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The maximum number of triangles in F_k -free graphs



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ABSTRACT

The generalized Turán number $\text{ex}(n, K_s, H)$ is the maximum number of complete graph K_s in an H -free graph on n vertices. Let F_k be the friendship graph consisting of k triangles. Erdős and Sós (1976) determined the value of $\text{ex}(n, K_3, F_2)$. Alon and Shikhelman (2016) proved that $\text{ex}(n, K_3, F_k) \leq (9k - 15)(k + 1)n$. In this paper, by using a method developed by Chung and Frankl in hypergraph theory, we determine the exact value of $\text{ex}(n, K_3, F_k)$ and the extremal graph for any F_k when $n \geq 4k^3$.

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1. Introduction

One of the most basic problems in extremal Combinatorics is the study of the *Turán number* $\text{ex}(n, F)$, that is the largest number of edges an n -vertex F -free graph can have. A natural generalization is to count other subgraphs instead of edges. Given graphs H and G , we let $\mathcal{N}(H, G)$ denote the number of copies of H in G . The *generalized Turán number* $\text{ex}(n, H, F)$ is the largest $\mathcal{N}(H, G)$ among n -vertex F -free graphs G .

Let $T_p(n)$ denote the *Turán graph*; a balanced complete p -partite graph on n vertices. Turán [22] proved that $T_p(n)$ is the unique extremal graph of $\text{ex}(n, K_{p+1})$, which is regarded as the beginning

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of the extremal graph theory. The famous Erdős–Stone–Simonovits Theorem [10,11] states if H is a graph with chromatic number $\chi(H) = \chi \geq 3$, then

$$\text{ex}(n, H) = \left(\frac{\chi - 2}{\chi - 1} + o(1) \right) \binom{n}{2}.$$

That is, the Turán number $\text{ex}(n, H)$ is determined asymptotically for any nonbipartite graph H . However, it is still a challenging problem to determine the exact value of the Turán number and the extremal graphs for many nonbipartite graphs.

The *friendship graph* or k -*fan* F_k consists of k triangles all intersecting in one common vertex v . Obviously, F_k is nonbipartite. Erdős, Füredi, Gould and Gunderson determined the Turán number of it.

Theorem 1 (Erdős, Füredi, Gould and Gunderson [9]). *For every $k \geq 1$ and $n \geq 50k^2$,*

$$\text{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

Recently, the problem of estimating generalized Turán number has received a lot of attention, some classical results have been extended to the generalized Turán problem. One can find them in [3,4,14,16,17,19,20,24,25]. A particular line of research is to determine for a given graph H , what graphs F have the property that $\text{ex}(n, H, F) = O(n)$. This was started by Alon and Shikhelman [1], who dealt with the case $H = K_3$, and was continued for other graphs in [12,15].

An *extended friendship graph* consists of F_k for some $k \geq 0$ and any number of additional vertices or edges that do not create any additional cycles. Alon and Shikhelman [1] showed that $\text{ex}(n, K_3, F) = O(n)$ if and only if F is an extended friendship graph. We remark that known results easily imply that if F is not an extended friendship graph, then $\text{ex}(n, K_3, F) = \omega(n)$ and it is also easy to see that adding further edges to F without creating any cycle does not change linearity of $\text{ex}(n, K_3, F)$. Hence the key part of their proof is the following theorem.

Theorem 2 (Alon and Shikhelman [1]). *For any k we have $\text{ex}(n, K_3, F_k) < (9k - 15)(k + 1)n$.*

This upper bound for $\text{ex}(n, K_3, F_k)$ is not tight. For instance, for $k = 2$, it was observed by Liu and Wang [18] that a hypergraph Turán theorem of Erdős and Sós [21] gives the exact result for $\text{ex}(n, K_3, F_2)$. Let \mathcal{F}_k denote the 3-uniform hypergraph (k -star) consisting of k hyperedges sharing exactly one vertex. Let $\text{ex}_3(n, \mathcal{F}_k)$ denote the largest number of hyperedges that an \mathcal{F}_k -free n -vertex 3-uniform hypergraph can contain.

