{y} \in \mathcal{F} \cap \Delta_r(\overline{\mathcal{F}})$. On the other hand if $\mathbf{x}, \mathbf{y} \in \mathcal{F} \cap \Delta_r(\overline{\mathcal{F}})$ and \mathcal{F} is intersecting, then by the above, we cannot have $\mathbf{x} \leq \bar{\mathbf{y}}$. Therefore, we must have $\mathbf{x} \leq \bar{\mathbf{x}}$ and $\mathbf{y} \leq \bar{\mathbf{y}}$. But as $|\mathbf{x} \cap_{q+1} \mathbf{y}| > 0$, there must exist an index i with $x_i + y_i \geq q+1$, so either x_i or y_i , say x_i , is at least $\frac{q+1}{2}$. But then $x_i > q - x_i = \bar{x}_i$, a contradiction.

The reasoning of Daykin stays valid with a little modification, if for the maximal $(q+1)$ -sum intersecting family $\mathcal{F}_0 \subseteq Q(n, r)$ consisting of largest vectors with respect to the colex ordering we have both $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = Q(n, r)$ and $|\Delta_r(\overline{\mathcal{F}_0})| < |\Delta_r(\overline{\mathcal{F}_0}^+)|$, where $\overline{\mathcal{F}_0}^+$ is the initial segment of the colex ordering of $Q(n, qn-r)$ one larger than $\overline{\mathcal{F}_0}$. Indeed, if \mathcal{F} was an r -rank uniform $(q+1)$ -sum intersecting family larger than \mathcal{F}_0 , then we would get a contradiction by the following series of inequalities:

$$|\mathcal{F} \cup \Delta_r(\overline{\mathcal{F}})| \geq |\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| - 1 \geq |\mathcal{F}_0| + 1 + |\Delta_r(\overline{\mathcal{F}_0})| + 1 - 1 = |Q(n, r)| + 1.$$

And this is exactly the case: for the maximal $(q+1)$ -sum intersecting family $\mathcal{F}_0 \subseteq Q(n, r)$ consisting of largest vectors with respect to the colex ordering, we prove that we have both $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = Q(n, r)$ and $|\Delta_r(\overline{\mathcal{F}_0})| < |\Delta_r(\overline{\mathcal{F}_0}^+)|$, where $\overline{\mathcal{F}_0}^+$ is the one larger initial segment of the colex ordering of $Q(n, qn-r)$ than $\overline{\mathcal{F}_0}$.

Suppose first that $q + 1 = 2k$. Then $\mathcal{F}_0 = \{\mathbf{x} \in Q(n, r) : x_n \geq k\}$, $\overline{\mathcal{F}}_0 = \{\mathbf{x} \in Q(n, qn - r) : x_n < k\}$ and clearly $\Delta_r(\overline{\mathcal{F}}_0) = \{\mathbf{x} \in Q(n, r) : x_n < k\} = Q(n, r) \setminus \mathcal{F}_0$; and since $\overline{\mathcal{F}}_0^+$ contains a vector \mathbf{x} with $x_n = k$, its r -shadow is strictly larger than that of $\overline{\mathcal{F}}_0$.

Suppose next $q + 1 = 2k + 1$. Then

$$\mathcal{F}_0 = \bigcup_{j=0}^{\lfloor \frac{r-1}{k} \rfloor} \{\mathbf{x} \in Q(n, r) : x_n = x_{n-1} = \dots = x_{n-j+1} = k, x_{n-j} > k\} \cup \{\mathbf{x}^*\},$$

where $x_n^* = x_{n-1}^* = \dots = x_{n-\lfloor \frac{r-1}{k} \rfloor}^* = k$, $x_{n-\lfloor \frac{r-1}{k} \rfloor - 1}^* \equiv r \pmod{k}$ and all other entries are 0. Observe that \mathbf{x}^* does not $(q + 1)$ -sum intersect itself. To see that $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}}_0) = Q(n, r)$ holds, one only has to observe that any vector $\mathbf{y} \in Q(n, r) \setminus \mathcal{F}_0$ with $y_n = y_{n-1} = \dots = y_{n-\lfloor \frac{r-1}{k} \rfloor} = k$ belongs to $\Delta_r(\overline{\mathbf{x}}^*)$. Also, any vector $\mathbf{y} \in Q(n, r)$ with $\overline{\mathbf{x}}^* <_{\text{colex}} \mathbf{y}$ has an entry larger than k in the last $\lfloor \frac{r-1}{k} \rfloor$ coordinates, so $|\Delta_r(\overline{\mathcal{F}}_0^+)| > |\Delta_r(\overline{\mathcal{F}}_0)|$.

This completes the proof of Theorem 1.4. \square

We continue with the proof of Theorem 1.3. Before doing so, we cite two well-known stability results that we use during the proof of Theorems 1.3 and 2.11.

Theorem 2.8 (Hilton, Milner [13]). If $\mathcal{F} \subseteq \binom{[n]}{r}$ is an intersecting family with $n \geq 2r + 1$ and $\cap_{F \in \mathcal{F}} F = \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

Theorem 2.9 (Frankl [7]). Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be a t -intersecting family with $|\cap_{F \in \mathcal{F}} F| < t$. If n is large enough, then $|\mathcal{F}| \leq \max\{|\mathcal{F}_1|, |\mathcal{F}_2|\}$, where

$$\mathcal{F}_1 = \left\{ F \in \binom{[n]}{r} : [t] \subset F, F \cap [t + 1, r + 1] \neq \emptyset \right\} \cup \binom{[r + 1]}{r}$$

and

$$\mathcal{F}_2 = \left\{ F \in \binom{[n]}{r} : |F \cap [t + 2]| \geq t + 1 \right\}.$$

Now let us restate Theorem 1.3.

Theorem 1.3. For any $s > q \geq 2$ and integer $r \geq 1$, if $\mathcal{F} \subseteq Q^n$ is r -support uniform s -sum intersecting with $n \geq qr^2$, then

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)q^{r-1} \binom{n-1}{r-1} & \text{if } s \text{ is even,} \\ 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^r \binom{n-i}{r-i} q^{r-i} & \text{if } s \text{ is odd,} \end{cases} \quad (4)$$

and these bounds are best possible.

Proof of Theorem 1.3. Suppose first that s is even. The constructions showing that the bound is best possible are $\mathcal{F}_{n,q,s,r,i} = \{\mathbf{x} \in Q^n : \frac{s}{2} \leq x_i\}$. To see the upper bound, let \mathcal{F} be an r -support uniform s -sum intersecting family and let $\mathcal{S}_{\mathcal{F}}$ denote the family of supports in \mathcal{F} . For a fixed support S , the number of vectors having S as support is bounded by a constant (depending on r and q), therefore, by Theorem 2.8, unless all supports in $\mathcal{S}_{\mathcal{F}}$ share a common element i , we have $|\mathcal{F}| \leq q^r r \binom{n-2}{r-2} < (q - \frac{s}{2} + 1)q^{r-1} \binom{n-1}{r-1}$ if $n \geq qr^2$. So we can suppose that there exists an index i that belongs to all supports. Assume next that there exists $\mathbf{x} \in \mathcal{F}$ with $x_i < \frac{s}{2}$. Then consider the subfamily $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_i \leq \frac{s}{2}\}$. As vectors in \mathcal{F}' must all s -sum intersect \mathbf{x} , but they do not s -sum intersect it at coordinate i , therefore their supports must intersect the support of \mathbf{x} in some coordinate other than i . Therefore, we obtain $|\mathcal{S}_{\mathcal{F}'}| \leq (r - 1) \binom{n-2}{r-2}$ and thus $|\mathcal{F}'| \leq \frac{s}{2} q^{r-1} (r - 1) \binom{n-2}{r-2}$. But then

$$|\mathcal{F}| \leq |\mathcal{F}'| + (q - \frac{s}{2})q^{r-1} \binom{n-1}{r-1} < (q - \frac{s}{2} + 1)q^{r-1} \binom{n-1}{r-1}$$

if $n \geq r^2 \frac{s}{2}$. We obtained that either \mathcal{F} is smaller than the claimed bound or $\mathcal{F} \subseteq \mathcal{F}_{n,q,s,r,i}$ for some index i .

