Generalized Turán numbers for the edge blow-up of a graph

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Abstract

Let H be a graph and p be an integer. The edge blow-up H^p of H is the graph obtained from replacing each edge in H by a copy of K_p where the new vertices of the cliques are all distinct. Let C_k and P_k denote the cycle and path of length k, respectively. In this paper, we find sharp upper bounds for $\mathrm{ex}(n, K_3, C_3^3)$ and the exact value for $\mathrm{ex}(n, K_3, P_3^3)$. Moreover, we determine the graphs attaining these bounds.

1 Introduction

Notation. In this paper, we use C_k, P_k, M_k and S_k to denote the cycle, path, matching and star with k edges, respectively. Let K_t be the complete graph on t vertices and $K_{s,t}$ be the complete bipartite graph with parts of size s and t. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Also we denote the number of edges in G by $e(G)$. For two graph G and H, let $G \cup H$ denote the disjoint union of G and H. Let $G + H$ denote the join of G and H, which is obtained from $G \cup H$ by adding all edges with one endvertex in $V(G)$ and the other endvertex in $V(H)$. Let $T(G)$ denote the set of all triangles in G and $t(G) = |T(G)|$. For a vertex v in $V(G)$, let $t(v)$ denote the number of triangles containing v. For an edge uv, let $N(uv) = N(u) \cap N(v)$. Hence, $|N(uv)|$ is the number of triangles containing uv. For a set of vertices $S \subseteq V(G)$ we denote by $G[S]$ the induced subgraph of G on S and we set $G - S = G[V(G) - S]$. For two disjoint sets of vertices $U, W \subseteq V(G)$ we denote by $G[U, W]$ the bipartite subgraph of G consisting of those edges with one endvertex in U and the other in W.

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Let H be a given graph and p be an integer greater than 2. The edge blow-up H^p of H is the graph obtained from replacing each edge in H by a copy of K_p where the new vertices of the cliques are all distinct. The problem of finding the Turán number of H^p for various graphs H has attracted a lot of attention. The first results on the topic can be dated back to 1960s. Moon [6], and independently Simonovits [7] determined the Turán number $\mathrm{ex}(n, M_k^p)$ for $p \geq 3$. Much later Erdős, Füredi, Gould and Gunderson [2] determined the Turán number $\mathrm{ex}(n, S_k^p)$ for $p = 3$, and then Chen, Gould, Pfender and Wei [1] extended this result to any $p \geq 3$. Glebov [4] determined the Turán number of P_k^p. More recently, Liu extended Glebov's result to the edge blow-up of a family of trees and also determined C_k^p for sufficiently large n. Wang, Hou, Liu and Ma [8] determined the $\mathrm{ex}(n, T^P)$ for a larger family of trees and Yuan [10] determined $\mathrm{ex}(n, H^P)$ for any non-bipartite graph H and $p \geq \chi(H) + 1$.

We will make use of the following result of Xiao, Katona, Xiao and Zamora [9], which determined the value of $\mathrm{ex}(n, C_3^3)$ for all $n \geq 6$.

Theorem 1. *(Xiao, Katona, Xiao and Zamora [9]) Let $n \geq 6$ be an integer, then*

$$\mathrm{ex}(n, C_3^3) = \begin{cases} \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor & \text{if } n \not\equiv 2 \pmod 4, \\ \frac{n^2}{4} + \frac{n}{2} - 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

When $n = 4k$, $M_{\frac{n}{4}} + M_{\frac{n}{4}}$ is the unique extremal graph.
When $n = 4k+1$, $(M_{\lfloor \frac{n}{4} \rfloor} \cup K_1) + M_{\frac{n-1}{4}}$ and $S_{\lfloor \frac{n}{2} \rfloor} + \overline{K_{\lfloor \frac{n}{2} \rfloor}}$ are the extremal graphs.
When $n = 4k+2$, $(M_{\lfloor \frac{n}{4} \rfloor} \cup K_1) + (M_{\lfloor \frac{n}{4} \rfloor} \cup K_1)$, $M_{\lceil \frac{n}{4} \rceil} + M_{\lfloor \frac{n}{4} \rfloor}$ and $S_{\frac{n}{2}-1} + \overline{K_{\frac{n}{2}}}$ are the extremal graphs.
When $n = 4k+3$, $(M_{\lfloor \frac{n}{4} \rfloor} \cup K_1) + M_{\lceil \frac{n}{4} \rceil}$ and $S_{\lfloor \frac{n}{2} \rfloor} + \overline{K_{\lfloor \frac{n}{2} \rfloor}}$ are the extremal graphs.