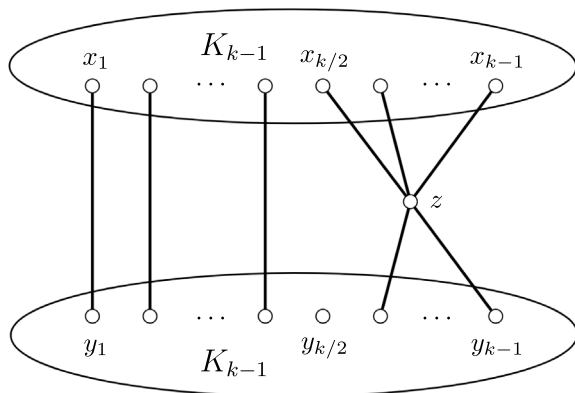
Theorem 3 (Erdős and Sós [21]). *For all $n \geq 3$,*

$$\text{ex}_3(n, \mathcal{F}_2) = \begin{cases} n & \text{if } n = 4m, \\ n - 1 & \text{if } n = 4m + 1, \\ n - 2 & \text{if } n = 4m + 2 \text{ or } n = 4m + 3. \end{cases}$$

Hence, it is interesting to determine the exact value of $\text{ex}(n, K_3, F_k)$ for any F_k ($k \geq 3$).

Let $G = (V(G), E(G))$ be a connected simple graph and $e(G) = |E(G)|$. For any vertex $v \in V(G)$ and subset $S \subseteq V(G)$, let $N_S(v)$ denote the neighbors of v in S and $d_S(v) = |N_S(v)|$. If $S = V(G)$, then $N(v) = N_S(v)$ and $d(v) = d_S(v)$. For $X, Y \subseteq V(G)$, $[X, Y]$ denotes the set of edges with one end in X and another in Y and $[x, Y] = [X, Y]$ if $X = \{x\}$. Let $\pi(G)$ denote the degree sequence of G . For two graphs G_1 and G_2 , $G_1 \cup G_2$ is the vertex disjoint union of G_1 and G_2 and kG consists of k copies of vertex disjoint union of G , $G_1 + G_2$ is the graph obtained by taking $G_1 \cup G_2$ and joining all pairs v_1, v_2 with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let K_n and \bar{K}_n denote the complete graph and the empty graph on n vertices, respectively.

We first define two graphs. Let $k \geq 4$ be even, $X = \{x_1, \dots, x_{k-1}\}$ and $Y = \{y_1, \dots, y_{k-1}\}$. The graph H'_k is a graph obtained from a complete bipartite graph with vertex classes X and Y . We

Fig. 1. The graph H_k .

subdivide the edge $x_i y_i$ once for $i \leq \frac{k}{2} - 1$, and then identify the $\frac{k}{2} - 1$ inserted vertices into one vertex z . The graph H_k is the complement of H'_k deleting the edge $zy_{k/2}$, which is shown in Fig. 1.

It is clear that $|H_k| = |H'_k| = 2k - 1$ and $\pi(H_k) = \pi(H'_k) = (k - 1, \dots, k - 1, k - 2)$.

The main result of this paper is the following.

Theorem 4. Let $k \geq 3$ be an integer and $n \geq 4k^3$. If k is odd, then

$$\text{ex}(n, K_3, F_k) = (n - 2k)k(k - 1) + 2\binom{k}{3},$$

and $\bar{K}_{n-2k} + 2K_k$ is the unique extremal graph, and if k is even, then

$$\text{ex}(n, K_3, F_k) = (n - 2k + 1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2,$$

and $\bar{K}_{n-2k+1} + H_k$ is the unique extremal graph.

Given a graph G , we let $\mathcal{T}(G)$ denote the 3-uniform hypergraph on the vertex set $V(G)$ where $\{u, v, w\}$ form a hyperedge if and only if uvw is a triangle in G . The key observation is that if G is F_k -free, then $\mathcal{T}(G)$ is \mathcal{F}_k -free. Therefore, $\text{ex}(n, K_3, F_k) \leq \text{ex}_3(n, \mathcal{F}_k)$. In the case $k = 2$, the upper bound obtained this way matches the lower bound provided by $\lfloor n/4 \rfloor$ vertex-disjoint copies of K_4 , and in the case $n = 4m + 3$ we also have a triangle on the remaining vertices. This gives the exact value of $\text{ex}(n, K_3, F_2)$.

The result of Erdős and Sós [21] was extended to arbitrary k by Chung and Frankl [7], after partial results [5,6,8].