Suppose next that s is odd. The extremal families are defined via ordered r -tuples (i_1, i_2, \dots, i_r) in the following way:

$$\mathcal{F}_{n,q,s,(i_1,i_2,\dots,i_r)} = \{\mathbf{x}\} \cup \bigcup_{j=1}^r \{\mathbf{y} \in Q^n : y_{i_1} = y_{i_2} = \dots = y_{i_{j-1}} = \lfloor \frac{s}{2} \rfloor, y_{i_j} \geq \frac{s}{2}\},$$

where \mathbf{x} is the vector with $x_{i_j} = \lfloor \frac{s}{2} \rfloor$ for all $1 \leq j \leq r$ and $x_i = 0$ otherwise. To prove the upper bound, we proceed by induction on r . If $r = 1$, then all supports of an r -support uniform s -sum intersecting family \mathcal{F} must be the same singleton

{i}. If m is the minimum entry over all vectors in \mathcal{F} at coordinate i , then all other entries must be at least $s - m$, so the number of vectors is at most $\min\{q - m + 1, q - (s - m)\}$. This is maximized if $m = \lfloor \frac{s}{2} \rfloor$ and the claimed bound follows. Let $r > 1$, and $\mathcal{F} \subseteq Q^n$ be an r -support uniform, s -sum intersecting family. Then just as in the even s case, using Theorem 2.8, we obtain that $|\mathcal{F}| \leq q^r r \binom{n-2}{r-2} < q^{r-1} \binom{n-1}{r-1}$ unless all sets in $\mathcal{S}_{\mathcal{F}}$ share a common element i_1 or $n \leq qr^2$. If there exists a vector $\mathbf{z} \in \mathcal{F}$ with $z_{i_1} < \lfloor \frac{s}{2} \rfloor$, then also just as in the even s case, we obtain that $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_{i_1} \leq \lceil \frac{s}{2} \rceil\}$ is of size at most $\lceil \frac{s}{2} \rceil q^{r-1} (r-1) \binom{n-2}{r-2}$ and thus \mathcal{F} is smaller than the claimed bound if $n \geq sr^2$. So we can assume that for all vectors $\mathbf{z} \in \mathcal{F}$, we have $z_{i_1} \geq \lfloor \frac{s}{2} \rfloor$. The number of those vectors \mathbf{z} with $z_{i_1} \geq \lceil \frac{s}{2} \rceil$ is $(q - \lceil \frac{s}{2} \rceil + 1) q^{r-1} \binom{n-1}{r-1}$, while the family $\mathcal{F}^* = \{\mathbf{z}' \in \mathcal{F} : z_{i_1} = \lfloor \frac{s}{2} \rfloor\}$ is $(r-1)$ -support uniform, s -sum intersecting, where \mathbf{z}' is the vector obtained from \mathbf{z} by removing its i_1 st entry. As $n-1 \geq qr^2 - 1 \geq q(r-1)^2$, by induction, we obtain

$$|\mathcal{F}^*| \leq 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r-1} q^{r-1-i} \binom{n-1-i}{r-1-i}$$

and so

$$\begin{aligned} |\mathcal{F}| &\leq |\mathcal{F}^*| + (q - \lceil \frac{s}{2} \rceil + 1) q^{r-1} \binom{n-1}{r-1} \\ &\leq 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r-1} q^{r-1-i} \binom{n-1-i}{r-1-i} + (q - \lceil \frac{s}{2} \rceil + 1) q^{r-1} \binom{n-1}{r-1} \\ &= 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^r q^{r-i} \binom{n-i}{r-i}, \end{aligned}$$

as claimed. \square

In the remainder of this section, we consider s -sum t -intersecting families with $t \geq 2$.

Construction 2.10. For any $n, q, r, t \geq 1$ with $n \geq r \geq t$ and s even with $q < s \leq 2q$ and for any $T \in \binom{[n]}{t}$ let us define

$$\mathcal{F}_{n,q,s,r,T} := \left\{ \mathbf{x} \in Q^n : x_i \geq \frac{s}{2} \text{ for all } i \in T, |S_{\mathbf{x}}| = r \right\}.$$

Observe that the size of $\mathcal{F}_{n,q,s,r,T}$ is $(q - \frac{s}{2} + 1)^t q^{r-t} \binom{n-t}{r-t}$.

For $n, r, q, t \geq 1$ with $n \geq r \geq t$ and s odd with $q < s < 2q$ let us define the following r -support uniform families: for any $T' \in \binom{[r]}{t-1}$ we pick a vector $\mathbf{x}_{T'}$ with $S_{\mathbf{x}_{T'}} = [r]$ such that $(x_{T'})_i > \frac{s}{2}$ for all $i \in T'$, and $(x_{T'})_i = \lfloor \frac{s}{2} \rfloor$ for all $i \in [r] \setminus T'$. Then we have

$$\begin{aligned} \mathcal{F}_{n,q,s,r,t} &:= \left\{ \mathbf{x}_{T'} : T' \in \binom{[r]}{t-1} \right\} \cup \\ &\bigcup_{T \in \binom{[r]}{t}} \left\{ \mathbf{y} \in Q^n : (\forall i \in T) (y_i > \lfloor \frac{s}{2} \rfloor) \wedge (\forall i \in [\max T] \setminus T) (y_i = \lfloor \frac{s}{2} \rfloor) \wedge |S_{\mathbf{y}}| = r \right\}. \end{aligned}$$

In words, the vectors belonging to the second row of the definition have at least t indices $i_1 < i_2 < \dots < i_t \leq r$ such that their corresponding entries have value strictly greater than $\lfloor s/2 \rfloor$, and all entries with indices smaller than i_t have value at least $\lfloor s/2 \rfloor$.

The family $\mathcal{F}_{n,q,s,r,t}$ is s -sum t -intersecting as for any pair $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{n,q,s,r,t}$ there exist at least t coordinates $i \in [r]$, where one of x_i, y_i is at least $\lfloor \frac{s}{2} \rfloor$ while the other is at least $\lceil \frac{s}{2} \rceil$.

Let $f(n, q, s, r, t)$ denote the size of $\mathcal{F}_{n,q,s,r,t}$.

Theorem 2.11. For any $2 \leq q < s \leq 2q$ and $r \geq t \geq 1$, there exists $n(q, s, r, t)$ such that if $\mathcal{F} \subseteq Q^n$ is r -support uniform s -sum t -intersecting with $n \geq n(q, s, r, t)$, then

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)^t q^{r-t} \binom{n-t}{r-t} & \text{if } s \text{ is even,} \\ f(n, q, s, r, t) & \text{if } s \text{ is odd,} \end{cases} \quad (5)$$

and these bounds are best possible as shown by the families of Construction 2.10.

We will need the following simple observations on $f(n, q, s, r, t)$.

Proposition 2.12. Suppose that n, q, s, r, t are integers with the assumptions on them as in Construction 2.10.