In this paper, we will consider the generalized Turán number. Let T and H be graphs, then the generalized Turán number $\mathrm{ex}(n, T, H)$ is the maximum number of copies of T that an n-vertex H-free graph G can contain. If $T = K_2$, then $\mathrm{ex}(n, T, H)$ is the classical Turán number of H.

Although several results about the Turán number of an edge blow-up of a graph have been obtained, less is known about the generalized Turán number of such graphs. However, there have been some results in this direction. Liu and Wang [5] determined the value of $\mathrm{ex}(n, K_p, S_2^p)$ and $\mathrm{ex}(n, K_p, M_2^p)$. Later Gerbner and Patkós [3] determined $\mathrm{ex}(n, K_r, S_2^p)$ and $\mathrm{ex}(n, K_r, M_2^p)$ for any r, p, and Yuan and Yang [11] determined $\mathrm{ex}(n, K_3, M_2^3)$ for all n. Recently, Zhu, Chen, Gerbner, Győri and Hama Kairm [12] determined $\mathrm{ex}(n, K_3, S_k^3)$ for any k.

Our results concern the edge blow-ups of cycles and paths. We prove the following theorems.

Theorem 2. *Let $n \geq 22$ be an integer, we have*

$$\mathrm{ex}(n, K_3, C_3^3) \leq \frac{n^2}{4} - 1 + \mathbb{1}_{4|n},$$

where $\mathbb{1}_{4|n}$ is the indicator function for $4|n$. Furthermore, equality holds when n is even and $M_{\lceil \frac{n}{4} \rceil} + M_{\lfloor \frac{n}{4} \rfloor}$ is the unique extremal graph.

Theorem 3. *Let $n \geq 300^3$ be an integer. We have*

$$\mathrm{ex}(n, K_3, P_3^3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor,$$

and the unique extremal graph is $K_1 + K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 2. In Section 3, we prove Theorem 3. In Section 4, we mention some problems about the general case: $\mathrm{ex}(n, K_3, C_k^3)$ and $\mathrm{ex}(n, K_3, P_k^3)$.

2 Proof of Theorem 2

One can see that when n is even, the graph $M_{\lceil \frac{n}{4} \rceil} + M_{\lfloor \frac{n}{4} \rfloor}$ contains $\frac{n^2}{4} - 1 + \mathbb{1}_{4|n}$ triangles. So our aim is to show that $\mathrm{ex}(n, K_3, C_3^3) \le \frac{n^2}{4} - 1 + \mathbb{1}_{4|n}$.

Let $n \ge 22$ be an integer and G be an n-vertex C_3^3-free graph with $t(G)$ being the maximum. We may assume every edge is contained in at least one triangle, otherwise we delete this edge.

We define the weight of uv by

$$w(uv) := \frac{1}{|N(uv)|}.$$

For a triangle xyz, its weight is defined by $w(xyz) = w(xy) + w(xz) + w(yz)$. We will prove the upper bound by making use of the following claims.

Claim 1. *For any triangle xyz in G,*

$$1 + \frac{1}{n-2} \le w(xyz) \le 3,$$

or $w(xy) = w(yz) = w(xz) = \frac{1}{3}$ and there exists another two vertices u, v such that $\{x, y, z, u, v\}$ induces a copy of K_5^- or K_5.

Proof. Since each edge is contained in at least one triangle, without loss of generality, we have

$$\frac{1}{n-2} \le w(yz) \le w(xz) \le w(xy) \le 1.$$

If $w(xy) = 1$, then $w(xyz) \ge 1 + \frac{2}{n-2}$ and we are done. Next we may assume $w(xy) \le \frac{1}{2}$ and we distinguish two cases based on whether $w(xy) = \frac{1}{2}$ or $w(xy) \le \frac{1}{3}$.