Theorem 5 (Chung and Frankl [7]). Let $k \geq 3$. If n is sufficiently large, then

$$\text{ex}_3(n, \mathcal{F}_k) = \begin{cases} (n - 2k)k(k - 1) + 2\binom{k}{3} & \text{if } k \text{ is odd,} \\ (n - 2k + 1)\frac{(2k-1)(k-1)-1}{2} + (2k-2)\binom{k-1}{2} + \binom{k-2}{2} - \frac{(k-2)(k-4)}{2} + \frac{k}{2} & \text{if } k \text{ is even.} \end{cases}$$

and $\mathcal{F}_k := \mathcal{T}(\bar{K}_{n-2k} + 2K_k)$ is the unique extremal 3-uniform hypergraph, when k is odd.

For odd k , this completes the proof of the upper bound. However, for even k , the construction giving the lower bound in the above theorem is not $\mathcal{T}(G)$ for some F_k -free graph G . Still, the upper bound differs from the lower bound only by an additive constant $c(k)$. We will heavily use the tools provided by Chung and Frankl [7] to obtain the improvement needed in Theorem 4.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we study the local structure within the neighborhood of a vertex in an F_k -free graph, and some properties of a weight function defined on the vertices of triangles which is our main method for counting the number of triangles. These results can be used to prove Theorem 4 with the best coefficient of n but a weak constant $f(k)$. The very technical Section 4 is devoted to prove Theorem 4 precisely. In Section 5, we give some concluding remarks.

2. Preliminaries

As a preparation for proving our result, we first present some known theorems, and then we count the number of triangles in a graph with given degree sequence, which are interesting of their own right.

Let $\nu(G)$ denote the number of edges of a maximum matching in a graph G . The following is the famous result about the maximum matching,

Theorem 6 (Berge [2]). Let $o(G - X)$ denote the number of odd components of $G - X$, then

$$\nu(G) = \frac{1}{2} \min \left\{ |G| - o(G - X) + |X| : X \subseteq V(G) \right\}.$$

Theorem 7 (Chung and Frankl [7]). Let k be an even integer and H be a graph on $2k - 1$ vertices and with $\pi(H) = (k - 1, \dots, k - 1, k - 2)$, then either

$$\mathcal{N}(K_3, H) \geq \left(\frac{k}{2} - 1 \right)^2 - 1,$$

or

$$\mathcal{N}(K_3, H) = \left(\frac{k}{2} - 2 \right) \left(\frac{k}{2} - 1 \right) \text{ and } H = H'_k.$$

Theorem 8. Let k be an even integer and H be a graph on $2k - 1$ vertices with $\pi(H) = (k - 1, \dots, k - 1, k - 2)$, then

$$\mathcal{N}(K_3, H) \leq 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2,$$

equality holds if and only if $H = H_k$.

Proof. The proof will be similar to the proof of Goodman. It is easy to see that

$$\mathcal{N}(K_3, H_k) = 2 \binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2} = 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2.$$

Let \overline{H} be the complement of H . For any triple (x, y, z) , if xyz is neither a triangle in H nor a triangle in \overline{H} , then it is easy to check exactly two of the three, say x, y such that $||x, \{y, z\}|| = 1$ and $||y, \{x, z\}|| = 1$ in H . Thus, we have

$$\begin{aligned} \mathcal{N}(K_3, H) &= \binom{2k-1}{3} - \mathcal{N}(K_3, \overline{H}) - \frac{1}{2} \sum_v d(v)(2k-2-d(v)) \\ &= \binom{2k-1}{3} - (k-1)^3 - \frac{1}{2} k(k-2) - \mathcal{N}(K_3, \overline{H}) \\ &= 2 \binom{k-1}{3} + \frac{(k-2)^2}{2} - \mathcal{N}(K_3, \overline{H}). \end{aligned}$$

Obviously, it is sufficient to show $\mathcal{N}(K_3, \overline{H}) \geq \left(\frac{k}{2} - 1 \right)^2$.

Note that \overline{H} is a graph on $2k - 1$ vertices with $\pi(\overline{H}) = (k - 1, \dots, k - 1, k)$. Let z be the vertex of degree k in \overline{H} .