(i) If $r \geq 2t$, then

$$f(n, q, s, r, t) = \binom{n-t}{r-t} q^{r-t} (q - \lfloor \frac{s}{2} \rfloor)^t + \sum_{S \subseteq [t]} (q - \lfloor \frac{s}{2} \rfloor)^{|S|} f(n-t, q, s, r-t, t-|S|).$$

(ii) If $t < r < 2t$, then

$$f(n, q, s, r, t) = \binom{n-t}{r-t} q^{r-t} (q - \lfloor \frac{s}{2} \rfloor)^t + \binom{t}{2t-r-1} + \sum_{S \subseteq [t], |S| \geq 2t-r} (q - \lfloor \frac{s}{2} \rfloor)^{|S|} f(n-t, q, s, r-t, t-|S|).$$

(iii) If $r > t$, then $f(n, q, s, r, t) > (q - \lfloor s/2 \rfloor)^t q^{r-t} \binom{n-t}{r-t}$.

Proof. In all of (i), (ii) and (iii), the first term of the right-hand side (in case of (iii), the only term) stands for those vectors for which $x_i > \frac{s}{2}$ for all $1 \leq i \leq t$. In (i), the big sum partitions the other vectors according to which of the first t entries have value greater than $s/2$.

In (ii), the big summation can neglect small subsets S of $[t]$, because if $|S| < 2t - r$, then for any $\mathbf{x} \in \mathcal{F}_{n,q,s,r,t}$ with $\{i \in [t] : x_i > s/2\} = S$, the number of indices i in $[r]$ for which $x_i > s/2$ is at most $|S| + r - t < t$. So if $|S| < 2t - r$, then to reach at least $t - 1$ such indices (the minimum for a vector in $\mathcal{F}_{n,q,s,r,t}$), we need exactly $2t - r - 1$ indices from $[t]$. The middle term stands for those $\mathbf{x}_{T',S}$ where T' contains exactly $2t - r - 1$ elements from $[t]$. \square

Now we continue with the proof of Theorem 2.11.

Proof of Theorem 2.11. Suppose first that s is even. To see the upper bound, let \mathcal{F} be an r -support uniform s -sum t -intersecting family and let $\mathcal{S}_{\mathcal{F}}$ denote the family of supports in \mathcal{F} . For a fixed support S , the number of vectors having S as support is bounded by a constant (depending on r and q), therefore, by Theorem 2.9, unless all supports in $\mathcal{S}_{\mathcal{F}}$ share all elements of a t -subset T of $[n]$, we have $|\mathcal{F}| = O(n^{r-t-1}) < \binom{n-t}{r-t}$ if n is large enough. So we can suppose that there exists a t -subset T that is contained in all supports. Assume next that there exists $\mathbf{x} \in \mathcal{F}$ with $x_i < \frac{s}{2}$ for some $i \in T$. Then consider the subfamily $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_i \leq \frac{s}{2}\}$. As vectors in \mathcal{F}' must all s -sum t -intersect \mathbf{x} , but they do not s -sum intersect it at coordinate i , therefore their supports must intersect the support of \mathbf{x} in some coordinate outside T . Therefore, we obtain $|\mathcal{F}'| = O(n^{r-t-1})$. But then

$$|\mathcal{F}| \leq |\mathcal{F}'| + (q - \frac{s}{2})(q - \frac{s}{2} + 1)^{t-1} q^{r-t} \binom{n-t}{r-t} < (q - \frac{s}{2} + 1)^t q^{r-t} \binom{n-t}{r-t}$$

if n is large enough. We obtained that either \mathcal{F} is smaller than the claimed bound or $\mathcal{F} \subseteq \mathcal{F}_{n,q,s,r,T}$ for some t -subset T .

Suppose next that s is odd. We proceed by induction on $r + t$ and observe that in all cases, the family of supports must be t -intersecting. The case $t = 1$ is covered by Theorem 1.3. Let $\mathcal{F} \subseteq Q^n$ be an s -sum t -intersecting r -support uniform family. We consider three cases according to the relationship of r and t .

CASE I: $r = t$.

The assumption $r = t$ implies that all supports in \mathcal{F} are identical, say the support is S . Therefore, for any $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ and $i \in S$ we must have $x_i + y_i \geq s$. In particular, for any $i \in S$ there is at most one $\mathbf{x} \in \mathcal{F}$ with $x_i < s/2$. So

$$|\mathcal{F}| \leq \binom{t}{t-1} + (q - \lfloor s/2 \rfloor)^t,$$

as claimed.

CASE II: $t < r$

The family $\mathcal{S}_{\mathcal{F}}$ of supports is t -intersecting, so unless all supports of \mathcal{F} share t elements, we have $|\mathcal{F}| = O(n^{r-t-1})$ by Theorem 2.9. Let T be the set of these t elements, and for any $S \subset T$ let \mathcal{F}_S denote the family of those vectors $\mathbf{x} \in \mathcal{F}$ for which $x_i \geq s/2$ for all $i \in S$, and $1 \leq x_i \leq s/2$ for all $i \in T \setminus S$. As all supports contain T , we have $\mathcal{F} = \cup_{S \subset T} \mathcal{F}_S$. Clearly, $|\mathcal{F}_T| \leq q^{r-t} \binom{n-t}{r-t} (q - \lfloor s/2 \rfloor)^t$.

We claim that if there exists $\mathbf{x} \in \mathcal{F}$ with $x_i < \lfloor s/2 \rfloor$ for some $i \in T$, then $|\mathcal{F}| < f(n, q, s, r, t)$. Indeed, the vectors $\mathbf{y} \in \mathcal{F}_T$ with $y_i = \lfloor s/2 \rfloor$ s -sum t -intersect \mathbf{x} , so $S_{\mathbf{x}} \cap S_{\mathbf{y}} \cap ([n] \setminus T) \neq \emptyset$. Therefore the number of such vectors is $O(n^{r-t-1})$ and thus $|\mathcal{F}_T| \leq (q - \lfloor s/2 \rfloor)(q - \lfloor s/2 \rfloor)^{t-1} q^{r-t} \binom{n-t}{r-t} + O(n^{r-t-1})$. Also, for any $j \in T$, the number of vectors $\mathbf{z} \in \mathcal{F}$ with $z_j \leq \lfloor s/2 \rfloor$ is

at most $O(n^{r-t-1})$ as to have $|\mathbf{z} \cap_s \mathbf{z}'| \geq t$ for two such vectors, $S_{\mathbf{z}}$ and $S_{\mathbf{z}'}$ must intersect outside T . Adding up for all $j \in T$, we obtain

$$|\mathcal{F}| \leq (q - \lceil s/2 \rceil)(q - \lfloor s/2 \rfloor)^{t-1} q^{r-t} \binom{n-t}{r-t} + O(n^{r-t-1}) + O(tn^{r-t-1}) < f(n, q, s, r, t)$$

as claimed. Here the last inequality follows from Proposition 2.12 (iii) (watch out for floor and ceiling signs!). So we can assume that $x_i \geq \lfloor s/2 \rfloor$ for all $i \in T$, thus $x_i = \lfloor s/2 \rfloor$ for all $i \in T \setminus S$ and $\mathbf{x} \in \mathcal{F}_S$. This implies $|\mathcal{F}_S| \leq (q - \lfloor s/2 \rfloor)^{|S|} |\mathcal{F}'_S|$ with $\mathcal{F}'_S = \{\mathbf{x}' : \mathbf{x} \in \mathcal{F}_S\}$, where \mathbf{x}' is the vector obtained from \mathbf{x} by deleting the coordinates belonging to T .

CASE IIA: $t < r < 2t$.