First suppose $w(xy) = \frac{1}{2}$ and let $N(xy) = \{z, z'\}$. If $w(xz) = \frac{1}{2}$, then $w(xyz) \ge 1 + \frac{1}{n-2}$ and we are done. Thus we may assume $w(xz) \le \frac{1}{3}$ and let $y' \in N(xz) - \{y, z'\}$. If $w(yz) \le \frac{1}{4}$, then we can find a vertex $x' \in N(yz) - \{x, y', z'\}$ and $\{x, y, z, x', y', z'\}$ contains a copy of C_3^3, a contradiction. Hence $w(yz) = w(xz) = \frac{1}{3}$ and $w(xyz) = \frac{7}{6} \ge 1 + \frac{1}{n-2}$, inequality holds since $n \ge 22$.

Now suppose $w(xy) \le \frac{1}{3}$. Let $u, v \in N(xy) - \{z\}$. If $w(yz) \le \frac{1}{4}$, then there is a vertex $x' \in N(yz) - \{u, v, x\}$. Also we can find a vertex $y' \in N(xz) - \{y, x'\}$ and another vertex in $\{u, v\}$ not equal to y' (say $u \ne y'$). Then $\{y', x', u, x, y, z\}$ contains a copy of C_3^3, a contradiction. It follows that $w(xz) = w(yz) = w(xy) = \frac{1}{3}$. Furthermore, if $N(yz) - \{x\}$ or $N(xz) - \{y\}$ is not equal to $\{u, v\}$, then one can check that we still can find a copy of C_3^3, a contradiction. Hence $\{x, y, z, u, v\}$ induces a copy of K_5^- or K_5. □

Claim 2. $t(G) \le e(G)$.

Proof. By Claim 1, we have

$$t(G) = \sum_{xyz \in T(G)} 1 \le \sum_{xyz \in T(G)} (w(xz) + w(yz) + w(xy)) = e(G),$$

as required. □

Claim 3. *For any triangle xyz, $w(xyz) \geq 1 + \frac{1}{n-2}$, i.e., there is no K_5^- in G.*

Proof. Suppose to the contrary that there is a subgraph H of G isomorphic to K_5 or K_5^- induced on the set $\{v_1, v_2, v_3, v_4, v_5\}$. If H is isomorphic to K_5^-, then we may assume without loss of generality v_4v_5 is not an edge.

One can check that for any edge v_iv_j in H, $N(v_iv_j) \subseteq V(H)$. Otherwise we can find a copy of C_3^3. Let $S = (N(v_4) \cap N(v_5)) - V(H)$ if H is isomorphic to K_5^- and $S = \emptyset$ otherwise.

If $|S| \leq n - 10$, then $e(G - V(H)) \leq \frac{(n-5)^2}{4} + \frac{n-5}{2}$ by Theorem 1, and we have

$$\begin{aligned} e(G) &\leq e(H) + e(G[V(H), V(G) \setminus V(H)]) + e(G - V(H)) \\ &\leq 10 + (n-5) + |S| + \frac{(n-5)^2}{4} + \frac{n-5}{2} \\ &< \frac{n^2}{4} - 1. \end{aligned}$$

By Claim 2, it follows that G is not the extremal graph.

If $|S| > n - 10$, then $G[S]$ is P_3-free, otherwise together with v_4, v_5, we can find a copy of C_3^3. Hence $e(G - V(H)) \leq (n - 5 - |S|)|S| + |S| + \binom{n-5-|S|}{2}$. When $n \geq 22$, we have

$$\begin{aligned} e(G) \leq\ & 9 + (n-5) + |S| + (n-5-|S|)|S| + |S| + \binom{n-5-|S|}{2} \\ \leq\ & 8n - 56 \\ <\ & \frac{n^2}{4} - 1. \end{aligned}$$

Again by Claim 2, we are done. □

Let $T_1(G) = \{xyz \in T(G) : w(xyz) \geq 1 + \frac{2}{n}\}$ and $T_2(G) = T(G) - T_1(G)$. We have the following bound on the average weight of a triangle in G.