If there is an edge $zz' \in E(\overline{H})$ such that zz' is contained in at least two triangles, then $\mathcal{N}(K_3, \overline{H}) \geq \mathcal{N}(K_3, \overline{H} - zz') + 2$. Because of $\pi(\overline{H} - zz') = (k - 1, \dots, k - 1, k - 2)$, by Theorem 7, either $\mathcal{N}(K_3, \overline{H} - zz') \geq \left(\frac{k}{2} - 1\right)^2 - 1$ or $\overline{H} - zz' = H'_k$. In the former case, we have

$$\mathcal{N}(K_3, H) \leq 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2 - 1.$$

In the latter case, it is easy to check that $\overline{H} = \overline{H}_k$, and hence $H = H_k$.

If each edge $zz' \in E(\overline{H})$ is contained in at most one triangle, then $\Delta(\overline{H}[N(z)]) \leq 1$. Let $V_1 = N(z)$, $V_2 = V(H) - V_1$, $s = e(\overline{H}[V_1])$ and $t = e(\overline{H}[V_2])$. Count the edges between V_1 and V_2 in two ways, we have

$$k(k-1) - 2s = (k-1)^2 + 1 - 2t,$$

which implies $s - t = \frac{k}{2} - 1$. However, because $s \leq \frac{k}{2}$, we must have $s = \frac{k}{2}$ and $t = 1$, or $s = \frac{k}{2} - 1$ and $t = 0$. Since each edge in $\overline{H}[V_1]$ can form a triangle with at least $k - 3$ vertices in V_2 , we get

$$\mathcal{N}(K_3, \overline{H}) \geq \frac{k}{2}(k-3) + (k-4) > \left(\frac{k}{2} - 1\right)^2$$

in the former case, and

$$\mathcal{N}(K_3, \overline{H}) \geq \left(\frac{k}{2} - 1\right)(k-3) \geq \left(\frac{k}{2} - 1\right)^2$$

in the latter case with equality only if $k = 4$. In this case, it is not difficult to check that $\overline{H} = \overline{H}_4$, and so $H = H_4$. ■

Theorem 9. Let k be an even integer and H be a graph on $2k - 1 - 2s$ vertices with $\pi(H) = (k - 1, \dots, k - 1, k - 2)$, then

$$\mathcal{N}(K_3, H) \leq \frac{1}{6}(2k - 1 - 2s)((k - 1)(k - 2) - (k - 1 - 2s)(2s + 1)) + \frac{1}{2} - s.$$

Proof. Let $\Lambda(H)$ denote the number of triples (x, y, z) having exactly two edges in H , say $xy, xz \in E(H)$ and $yz \notin E(H)$. Because $|N(y) \cap N(z)| \geq d(y) + d(z) - (|H| - 2)$ for every nonadjacent pair (y, z) , and there are $|H| - (d(y) + 1)$ nonadjacent pairs containing y for any $y \in V(H)$, we have

$$\Lambda(H) \geq \frac{1}{2}((2k - 1 - 2s)(k - 1 - 2s) + 1)(2s + 1) - (k - 2s).$$

On the other hand, since

$$(2k - 2 - 2s) \binom{k-1}{2} + \binom{k-2}{2} = \Lambda(H) + 3\mathcal{N}(K_3, H),$$

we get

$$3\mathcal{N}(K_3, H) \leq \frac{1}{2}(2k - 1 - 2s)((k - 1)(k - 2) - (k - 1 - 2s)(2s + 1)) + \frac{3}{2} - 3s.$$

This completes the proof. ■

3. Some properties of F_k -free graphs and a weight function

Let G be an F_k -free graph, $uv \in E(G)$ and $N(uv) = N(u) \cap N(v)$. Clearly, $|N(uv)|$ is the number of triangles containing the edge uv in G . We classify the edges into the following three classes:

- Heavy edges: $\mathcal{H} = \{uv : |N(uv)| \geq 2k - 1\}$,

- Medium edges: $\mathcal{M} = \{uv : k \leq |N(uv)| \leq 2k - 2\}$, and
- Light edges: $\mathcal{L} = \{uv : 1 \leq |N(uv)| \leq k - 1\}$.

For a fixed vertex $u \in V(G)$, let $G_u = G[N(u)]$ and

- $\mathcal{H}(u) = \{v : v \in N(u) \text{ and } uv \in \mathcal{H}\}$.
- $\mathcal{M}(u) = \{v : v \in N(u) \text{ and } uv \in \mathcal{M}\}$,
- $\mathcal{L}(u) = \{v : v \in N(u) \text{ and } uv \in \mathcal{L}\}$.