Consider families \mathcal{F}_S for all subsets S with $2t - r \leq |S| < t$. Observe \mathcal{F}'_S is $(r - t)$ -support uniform s -sum $(t - |S|)$ -intersecting, and thus by induction, we have

$$|\mathcal{F}_S| \leq (q - \lfloor s/2 \rfloor)^{|S|} |\mathcal{F}'_S| \leq (q - \lfloor s/2 \rfloor)^{|S|} f(n - t, q, s, r - t, t - |S|).$$

Finally, consider all subsets $S \subset T$ with $|S| < 2t - r$. As for two vectors $\mathbf{x}', \mathbf{x}'' \in \mathcal{F}_S$, we have $|\mathbf{x}' \cap_s \mathbf{x}''| \leq |S| + r - t < 2t - r + r - t = t$, we must have $|\mathcal{F}_S| \leq 1$ for all such S . Observe that for any $(r + 1 - t)$ -subset $Z \subset T$ there exists at most one subset $S \subset T$ with $Z \cap S = \emptyset$ and $\mathcal{F}_S \neq \emptyset$. Indeed, if $\mathbf{x} \in \mathcal{F}_S$, $\mathbf{y} \in \mathcal{F}_{S'}$, then \mathbf{x} and \mathbf{y} can only s -sum intersect in at most $r - t$ coordinates outside T and in at most $t - (r + 1 - t) = 2t - r - 1$ coordinates within T , so $|\mathbf{x} \cap_s \mathbf{y}| \leq t - 1$, a contradiction. Therefore

$$\sum_{S \subset T, |S| < 2t-r} |\mathcal{F}_S| \leq \binom{t}{r+1-t} = \binom{t}{2t-r-1}.$$

Adding up these bounds for all $|\mathcal{F}_S|$ together with the bound on $|\mathcal{F}_T|$, we obtain the desired bound on $|\mathcal{F}|$ by Proposition 2.12 (ii).

CASE IIB: $2t \leq r$.

In this case, for any $S \subsetneq T$, the family \mathcal{F}'_S is $(r - t)$ -support uniform s -sum $(t - |S|)$ -intersecting, and thus by induction, we have

$$|\mathcal{F}_S| \leq (q - \lfloor s/2 \rfloor)^{|S|} |\mathcal{F}'_S| \leq (q - \lfloor s/2 \rfloor)^{|S|} f(n - t, q, s, r - t, t - |S|).$$

Adding up these bounds for all $|\mathcal{F}_S|$ together with the bound on $|\mathcal{F}_T|$, we obtain the desired bound on $|\mathcal{F}|$ by Proposition 2.12 (i). \square

We did not elaborate on the value of the threshold $n(q, s, r, t)$. The statement of Theorem 2.9 was proved by Ahlswede and Khachatrian [1] under the weaker condition $n \geq (t + 1)(k - t + 1) + 1$. Computations similar to the one in the proof of Theorem 1.3, one would obtain that the choice $n(q, s, r, t) = q^t r(r + t)$ works.

3. Intersecting vector pairs

In this section we provide proofs for Theorem 1.8 and Theorem 1.9.

Let us start with a general construction.

Construction 3.1. Let $c \leq a \leq b$ and $3 \leq s < 2q$ be integers and fix a set X of size $a + b - c$. For any 3-partition $A \cup B \cup C = X$ with $|A| = a - c$, $|B| = b - c$, $|C| = c$, we define the pairs $\mathbf{x}^{A,B,C}$ and $\mathbf{y}^{A,B,C}$ with

$$\begin{aligned} x_i^{A,B,C} &= y_i^{A,B,C} = \lceil s/2 \rceil - 1 \text{ if } i \in C, \\ x_i^{A,B,C} &= \lfloor s/2 \rfloor + 1, \quad y_i^{A,B,C} = 0 \text{ if } i \in A \end{aligned}$$

and

$$x_i^{A,B,C} = 0, \quad y_i^{A,B,C} = \lfloor s/2 \rfloor + 1 \text{ if } i \in B.$$

Note that $\{(\mathbf{x}^{A,B,C}, \mathbf{y}^{A,B,C}) : A \cup B \cup C = X, |A| = a - c, |B| = b - c, |C| = c\}$ is a strong s -sum IVP-system of cardinality $\binom{a+b-c}{b-c} \binom{a}{c}$.

More generally, let $\alpha_0, \alpha_1, \dots, \alpha_q$ be positive integers with $\sum_{i=0}^{q-1} \alpha_i \leq a$ and $\sum_{i=1}^q \alpha_i \leq b$. Set $N = \sum_{i=0}^q \alpha_i \leq a$, and define

$$\{(\mathbf{x}^{A_0, A_1, \dots, A_q}, \mathbf{y}^{A_0, A_1, \dots, A_q}) : [N] = \bigsqcup_{i=0}^q A_i, |A_i| = \alpha_i\},$$

where $x_j^{A_0, A_1, \dots, A_q} = q - y_j^{A_0, A_1, \dots, A_q} = i$ if and only if $j \in A_i$.

Observe that the above is a strong (a, b) -system. Indeed, by definition we have that $x_j^{A_0, A_1, \dots, A_q} + y_j^{A_0, A_1, \dots, A_q} = q$ for any $j \in N$, and A_0, A_1, \dots, A_q partition N with $|A_i| = \alpha_i$, and so $|x^{A_0, A_1, \dots, A_q} \cap_{q+1} y^{A_0, A_1, \dots, A_q}| = 0$. Furthermore, if $(A_0, A_1, \dots, A_q) \neq (B_0, B_1, \dots, B_q)$, then there exists j such that $A_j \neq B_j$. We consider such j that minimizes $\min\{j, q - j\}$ and we can suppose without loss of generality that $j \leq q/2$. By the assumption on j , there exist $i \in B_j \setminus A_j$ and $i' \in A_j \setminus B_j$. Then we have $x_i^{A_0, A_1, \dots, A_q} + y_i^{B_0, B_1, \dots, B_q} > j + q - j$, as $y_i^{B_0, B_1, \dots, B_q} = j$, $i \in B_j \setminus A_j$ and j is minimal; and we also have $x_{i'}^{B_0, B_1, \dots, B_q} + y_{i'}^{A_0, A_1, \dots, A_q} > q - j + j$ by similar reasons. This proves that we indeed defined a strong (a, b) -system.

3.1. Upper bound for strong 3-sum IVP-systems in $\{0, 1, 2\}^{<\mathbb{N}}$

In this subsection we will prove Theorem 1.8. Let $\{(\mathbf{x}^j, \mathbf{y}^j) \mid 1 \leq j \leq m\}$ be a strong 3-sum (k, k) -system in $\{0, 1, 2\}^{<\mathbb{N}}$. Let us also introduce the following further notation for $j = 1, \dots, m$:

- $a_j = |S_{\mathbf{x}^j} \setminus S_{\mathbf{y}^j}|$,
- $b_j = |S_{\mathbf{y}^j} \setminus S_{\mathbf{x}^j}|$,
- $c_j = |S_{\mathbf{x}^j} \cap S_{\mathbf{y}^j}|$.

First we prove the following LYM-type theorem for 3-sum (a, b) -systems.

Theorem 3.2. Suppose that $a, b, m \geq 1$ and $\{(\mathbf{x}^j, \mathbf{y}^j) \mid 1 \leq j \leq m\}$ is a strong 3-sum (a, b) -system in $\{0, 1, 2\}^{<\mathbb{N}}$. Then

$$\sum_{j=1}^m \frac{a_j! b_j! c_j!}{(a_j + b_j + c_j)!} = \sum_{j=1}^m \frac{1}{\binom{a_j + b_j + c_j}{a_j} \binom{a_j + b_j}{a_j}} \leq \min(a, b). \quad (6)$$

Proof. Essentially we apply induction on n , that is the size of union of the supports of elements in $\{(\mathbf{x}^j, \mathbf{y}^j) \mid 1 \leq j \leq m\}$.