Claim 4. *The average weight of each triangle in G is at least $1 + \frac{2}{n}$.*

Proof. If $T_2(G)$ is empty, then there is nothing to prove. Hence we may assume $T_2(G) \neq \emptyset$.

Let xyz be a triangle in $T_2(G)$ with $w(xy) \geq w(xz) \geq w(yz)$. By Claim 1 and 3, we have $w(xy) \in \{1, \frac{1}{2}\}$. If $w(xy) = 1$, then

$$w(xyz) \geq 1 + \frac{2}{n-2}$$

which means $xyz \in T_1(G)$, a contradiction. So $w(xy) = \frac{1}{2}$. Let $N(xy) = \{z,\ x'\}$. Suppose $N(xz) - \{y, x'\} \neq \emptyset$ and $y' \in N(xz) - \{y, x'\}$. Then either $N(yz) - \{x,\ x',\ y'\} \neq \emptyset$ and we can find a copy of C_3^3, or $N(yz) = \{x,\ x',\ z'\}$ which means $\{x, y, z, x', y'\}$ contains a copy of K_5^-, or $w(yz) \geq \frac{1}{2}$ which means $w(xyz) = \frac{3}{2} \geq 1 + \frac{2}{n}$. In all of these cases, we get a contradiction. Hence, $N(xz) = \{y,\ x'\}$ and $w(xy) = w(xz) = \frac{1}{2}$ and so $w(yz) \leq \frac{2}{n}$. For the edges $x'y,\ x'z$, we deduce that $w(x'y) = w(x'z) = \frac{1}{2}$. If not, suppose $w(x'y) < \frac{1}{2}$. Let u be in $N(x'y) - \{x, z\}$. Since $|N(yz)| > \frac{n}{2}$, let v be in $N(yz) - \{x, x', u\}$. Then $\{u, v, x', x, y, z\}$ contains a copy of C_3^3, a contradiction. It follows that the triangle $x'yz$ is also in $T_2(G)$.

Therefore, for any triangle xyz in $T_2(G)$, there is a unique triangle $x'yz$ in $T_2(G)$ such that $\{x, x', y, z\}$ induces a copy of K_4 and $w(xy) = w(xz) = w(x'y) = w(x'z) = \frac{1}{2}$. Hence, we can partition the set of triangles in $T_2(G)$ into pairs $(xyz,\ x'yz)$. For each such pair, we define a mapping

$$\phi(xyz,\ x'yz) = \{xyz,\ x'yz,\ xx'y,\ xx'z\}.$$

Note that since $w(x'z) = \frac{1}{2}$ and $N(x'z) = \{x, y\}$, then $N(xx') \cap N(yz) = \emptyset$ and $|N(xx')| < n - |N(yz)| \leq \frac{n}{2}$. This means $xx'y, xx'z$ are in $T_1(G)$. Furthermore, by Claim 3, each triangle is contained in at most one copy of K_4, so $xx'y, xx'z$ do not belong to any other $\phi(uvw,\ u'vw)$. Since

$$\begin{aligned}
&w(xyz) + w(x'yz) + w(xx'y) + w(xx'z)\\
&= 4 + \frac{2}{|N(yz)|} + \frac{2}{N(xx')}\\
&\geq 4 + \frac{8}{|N(xx')| + |N(yz)|} \geq 4 + \frac{8}{n},
\end{aligned}$$

we can transfer the weight of $xx'y,\ xx'z$ to $xyz,\ x'yz$ and ensure the average weight is at least $1 + \frac{2}{n}$. □

Now by the Claim 4 and Theorem 1, we have

$$\begin{aligned}
t(G) &= \sum_{xyz \in T(G)} 1 \leq \frac{n}{n+2} \sum_{xyz \in T(G)} 1 + \frac{2}{n}\\
&\leq \frac{n}{n+2} \sum_{xyz \in T(G)} (w(xz) + w(yz) + w(xy))\\
&\leq \frac{n}{n+2} e(G) \leq \frac{n^2}{4} - 1 + \mathbb{1}_{4|n}.
\end{aligned}$$