This notation will be used throughout the rest of this paper.

Since G is F_k -free, then $v(G_u) \leq k - 1$ for any u . Thus, [Theorem 6](#) implies

Observation 1. *There exists some $X \subseteq V(G_u)$ such that*

$$\sum_{i=1}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X| \leq k - 1, \quad (3.1)$$

where C_1, \dots, C_{ℓ} are all the components of $G_u - X$.

Lemma 1. *Let G be an F_k -free graph, $u \in V(G)$ and X a subset of $V(G_u)$ satisfying Eq. (3.1). Then we have the following:*

- $\mathcal{H}(u) \subseteq X$. Moreover, $|\mathcal{H}(u)| \leq k - 1$ and if equality holds, then $\mathcal{M}(u) = \emptyset$.
- $|\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u)| \leq k - \frac{1}{2}$.

Proof. Let C_1, \dots, C_{ℓ} be the components of $G_u - X$.

(i) Let $v \in \mathcal{H}(u)$, we know that $|N(uv)| \geq 2k - 1$. If v lies in some component C_i , then $N(uv) \subseteq V(C_i) \cup X$ and so

$$\frac{1}{2} |(C_i \cap N(uv)) \cup \{v\}| + |X \cap N(uv)| \geq k,$$

which contradicts (3.1). Hence we have $\mathcal{H}(u) \subseteq X$.

By (3.1), we have $|\mathcal{H}(u)| \leq k - 1$, and if $|\mathcal{H}(u)| = k - 1$, then $X = \mathcal{H}(u)$ and $|C_i| = 1$ for $1 \leq i \leq \ell$. Let v be any vertex of $G_u - X$, then $N(uv) \subseteq X$ and hence $uv \in \mathcal{L}$, and so $\mathcal{M}(u) = \emptyset$.

(ii) Clearly, $N(uv) \subseteq X \cup V(C_i)$ if $v \in V(C_i)$. Thus, if there are two components, say C_1, C_2 , such that $\mathcal{M}(u) \cap V(C_i) \neq \emptyset$, then $|X| + |C_i| \geq k + 1$ for $i = 1, 2$. Hence we have $|X| + \lfloor |C_1|/2 \rfloor + \lfloor |C_2|/2 \rfloor \geq k$, which contradicts (3.1). Thus, we may assume $\mathcal{M}(u) \subseteq X \cup V(C_i)$. Note that $\mathcal{H}(u) \subseteq X$ as shown in (i), and

$$\begin{aligned} |\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u)| &= |\mathcal{H}(u)| + \frac{1}{2}|\mathcal{M}(u) \cap (X - \mathcal{H}(u))| + \frac{1}{2}|\mathcal{M}(u) \cap C_i| \\ &\leq |X| + \frac{1}{2}|C_i| \leq |X| + \left\lfloor \frac{|C_i|}{2} \right\rfloor + \frac{1}{2} \leq k - \frac{1}{2}. \end{aligned}$$

The proof of the lemma is complete. ■

For each triangle $T = xyz$ in G , assign T of weight 1 and define a distribution rule $w(T, \cdot)$ to distribute the weight 1 to its three vertices as below (suppose $|N(xy)| \geq |N(yz)| \geq |N(xz)|$): $w(T, x) = w(T, y) = w(T, z) = \frac{1}{3}$, if $E(T) \cap \mathcal{H} = \emptyset$ or $E(T) \cap \mathcal{L} = \emptyset$, $w(T, x) = w(T, z) = \frac{1}{2}$, $w(T, y) = 0$, if $xy \in \mathcal{H}$, $yz \in \mathcal{H} \cup \mathcal{M}$ and $xz \in \mathcal{L}$, $w(T, x) = w(T, y) = 0$, $w(T, z) = 1$, if $xy \in \mathcal{H}$ and $yz, xz \in \mathcal{L}$. Now, we define a weight function $f(u)$ for each vertex u of G as follows.

$$f(u) = \sum_{vx \in E(G_u)} w(uvx, u)$$

if u lies in triangles, and $f(u) = 0$ otherwise. It is clear

$$\mathcal{N}(K_3, G) = \sum_{u \in V(G)} f(u).$$

Now, we first discuss some properties of the weight functions $w(T, \cdot)$ and $f(u)$.