- (1) Note first that $a_i = 0$ and $b_j = 0$ cannot hold simultaneously for any $1 \leq i \neq j \leq m$. Indeed, if $S_{\mathbf{x}^i} \subset S_{\mathbf{y}^j}$ and $S_{\mathbf{y}^j} \subset S_{\mathbf{x}^i}$ then all nonzero entries in \mathbf{x}^i are equal to 1, and the same holds for all nonzero entries in \mathbf{y}^j as well, hence $\mathbf{x}^i \cap_3 \mathbf{y}^j = \emptyset$, a contradiction. As a consequence, either $a_j > 0$ for all j or $b_j > 0$ for all j (or both), or there is exactly one j with $a_j = b_j = 0$.

- (2) As long as $S_{\mathbf{y}^j} \not\subseteq S_{\mathbf{x}^i}$ holds for all j :

For every $t \in [n]$, consider the systems

$$\{(\mathbf{x}^j, (\mathbf{y}^j)') \mid 1 \leq j \leq m, t \notin S_{\mathbf{x}^j}\},$$

where $((\mathbf{y}^j)')_i = (\mathbf{y}^j)_i$ for all $i \in [n] \setminus \{t\}$ and $((\mathbf{y}^j)')_t = 0$.

These systems keep the required intersections. Denoting $b'_j = |S_{(\mathbf{y}^j)'} \setminus S_{\mathbf{x}^j}|$ we have $b'_j = b_j - 1$ exactly $b_j > 0$ times, and $b'_j = b_j$ exactly $n - (a_j + b_j + c_j)$ times. Taking the sum of (6) over all t , for the term belonging to j we have

$$b_j \cdot \frac{a_j! (b_j - 1)! c_j!}{(a_j + (b_j - 1) + c_j)!} + (n - a_j - b_j - c_j) \cdot \frac{a_j! b_j! c_j!}{(a_j + b_j + c_j)!} = n \cdot \frac{a_j! b_j! c_j!}{(a_j + b_j + c_j)!},$$

hence the overall sum for all j is n times the left-hand side of (6). Certainly the right-hand side is also multiplied by n , and the inequality follows by induction.

This step is applicable unless $b_j = 0$ holds for some j . Hence from now on assume $S_{\mathbf{y}^j} \subseteq S_{\mathbf{x}^j}$.

- (3) As long as $S_{\mathbf{x}^j} \not\subseteq S_{\mathbf{y}^i}$ holds for all j , also including $j = i$:

For every t consider the systems

$$\{((\mathbf{x}^j)', \mathbf{y}^j) \mid 1 \leq j \leq m, t \notin S_{\mathbf{y}^j}\},$$

where $((\mathbf{x}^j)')_i = (\mathbf{x}^j)_i$ for all $i \in [n] \setminus \{t\}$ and $((\mathbf{x}^j)')_t = 0$.

The argument analogous to the previous case yields the required inequality unless $a_j = 0$ holds for some j . However, then we have $a_j = b_i = 0$ which implies $j = i$.

Hence for the rest of the proof assume $S_{\mathbf{x}^1} = S_{\mathbf{y}^1}$, as we can choose $i = 1$, without loss of generality. Recall that in this situation $(\mathbf{x}^1)_i = (\mathbf{y}^1)_i$ for all $i \in S_{\mathbf{x}^1} = S_{\mathbf{y}^1}$.

- (4) If we omit $(\mathbf{x}^1, \mathbf{y}^1)$ from the system, the left-hand side of (6) decreases by exactly 1, as currently $c_1 = |S_{\mathbf{x}^1}|$ and $a_1 = b_1 = 0$. For every $j \neq 1$ in the remaining subsystem we have $a_j, b_j > 0$ because each \mathbf{x}^j needs an entry of 2 to intersect \mathbf{y}^1 , and each \mathbf{y}^j needs an entry of 2 to intersect \mathbf{x}^1 , while those two elements cannot be the same as \mathbf{x}^j must not sum-intersect \mathbf{y}^j .

Consequently when we repeat Step (2) and (3) for the remaining system, once the procedure halts, the elements of $S_{\mathbf{x}^j} \setminus S_{\mathbf{y}^j}$ and of $S_{\mathbf{y}^j} \setminus S_{\mathbf{x}^j}$ will not remain there, i.e. the value of the corresponding c_j will be at most $\min(a, b) - 1$ when $a_j = b_j = 0$.

- (5) The last halt occurs when the system contains a single vector-pair $(\mathbf{x}^j, \mathbf{y}^j)$ with $a_j = b_j = 0$ and $c_j \geq 1$. This situation is reached after performing the above procedure at most $\min(a, b) - c_j + 1 \leq \min(a, b)$ times. Note that if $c_j = 1$ then the intersection conditions exclude the presence of any other vector-pair. \square

Let us repeat that $m(k)$ denotes the maximum number of vector pairs in a strong 3-sum (k, k) -system and

$$f(k) := \max \frac{(x + y + z)!}{x! y! z!},$$

where the maximum is taken over all nonnegative integers x, y, z such that $x + z \leq k$ and $y + z \leq k$. Now we are ready to prove

Theorem 1.8. *For every $k \geq 1$ we have*

$$f(k) \leq m(k) \leq k \cdot f(k).$$

Proof of Theorem 1.8. The upper bound is a consequence of Theorem 3.2 as all terms on the left-hand side of (6) are at least $(f(k))^{-1}$. To obtain the lower bound we choose x, y, z for which $f(k)$ is attained, and choose $a = x + z$, $b = y + z$ and $c = z$ in Construction 3.1. \square

3.2. Upper bound for weak $(q + 1)$ -sum IVP-systems in $\{0, 1, \dots, q\}^{<\mathbb{N}}$

Let $\{(\mathbf{x}^j, \mathbf{y}^j) \mid 1 \leq j \leq m\}$ be a weak $(q + 1)$ -sum IVP-system in $\{0, 1, \dots, q\}^{<\mathbb{N}}$.

Observation 3.3. (i) *For any weak $(q + 1)$ -sum (a, b) -system \mathcal{F} there exists another one \mathcal{F}' with $|\mathcal{F}| = |\mathcal{F}'|$ such that for any $(\mathbf{x}^j, \mathbf{y}^j) \in \mathcal{F}$ and i with $x_i^j + y_i^j > 0$ we have $x_i^j + y_i^j = q$.*

(ii) *For any strong $(q + 1)$ -sum (a, b) -system \mathcal{F} there exists another one \mathcal{F}' with $|\mathcal{F}| = |\mathcal{F}'|$ such that for any $(\mathbf{x}^j, \mathbf{y}^j) \in \mathcal{F}$ and i with $x_i^j + y_i^j > 0$ we have $x_i^j + y_i^j = q$.*

Proof. As $|\mathbf{x}^j \cap_{q+1} \mathbf{y}^j| = 0$ implies $x_i^j + y_i^j \leq q$, and increasing a coordinate helps to intersect other vectors, we can replace \mathbf{y}^j by $\mathbf{y}^{j'}$ with $y_i^{j'} = q - x_i^j$. \square

We will say that a weak/strong $(q + 1)$ -sum (k, k) -system is *saturated* if it satisfies the property of Observation 3.3. For such $\mathcal{F} = \{(\mathbf{x}^j, \mathbf{y}^j) : 1 \leq j \leq m\}$, let us write A_i^j to denote $\{t : x_t^j = i\}$ and α_i^j to denote $|A_i^j|$.