Equality holds if and only if $e(G)$ attains the maximum and the average weight of each triangle is exactly $1 + \frac{2}{n}$. Hence, by the characterization of the extremal graphs for $\text{ex}(n, C_3^3)$ in Theorem 1, we have $G = M_{\lceil \frac{n}{4} \rceil} + M_{\lfloor \frac{n}{4} \rfloor}$ when n is even. ■

3 Proof of Theorem 3

Let $t(u, v)$ denote the number of triangles containing u or v or both. First, we use a technique to reduce the proof of Theorem 3 to the case that each vertex is contained in many triangles. To this end we use the following lemma.

Lemma 1. *Suppose G is a P_3^3-free graph on at least 300 vertices. If for any two different vertices u, v, we have $t(u), t(v) \geq \frac{n}{2} - 1$ and $t(u, v) \geq n - 2$, then $t(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ and equality holds if and only if $G = K_1 + K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.*

First we will show how to deduce Theorem 3 from Lemma 1, then we will prove Lemma 1.

3.1 Proof of Theorem 3 using Lemma 1.

Let G be a P_3^3-free graph on n vertices with $n \geq 300^3$ and $t(G) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. We initialize $G_n = G$ and define a process of as follows: for $j < n$, let $G_j = G_{j+1} - v_1$ if $t(v_1) < \frac{j+1}{2} - 1$

in G_{j+1}, or $G_{j-1} = G_{j+1} - \{v_1, v_2\}$ if $t(v_1, v_2) < (j+1) - 2$ in G_{j+1}. Suppose the process ends at G_ℓ and for any two vertices u, v in G_ℓ, we have $t(u), t(v) \geq \frac{\ell}{2} - 1$ and $t(u, v) \geq \ell - 2$. Note that

$$\binom{\ell}{3} \geq t(G_\ell) \geq \left\lfloor \frac{(\ell-1)^2}{4} \right\rfloor + \frac{n-\ell}{2}$$

Hence $\ell \geq \sqrt[3]{3n} \geq 300$ and by Lemma 1, G_ℓ contains a copy of P_3^3, a contradiction.

That is to say, G_n satisfies the conditions in Lemma 1 and we are done. ■

3.2 Proof of Lemma 1.

Let $G = G_1 \cup \cdots \cup G_c$ be a P_3^3-free graph on $n \geq 300$ vertices, where G_i are the connected components of G, for $1 \leq i \leq c$. We may assume each edge of G is contained in at least one triangle, otherwise we delete it and the conditions still hold in the resulting graph. For any two distinct vertices u, v, we have $t(u), t(v) \geq \frac{n}{2} - 1$ and $t(u, v) \geq n - 2$. It follows that $v(G_i) \geq \delta(G) \geq \sqrt{n}$.

As mentioned in the introduction, Yuan and Yang [11] determined $\mathrm{ex}(n, K_3, M_2^3)$ for all n.

Theorem 4. *(Yuan and Yang [11]) For $n \geq 7$, we have*

$$\mathrm{ex}(n, K_3, M_2^3) = \max\left\{3n - 8, \left\lfloor \frac{(n-1)^2}{4} \right\rfloor\right\}.$$

Furthermore, $K_3 + \overline{K}_{n-3}$ or $K_1 + K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ is the unique extremal graph.

If no G_i contains two vertex-disjoint triangles, then since $v(G_i) \geq \sqrt{n} \geq \sqrt{300}$, we have $t(G_i) \leq \left\lfloor \frac{(v(G_i)-1)^2}{4} \right\rfloor$ by Theorem 4 and

$$t(G) = \sum_{i=1}^{c} t(G_i) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

Equality holds if and only if G is connected and $G = K_1 + K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