Lemma 2. Let uv be an edge of an F_k -free graph with $k \geq 3$, then either

$$\sum_{x \in N(uv)} w(uvx, u) = k - 1,$$

if $uv \in \mathcal{L}$, $|N(uv)| = k - 1$, $vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$, or

$$\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2}$$

otherwise.

Proof. Let $\mathcal{H}'(v)$, $\mathcal{M}'(v)$ and $\mathcal{L}'(v)$ be $\mathcal{H}(v)$, $\mathcal{M}(v)$ and $\mathcal{L}(v)$ intersecting with $N(uv)$, respectively. It is clear

$$\sum_{x \in N(uv)} w(uvx, u) = \sum_{x \in \mathcal{H}'(v)} w(uvx, u) + \sum_{x \in \mathcal{M}'(v)} w(uvx, u) + \sum_{x \in \mathcal{L}'(v)} w(uvx, u).$$

We distinguish three cases on the number of $|N(uv)|$.

Case 1. $uv \in \mathcal{H}$

By the definition of $w(T, \cdot)$, $w(uvx, u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v) \cup \mathcal{M}'(v)$ and $w(uvx, u) = 0$ if $x \in \mathcal{L}'(v)$. Noting that $u \in \mathcal{H}(v) - \mathcal{H}'(v)$, we have $|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2}$ by Lemma 1(ii), and hence

$$\sum_{x \in N(uv)} w(uvx, u) \leq \frac{1}{2}|\mathcal{H}'(v)| + \frac{1}{2}|\mathcal{M}'(v)| \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)|, \quad (3.2)$$

which implies the result holds.

Case 2. $uv \in \mathcal{M}$

In this case, $|N(uv)| \leq 2k - 2$. Since $uv \in \mathcal{M}$ implies $\mathcal{M}(v) \neq \emptyset$, by Lemma 1(i), we have $|\mathcal{H}'(v)| \leq |\mathcal{H}(v)| \leq k - 2$. By the definition of $w(T, \cdot)$, $w(uvx, u) \leq \frac{1}{2}$ if $x \in \mathcal{H}'(v)$ and $w(uvx, u) \leq \frac{1}{3}$ if $x \in \mathcal{M}'(v) \cup \mathcal{L}'(v)$. Thus, we have

$$\sum_{x \in N(uv)} w(uvx, u) \leq \frac{1}{2}|\mathcal{H}'(v)| + \frac{1}{3}|\mathcal{M}'(v)| + \frac{1}{3}|\mathcal{L}'(v)| \leq k - 1 - \frac{k}{6}. \quad (3.3)$$

The upper bound $k - \frac{3}{2}$ follows from the assumption $k \geq 3$.

Case 3. $uv \in \mathcal{L}$

Because $uv \in \mathcal{L}$, we have $|N(uv)| \leq k - 1$. By the definition of $w(T, \cdot)$, some triangles satisfy $w(uvx, u) = 1$ and other triangles satisfy $w(uvx, u) \leq \frac{1}{2}$. Thus we have

$$\sum_{x \in N(uv)} w(uvx, u) \leq (k - 1) - \frac{1}{2} \left| \left\{ uvx : w(uvx, u) \leq \frac{1}{2} \right\} \right|, \quad (3.4)$$

which implies $\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2}$ or $\sum_{x \in N(uv)} w(uvx, u) = k - 1$, and the latter holds if and only if $|N(uv)| = k - 1$, and all triangles uvx satisfy $w(uvx, u) = 1$, that is, $vx \in \mathcal{H}$ and $ux \in \mathcal{L}$ for any $x \in N(uv)$. ■

Lemma 3. Suppose G is an F_k -free graph and $k \geq 4$ is even. Let $u \in V(G)$, X be a subset of $V(G_u)$ satisfying (3.1) and C_1, \dots, C_ℓ be the components of $G_u - X$ with $|C_1| \geq \dots \geq |C_\ell|$. Then

$$f(u) \leq k \left(k - \frac{3}{2} \right) - \frac{1}{2},$$

or $f(u) = k \left(k - \frac{3}{2} \right)$ and the following hold:

- (i) If $X \neq \emptyset$, then X is an independent set and $d_{G_u}(v) = k - 1$ for any $v \in X$;
- (ii) $\pi(C_1) = (k - 1, \dots, k - 1, k - 2)$, and either $G_u = C_1 \cup K_{k-1}$ with $|C_1| = k + 1$, or $G_u - X = C_1 \cup (\ell - 1)K_1$ with $|C_1| = 2k - 1 - 2|X| \geq k + 1$;
- (iii) $E(G_u) \subseteq \mathcal{H}$, $[u, G_u] \subseteq \mathcal{L}$ and $\Delta(G_u) = k - 1$.