Theorem 3.4. *Let p_i for $i = 0, 1, \dots, q$ be non-negative reals with $\sum_{i=0}^q p_i = 1$. If $\mathcal{F} = \{(\mathbf{x}^j, \mathbf{y}^j) : 1 \leq j \leq m\}$ is a saturated weak $(q + 1)$ -sum IVP-system, then $\sum_{j=1}^m \prod_{i=0}^q p_i^{\alpha_i^j} \leq 1$ holds.*

Proof. Let (X_0, X_1, \dots, X_q) be a partition of $[n]$ taken at random by the rule

$$\mathbb{P}(t \in X_0) = p_0, \quad \mathbb{P}(t \in X_1) = p_1, \quad \dots, \quad \mathbb{P}(t \in X_q) = p_q,$$

applied independently for each $t \in [n] = \bigcup_{j=1}^m (S(\mathbf{x}^j) \cup S(\mathbf{y}^j))$. For $j = 1, \dots, m$ consider the events

$$E_j = \bigwedge_{i=0}^q (A_i^j \subseteq X_i).$$

We then have

$$\mathbb{P}(E_j) = \prod_{i=0}^q p_i^{\alpha_i^j}.$$

Observe that $\mathbb{P}(E_j \wedge E_{j'}) = 0$ holds for all $1 \leq j \neq j' \leq m$. Indeed, otherwise $A_i^j, A_i^{j'} \subseteq X_i$ holds for all $i = 0, 1, \dots, q$. But then for all $i = 0, 1, \dots, q$ we have that $z \in X_i$ and $\mathbf{x}_z^j = i$ implies $\mathbf{y}_z^{j'} = q - i$ or $\mathbf{y}_z^{j'} = 0$ and similarly $\mathbf{x}_z^{j'} = i$ implies $\mathbf{y}_z^j = q - i$

or $\mathbf{y}_z^{j'} = 0$. So $|\mathbf{x}^j \cap_{q+1} \mathbf{y}^{j9'}| = |\mathbf{x}^{j'} \cap_{q+1} \mathbf{y}^j| = 0$. This is a contradiction as the vectors are elements of a weak $(q+1)$ -sum IVP-system.

Consequently the events E_1, \dots, E_m mutually exclude each other, which implies that the sum of their probabilities is at most 1. \square

Now we prove

Theorem 1.9. For any $q \geq 1$ let $m(q, k)$ and $m'(q, k)$ denote the maximum size of a strong / weak $(q+1)$ -sum (k, k) -system. Then $\lim_{k \rightarrow \infty} \sqrt[k]{m(q, k)} = \lim_{k \rightarrow \infty} \sqrt[k]{m'(q, k)} = (\sqrt{q} + 1)^2$.

Proof. Let us prove the upper bound first. By Observation 3.3, we can assume that \mathcal{F} is saturated. Then we apply Theorem 3.4 with $p_0 = p_q = \frac{\sqrt{q}-1}{q-1}$ and $p_1 = p_2 = \dots = p_{q-1} = p_0^2 = \frac{q+1-2\sqrt{q}}{(q-1)^2}$. (Observe that $2p_0 + (q-1)p_0^2 = 1$ as required.) As the system is saturated we have $\alpha_0^j = k - \sum_{i=1}^{q-1} \alpha_i^j = \alpha_q^j$ and thus we obtain

$$\prod_{i=0}^q p_i^{\alpha_i^j} = p_0^{\alpha_0^j + \alpha_q^j} p_0^{2(k - \sum_{i=1}^{q-1} \alpha_i^j)} = p_0^{2k}.$$

Therefore, Theorem 3.4 implies $|\mathcal{F}| \leq (p_0^{-2})^k = ((\frac{q-1}{\sqrt{q}-1})^2)^k = (\sqrt{q} + 1)^{2k}$.

The lower bound is obtained using Construction 3.1. For fixed q and growing N , we let $\alpha_i = p_i N$ for $i = 0, 1, \dots, q$ with p_i as above in the proof of the lower bound, and so $k = \frac{p_0 + (q-1)p_0^2}{2p_0 + (q-1)p_0^2} N = (p_0 + (q-1)p_0^2)N$. Then the number of pairs in the construction is $\prod_{i=0}^q \binom{(1 - \sum_{j=0}^{i-1} p_j)N}{p_i N}$. Using Stirling's formula and omitting polynomial terms, this is

$$\left[\frac{1}{p_0^{2p_0} (p_0^2)^{(q-1)p_0^2}} \right]^N = (p_0^{-2})^{(p_0 + (q-1)p_0^2)N} = (p_0^{-2})^k = \left(\frac{q-1}{\sqrt{q}-1} \right)^{2k} = (\sqrt{q} + 1)^{2k}.$$

Taking k th root yields the claimed lower bound. \square

4. Concluding remarks

There exist lots of intersection theorems all waiting to be addressed in the sum-intersection setting. We just would like to point out one. Katona's intersection theorem [14] gives the maximum size of a non-uniform t -intersecting family $\mathcal{F} \subseteq 2^{[n]}$. The extremal family consists of all sets of size at least $\frac{n+t}{2}$ if $n+t$ is even, while if $n+t$ is odd, then the extremal family consists of all sets of size at least $\lceil \frac{n+t}{2} \rceil$ together with $\binom{[n-1]}{\lfloor \frac{n+t}{2} \rfloor}$. One would hope to see a similar result for non-uniform s -sum t -intersecting families. That is extremal families are expected to consist of vectors of large rank. This is not going to hold as for two such vectors \mathbf{x}, \mathbf{y} there might be coordinates where they s -sum 'intersect very much' (i.e. $x_i + y_i$ is much larger than s), but do not intersect anywhere else, so it is not a must that the support of the vectors are large.

To remedy this situation, we can define the size of the $multi$ - s -sum intersection of two vectors $\mathbf{x}, \mathbf{y} \in Q^n$ as $|\mathbf{x} \cap_{m,s} \mathbf{y}| = \sum_{i=1}^n (x_i + y_i - s + 1)^+$, where for any real z we define $z^+ := \max\{0, z\}$. A family $\mathcal{F} \subseteq Q^n$ is s -multisum t -intersecting if for any $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ we have $|\mathbf{x} \cap_{m,s} \mathbf{y}| \geq t$. Below, we show the first step towards such intersection theorems. Katona's tool was his intersecting shadow theorem and we will need a similar result.

We need to define the well-known shifting operation $\tau_{i,j}$ for the vector setting. For a vector \mathbf{x} of length n and integers $1 \leq i \neq j \leq n$ we let $\tau_{i,j}(\mathbf{x})$ be the vector obtained from \mathbf{x} by exchanging its i th and j th coordinates if $x_i < x_j$ and we let $\tau_{i,j}(\mathbf{x}) = \mathbf{x}$ otherwise. For a family \mathcal{F} of vectors we define $\tau_{i,j}(\mathcal{F}) = \{\tau_{i,j}(\mathbf{x}) : \mathbf{x} \in \mathcal{F}, \tau_{i,j}(\mathbf{x}) \notin \mathcal{F}\} \cup \{\mathbf{x} \in \mathcal{F} : \tau_{i,j}(\mathbf{x}) \in \mathcal{F}\}$.

The next lemma shows two basic properties of the shifting operation that are well-known for set systems.

Lemma 4.1. For any $\mathcal{F} \subseteq Q(n, r)$ and $1 \leq i, j \leq n$ we have $|\Delta(\tau_{i,j}(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$. Furthermore, if \mathcal{F} is s -multisum t -intersecting, then so is $\tau_{i,j}(\mathcal{F})$.