Therefore, we may assume without loss of generality that G_1 contains two vertex-disjoint triangles. We define the distance between two vertex-disjoint triangles as the minimum length of a path with endvertices in the respective triangles. Among all vertex-disjoint triangle pairs in G_1, let $x_1y_1z_1$, $x_2y_2z_2$ be two vertex disjoint triangles whose distance is the smallest and let $P = x_1 \cdots y_2$ be a path of minimal length between them. First suppose the length of P is at least 2. Let x_1^+ be the vertex adjacent to x_1 on the path P and let $x_1x_1^+w$ be a triangle containing the edge $x_1x_1^+$. Then we either find a copy of P_3^3 if $w \in \{x_2, y_2, z_2\}$, or we find another two vertex disjoint triangles whose distance is smaller, and in both cases we obtain a contradiction. Hence we have that $P = x_1y_2$ is a single edge.

Note that x_1y_2 is also contained in a triangle and the third vertex of this triangle must be in $\{y_1, z_1, x_2, z_2\}$. Without loss of generality, say $x_1y_2z_2$ is a triangle. Let $S = (N(y_2) \cap N(z_2)) - \{x_1, y_1, z_1\}$. Obviously, we have that S is nonempty and independent, since $x_2 \in S$ and G contains no copy of P_3^3.

Suppose u is a vertex in $N(y_2) - (S \cup \{x_1, y_1, z_1, z_2\})$ and uy_2w is a triangle containing the edge uy_2. If w does not belong to $\{x_1, y_1, z_1\}$, then $y_1z_1x_1$, $x_1z_2y_2$, y_2uw forms a copy of P_3^T. If $w \in \{x_1, y_1, z_1\}$, then $y_1z_1x_1$, wuy_2, $y_2z_2x_2$ forms a copy of P_3^T. In both cases we have a contradiction. It follows that $N(y_2) \subset S \cup \{x_1, y_1, z_1, z_2\}$ and analogously $N(z_2) \subset S \cup \{x_1, y_1, z_1, y_2\}$.

Since $\delta(G) \geq \sqrt{n}$, then $|S| \geq 2$. We may assume that for each edge of the form y_2u and z_2u with $u \in S$, is only contained in the triangle y_2uz_2. If not, suppose y_2uw is a new triangle distinct from y_2uz_2. Since S is independent, we have $w \in \{x_1, y_1, z_1\}$. Let $u' \in S - \{u\}$, then $y_1z_1x_1$, wuy_2, $y_2u'z_2$ forms a copy of P_3^3, a contradiction. Therefore, if we delete the two vertices y_2, z_2 we destroy at most $|S| + 9$ triangles. By the condition $t(u) + t(v) \geq n - 2$, we have $|S| \geq n - 11$. We obtain that the total number of triangles is at most

$$(n-11)\binom{11}{2} + \binom{11}{3} < \left\lfloor \frac{(n-1)^2}{4} \right\rfloor,$$

where the equality holds because $n \geq 300$. It follows that G is not the extremal graph, and we are done. ■

4 Concluding remarks

In this paper, we studied the generalized Turán number of edge blow-ups of the graphs C_3^3 and P_3^3. It would be interesting to consider the general case of C_k^3 and P_k^3. In this section, we pose two conjectures about the generalized extremal numbers of these graphs.

Let $H(n, p, t)$ denote the graph $K_{t-1} + T_p(n-t+1)$, where $T_p(n-t+1)$ is the balanced p-partite complete graph on $n - t + 1$ vertices, i.e., the Turán graph. Let $H^+(n, p, t)$ be the graph obtained from $H(n, p, t)$ by adding an extra edge in any class of $T_p(n - t + 1)$.

Based on the our results and the Turán number of $\mathrm{ex}(n, C_k^3)$ and $\mathrm{ex}(n, P_k^3)$, we pose the following conjecture for the generalized Turán number.

Conjecture 1. *When $k \geq 4$ and n is sufficiently large, $H(n, 2, \lfloor \frac{k-1}{2} \rfloor + 1)$ is the unique extremal graph for both $\mathrm{ex}(n, K_3, C_k^3)$ and $\mathrm{ex}(n, K_3, P_k^3)$ when k is odd, and $H^+(n, 2, \lfloor \frac{k-1}{2} \rfloor + 1)$ is the unique extremal graph when k is even.*

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