Proof. By (3.1), $|C_i| \leq k$ for all $i \neq 1$. Let uvx be any triangle. Then

$$f(u) = \sum_{i=1}^{\ell} \sum_{vx \subseteq E(C_i)} w(uvx, u) + \sum_{\{v,x\} \cap X \neq \emptyset} w(uvx, u).$$

If the edge vx satisfies $\{v, x\} \cap X \neq \emptyset$, then by Lemma 2, we have

$$\sum_{\{v,x\} \cap X \neq \emptyset} w(uvx, u) \leq \sum_{v \in X} \sum_{x \in N(uv)} w(uvx, u) \leq |X|(k-1). \quad (3.5)$$

If $vx \in E(C_i)$ with $|C_i| \leq k$, then noting that k is even, uvx is a triangle for each $vx \in E(C_i)$ and $w(uvx, u) \leq 1$, we have

$$\begin{aligned} \sum_{vx \in E(C_i)} w(uvx, u) &\leq \frac{1}{2} \sum_{v \in V(C_i)} \sum_{x \in N(uv) \cap C_i} w(uvx, u) \\ &\leq \frac{1}{2} |C_i| (|C_i| - 1) \leq \left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1). \end{aligned} \quad (3.6)$$

If $|C_1| \leq k$, that is, $|C_i| \leq k$ for $1 \leq i \leq \ell$, then by (3.1), (3.5) and (3.6), we have

$$f(u) \leq (k-1) \left(\sum_{i=1}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X| \right) \leq (k-1)^2 < k \left(k - \frac{3}{2} \right) - \frac{1}{2}.$$

The last inequality holds because of $k \geq 4$. So, we may assume that $|C_1| > k$.

If $\Delta(C_1) \geq k$, say $d_{C_1}(v) \geq k$ for some vertex v in C_1 , then $|N(uv)| \geq k$ and so $uv \notin \mathcal{L}$. Since $\mathcal{H}(u) \subseteq X$ by Lemma 1(i), we have $v \notin \mathcal{H}(u)$ and hence $uv \in \mathcal{M}$. By (3.3), we have $\sum_{x \in N(uv) \cap C_1} w(uvx, u) \leq (k-1) - \frac{k}{6}$. Meanwhile, because $d_{C_1}(v) \geq k \geq 4$, there exists $v_i \in N(uv) \cap C_1$ for $1 \leq i \leq 4$. Note that $v \in \mathcal{M}(u)$ and $v \in N(uv_i)$, by Lemma 2, we have $\sum_{x \in N(uv_i) \cap C_1} w(uv_i x, u) \leq (k-1) - \frac{1}{2}$ for $1 \leq i \leq 4$, and

$$\sum_{vx \in E(C_1)} w(uvx, u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_1} w(uvx, u) \leq \frac{1}{2} |C_1| (k-1) - \left(\frac{k}{12} + 1 \right). \quad (3.7)$$

If $\Delta(C_1) \leq k-1$, then

$$\sum_{vx \in E(C_1)} w(uvx, u) = \frac{1}{2} \sum_{v \in C_1} \sum_{x \in N(uv) \cap C_1} w(uvx, u) \leq \left\lfloor \frac{1}{2} |C_1| (k-1) \right\rfloor. \quad (3.8)$$

Set $\mu(C_1) = \frac{k}{12} + 1$ if $\Delta(C_1) \geq k$, $\mu(C_1) = \frac{1}{2}$ if $|C_1|$ is odd and $\Delta(C_1) \leq k-1$ and $\mu(C_1) = 0$ if $|C_1|$ is even and $\Delta(C_1) \leq k-1$, then (3.5)–(3.8) imply

$$f(u) \leq (k-1) \left(\frac{|C_1|}{2} + \sum_{i=2}^{\ell} \left\lfloor \frac{|C_i|}{2} \right\rfloor + |X| \right) - \mu(C_1). \quad (3.9)$$