Proof. Let us start with the proof of the claim concerning t -intersection. Suppose for $\mathbf{x}, \mathbf{y} \in \tau_{i,j}(\mathcal{F})$ we have $|\mathbf{x} \cap_{m,s} \mathbf{y}| < t$. We cannot have $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, as it is impossible by the s -multisum t -intersecting property of \mathcal{F} . If $\mathbf{x}, \mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, then $\tau_{j,i}(\mathbf{x}), \tau_{j,i}(\mathbf{y}) \in \mathcal{F}$ and $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\tau_{j,i}(\mathbf{x}) \cap_{m,s} \tau_{j,i}(\mathbf{y})|$ contradicts the s -multisum t -intersecting property of \mathcal{F} . Finally, if $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, then $\mathbf{y}' := \tau_{j,i}(\mathbf{y}) \in \mathcal{F} \setminus \tau_{i,j}(\mathcal{F})$. So if $\mathbf{x} = \tau_{i,j}(\mathbf{x})$, then $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\mathbf{x} \cap_{m,s} \mathbf{y}'|$ contradicts the s -multisum t -intersecting property of \mathcal{F} . If $\mathbf{x}' := \tau_{i,j}(\mathbf{x}) \neq \mathbf{x}$, then as $\mathbf{x} \in \tau_{i,j}(\mathcal{F})$, we must have $\mathbf{x}' \in \mathcal{F}$, and thus $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\mathbf{x}' \cap_{m,s} \mathbf{y}'|$ contradicts the s -multisum t -intersecting property of \mathcal{F} . This finishes the proof that shifting preserves multisum intersecting properties.

To see $|\Delta(\tau_{i,j}(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ we define an injection $\iota : \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F}) \rightarrow \Delta(\mathcal{F}) \setminus \Delta(\tau_{i,j}(\mathcal{F}))$ by letting $\iota(\mathbf{x})$ be the vector obtained from \mathbf{x} by interchanging its i th and j th coordinate. This is clearly an injection, all we need to verify is

that every image belongs to $\Delta(\mathcal{F}) \setminus \Delta(\tau_{i,j}(\mathcal{F}))$. So let $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$ be arbitrary. Then there exists $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$ with $\mathbf{x} \in \Delta(\mathbf{y})$ and $\mathbf{y}' := \tau_{j,i}(\mathbf{y}) \in \mathcal{F} \setminus \tau_{i,j}(\mathcal{F})$. Clearly, $\iota(\mathbf{x}) \in \Delta(\mathbf{y}') \subset \Delta(\mathcal{F})$. It remains to show $\iota(\mathbf{x}) \notin \Delta(\tau_{i,j}(\mathcal{F}))$. First we claim $x_i > x_j$. Indeed, as $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, we have $y_i > y_j$ showing $x_i \geq x_j$, and $x_i = x_j$ would mean $\mathbf{x} \in \Delta(\mathbf{y}')$ and $\mathbf{x} \in \Delta(\mathcal{F})$ contradicting $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$. Now, $x_i > x_j$ implies $\iota(\mathbf{x})_i < \iota(\mathbf{x})_j$. Assume for a contradiction that there exists $\mathbf{y}^* \in \tau_{i,j}(\mathcal{F})$ with $\iota(\mathbf{x}) \in \Delta(\mathbf{y}^*)$. Then we must have $y_i^* \leq y_j^*$. This is only possible if $\tau_{i,j}(\mathbf{y}^*) \in \mathcal{F}$. But then $\mathbf{x} \in \Delta(\tau_{i,j}(\mathbf{y}^*)) \subset \Delta(\mathcal{F})$ contradicting $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$. This finishes the proof. \square

Note that Lemma 4.1 is not valid for s -sum t -intersection instead of s -multisum t -intersection in the case of general t as, say, the family $\{(3, 2), (1, 3)\}$ is 4-sum 2-intersecting, while its $(1, 2)$ -shift $\{(3, 2), (3, 1)\}$ is only 4-sum 1-intersecting.

We say that \mathcal{F} is *left-shifted* if $\tau_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $i < j$. Whenever $\tau_{i,j}(\mathcal{F}) \neq \mathcal{F}$ for some $i < j$, then $w(\mathcal{F}) = \sum_{\mathbf{x} \in \mathcal{F}} \sum_{i=1}^n ix_i$ strictly decreases, so starting from any family \mathcal{F} , after a finite number of shift operations one obtains a left-shifted family. Furthermore, by Lemma 4.1, the size of the shadow does not increase and intersection properties are preserved. Therefore, when proving a lower bound on the size of shadows, one can assume that \mathcal{F} is left-shifted. We use the notation $2(n, r)$ for the set of vectors of rank r in $\{0, 1, 2\}^n$.

Theorem 4.2. *If $\mathcal{F} \subseteq 2(n, r)$ is 3-sum intersecting, then $|\Delta(\mathcal{F})| \geq |\mathcal{F}|$.*

Proof. We proceed by induction on n . If $n < r$, then for any $\mathbf{x} \in 2(n, r)$ we have $|\{i : x_i = 2\}| > |\{j : x_j = 0\}|$. Therefore for any $\mathcal{F} \subseteq 2(n, r)$ in the auxiliary bipartite graph B with parts \mathcal{F} and $\Delta(\mathcal{F})$ and edges between pairs $\mathbf{y} \in \Delta(\mathbf{x})$, we have that the degree of any vector \mathbf{x} in \mathcal{F} is at least as large as the degree of any of its neighbors $\mathbf{y} \in \Delta(\mathbf{x})$. Consequently, $|\Delta(\mathcal{F})| \geq |\mathcal{F}|$ as claimed.

If $n \geq r$, then, by Lemma 4.1, we can assume that \mathcal{F} is left-shifted. For $a = 0, 1, 2$ we introduce $\mathcal{F}_a := \{\mathbf{x} \in \mathcal{F} : x_n = a\}$ and $\mathcal{F}_a^- := \{\mathbf{x}^- : \mathbf{x} \in \mathcal{F}_a\}$, where \mathbf{x}^- is the vector obtained from \mathbf{x} by omitting its last coordinate. Observe that if $\mathbf{y} \in \Delta(\mathcal{F}_a^-)$, then $\mathbf{y}^{+a} \in \Delta(\mathcal{F})$, where \mathbf{y}^{+a} is the vector obtained from \mathbf{y} by concatenating a as a last coordinate. So, by induction, $|\Delta(\mathcal{F})| \geq \sum_{a=0}^2 |\Delta(\mathcal{F}_a^-)| \geq \sum_{a=0}^2 |\mathcal{F}_a^-| = |\mathcal{F}|$ if we can prove for the second inequality that \mathcal{F}_a^- is 3-sum intersecting for all $a = 0, 1, 2$. This is clear for $a = 0, 1$ as vectors in \mathcal{F}_a 3-sum intersect but as their last coordinate is 0 or 1, they must 3-sum intersect among the first $n - 1$ coordinates.

Finally, consider \mathcal{F}_2^- . Suppose for a contradiction that $\mathbf{x}^-, \mathbf{y}^- \in \mathcal{F}_2^-$ with $|\mathbf{x}^- \cap_3 \mathbf{y}^-| = 0$. If for some $i \in [n - 1]$ we have $x_i = 0$ and $y_i \leq 1$, then $|\tau_{i,n}(\mathbf{x}) \cap_3 \mathbf{y}| = 0$ holds, contradicting the 3-sum intersecting property of \mathcal{F}_2 . We derive the same contradiction if $x_i \leq 1$ and $y_i = 0$. But $\mathbf{x}^-, \mathbf{y}^- \in 2(n - 1, r - 2)$, so there are at most $r - 2$ coordinates from $i \in [n - 1]$ with $x_i, y_i \geq 1$, hence there exists at least one coordinate i for which we get the desired contradiction. \square

Theorem 4.3. *If $\mathcal{F} \subseteq 2^n$ is 3-multisum 2-intersecting, then $|\mathcal{F}| \leq |\cup_{r=n+1}^{2n} 2(n, r)|$.*

Proof. Let \mathcal{F} be a 3-multisum 2-intersecting family of maximum size. Clearly, \mathcal{F} is upward closed, i.e. $\mathbf{y} > \mathbf{x} \in \mathcal{F}$ implies $\mathbf{y} \in \mathcal{F}$. Observe that writing $\nabla(\mathbf{x}) = \{\mathbf{y} > \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x}) + 1\}$, we have that for any $\mathbf{x} \in \mathcal{F}$ the shade $\nabla(\mathbf{x})$ is disjoint from \mathcal{F} . Let r be the rank of a smallest ranked vector in \mathcal{F} and consider $\mathcal{F}_r = \{\mathbf{x} \in \mathcal{F} : r(\mathbf{x}) = r\}$. Observe that $\mathcal{F}' := (\mathcal{F} \setminus \mathcal{F}_r) \cup \nabla_2(\overline{\mathcal{F}_r})$ is 3-multisum 2-intersecting. Indeed, vectors from $\mathcal{F}' \setminus \mathcal{F}$ are all of rank $2n - r + 1$ and vectors from $\mathcal{F} \cap \mathcal{F}'$ are all of rank at least $r + 1$, so they must 3-multisum 2-intersect. As $|\nabla(\overline{\mathcal{F}_r})| = |\Delta(\mathcal{F})|$, by Theorem 4.2, $|\mathcal{F}'| \geq |\mathcal{F}|$ and we can repeat this procedure as long as $r \leq n$ and thus $2n - r + 1 > r$. We obtain that $|\mathcal{F}| \leq |\cup_{r=n+1}^{2n} 2(n, r)|$. \square

Theorem 4.4. *If $\mathcal{F} \subseteq 2^n$ is 3-multisum 3-intersecting, then $|\mathcal{F}| \leq |\cup_{r=n+2}^{2n} 2(n, r)| + M(n)$, where $M(n)$ denotes the maximum size of a 3-multisum 3-intersecting family in $2(n, n + 1)$.*

Proof. The proof is almost identical to that of Theorem 4.3. Let \mathcal{F} be a 3-multisum 3-intersecting family of maximum size. Observe that writing $\nabla_2(\mathbf{x}) = \{\mathbf{y} > \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x}) + 2\}$, we have that for any $\mathbf{x} \in \mathcal{F}$ the 2-shade $\nabla_2(\mathbf{x})$ is disjoint from \mathcal{F} . Let r be the rank of a smallest ranked vector in \mathcal{F} and consider $\mathcal{F}_r = \{\mathbf{x} \in \mathcal{F} : r(\mathbf{x}) = r\}$. Observe that $\mathcal{F}' := (\mathcal{F} \setminus \mathcal{F}_r) \cup \nabla_2(\overline{\mathcal{F}_r})$ is 3-multisum 2-intersecting. Indeed, vectors from $\mathcal{F}' \setminus \mathcal{F}$ are all of rank $2n - r + 2$ and vectors from $\mathcal{F} \cap \mathcal{F}'$ are all of rank at least $r + 1$, so they must 3-multisum 3-intersect. Note that if \mathcal{G} is s -multisum t -intersecting, then $\Delta(\mathcal{G})$ is s -multisum $(t - 2)$ -intersecting. So applying Theorem 4.2 twice and using $|\nabla_2(\overline{\mathcal{F}_r})| = |\Delta(\Delta(\mathcal{F}))|$, we obtain $|\mathcal{F}'| \geq |\mathcal{F}|$ and we can repeat this procedure as long as $r \leq n$ and thus $2n - r + 2 > r$. We obtain that there exists a maximum-sized 3-multisum 2-intersecting family \mathcal{F} consisting only of vectors of rank at least $n + 1$. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

- [1] R. Ahlswede, L.H. Khachatrian, The complete nontrivial-intersection theorem for systems of finite sets, *J. Comb. Theory, Ser. A* 76 (1) (1996) 121–138.
- [2] R. Ahlswede, L.H. Khachatrian, The complete intersection theorem for systems of finite sets, *Eur. J. Comb.* 18 (2) (1997) 125–136.
- [3] B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hung.* 16 (1965) 447–452.
- [4] G.F. Clements, B. Lindström, A generalization of a combinatorial theorem of Macaulay, *J. Comb. Theory* 7 (3) (1969) 230–238.
- [5] D.E. Daykin, Erdős-Ko-Rado from Kruskal-Katona, *J. Comb. Theory, Ser. A* 17 (2) (1974) 254–255.
- [6] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math.* 12 (1) (1961) 313–320.
- [7] P. Frankl, On intersecting families of finite sets, *J. Comb. Theory, Ser. A* 24 (2) (1978) 146–161.
- [8] P. Frankl, The Erdős-Ko-Rado theorem is true for $n = \text{ckt}$, in: *Combinatorics, Proc. Fifth Hungarian Colloq., Keszthely, 1976, vol. 1, 1978*, pp. 365–375.
- [9] P. Frankl, Z. Füredi, The Erdős-Ko-Rado theorem for integer sequences, *SIAM J. Algebraic Discrete Methods* 1 (4) (1980) 376–381.
- [10] P. Frankl, N. Tokushige, The Erdős-Ko-Rado theorem for integer sequences, *Combinatorica* 19 (1) (1999) 55–63.
- [11] P. Frankl, N. Tokushige, Intersection problems in the q -ary cube, *J. Comb. Theory, Ser. A* 141 (2016) 90–126.
- [12] Z. Füredi, D. Gerbner, M. Vizer, A discrete isodiametric result: the Erdős-Ko-Rado theorem for multisets, *Eur. J. Comb.* 48 (2015) 224–233.
- [13] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, *Q. J. Math.* 18 (1) (1967) 369–384.
- [14] Gy. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hung.* 15 (3–4) (1964) 329–337.
- [15] G. Katona, A theorem of finite sets, in: *Theory of Graphs, 1968*, pp. 187–207.
- [16] Z. Király, Z.L. Nagy, D. Pálvölgyi, M. Visontai, On families of weakly cross-intersecting set-pairs, *Fundam. Inform.* 117 (1–4) (2012) 189–198.
- [17] J.B. Kruskal, The number of simplices in a complex, *Math. Optim. Tech.* 10 (1963) 251–278.
- [18] Zs. Tuza, Inequalities for two set systems with prescribed intersections, *Graphs Comb.* 3 (1987) 75–80.
- [19] Zs. Tuza, Applications of the set-pair method in extremal hypergraph theory, in: *Extremal Problems for Finite Sets, Visegrád, 1991*, in: *Bolyai Society Mathematical Studies*, vol. 3, János Bolyai Mathematical Society, Budapest, 1994, pp. 479–514.
- [20] Zs. Tuza, Applications of the set-pair method in extremal problems, II, in: *Combinatorics, Paul Erdős Is Eighty, Keszthely, 1993*, in: *Bolyai Society Mathematical Studies*, vol. 2, János Bolyai Mathematical Society, Budapest, 1996, pp. 459–490.
- [21] A.Z. Wagner, Constructions in combinatorics via neural networks, *arXiv preprint, arXiv:2104.14516*, 2021.
- [22] R.M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, *Combinatorica* 4 (2) (1984) 247–257.