Assume that $f(u) > k(k - \frac{3}{2}) - \frac{1}{2}$. By (3.1) and (3.9), we have $\mu(C_1) = \frac{1}{2}$. In this case, $|C_1| \geq k+1$ is odd and $\Delta(C_1) \leq k-1$. Note that if one of the equalities in (3.5), (3.6) and (3.8) does not hold, then the upper bound in (3.9) can be reduced by at least an extra $\frac{1}{2}$. This implies the equalities in (3.5), (3.6) and (3.8) holds.

It is clear that the equalities in (3.5) hold if and only if X is an independent set and $\sum_{x \in N(uv)} w(uvx, u) = k-1$ for any $v \in X$. By Lemma 2, we get that $d_{G_u}(v) = k-1$, $uv, ux \in \mathcal{L}$ and $vx \in \mathcal{H}$ for any $v \in X$.

Since $|C_1| \geq k+1$ is odd, $\Delta(C_1) \leq k-1$ and the equality in (3.8) holds, we can deduce that $\pi(C_1) = (k-1, k-1, \dots, k-2)$, $E(C_1) \subseteq \mathcal{H}$ and $[u, C_1] \subseteq \mathcal{L}$.

Because equality (3.6) holds, recalling $|C_1| \geq k+1$ and k is even, by (3.1), we have $|C_i| \in \{1, k-1\}$ for $i \geq 2$ and each C_i is a clique with $E(C_i) \subseteq \mathcal{H}$ and $[u, C_i] \subseteq \mathcal{L}$. In addition, if $|C_i| = k-1$ for some

$i \geq 2$, then by (3.1), $|C_1| = k + 1$ and $X = \emptyset$, that is, $G_u - X = C_1 \cup K_{k-1} \cup (\ell - 2)K_1$. If $|C_i| = 1$ for all $i \geq 2$, then $|C_1| = 2k - 1 - 2|X|$.

So, the statements (i), (ii) and (iii) hold. ■

Remark. There is a similar lemma in Chung and Frankl's paper [7] when they deal with function $\text{ex}_3(n, \mathcal{F}_k)$. However, in their lemma, they overlooked the case $G_u = C_1 \cup K_{k-1}$. Using our method in Section 4, it is not difficult to complete the proof of this missed case, too.

Definition 1. For any vertex $u \in V(G)$, the loss of u is the number

$$k \left(k - \frac{3}{2} \right) - f(u).$$

See the following simple observations about the losses.

Observation 2. If some vertex $v \in X$ has $\sum_{x \in N(uv)} w(uvx, u) \leq (k-1) - c$, then the edge uv contributes c to the loss of u .

Proof. It is a direct consequence of (3.5). ■

Observation 3. An edge $uv \in \mathcal{H}$ contributes $\frac{1}{2}$ to the loss of u . Moreover, a triangle uvx with $uv, vx \in \mathcal{H}$ contributes another $\frac{1}{2}$ to the loss of u .

Proof. Since $uv \in \mathcal{H}$, by (3.2), $\sum_{x \in N(uv)} w(uvx, u) \leq k - \frac{3}{2} - \frac{1}{2}|\mathcal{H}'(v)|$. Because $v \in X$ by Lemma 1, the edge uv contributes $\frac{1}{2}$ to the loss of u by Observation 2. Moreover, since a triangle uvx with $uv, vx \in \mathcal{H}$ satisfies $x \in \mathcal{H}'(v)$, so it contributes another $\frac{1}{2}$ to the loss of u by (3.2). ■

Observation 4. Let $uv \in \mathcal{M}$. If $\sum_{x \in N(uv)} w(uvx, u) \leq (k-1) - c$, then the edge uv contributes at least $\min \left\{ \frac{c}{2}, \frac{k}{4} - \frac{1}{2} \right\}$ to the loss of u . Moreover, the edge uv contributes at least $\frac{k}{12}$ to the loss of u .

Proof. If $v \in X$, then by (3.3) and Observation 2, uv contributes c to the loss of u . If $v \in V(C_1)$ and $|C_1| \geq k + 1$, then by (3.7) and (3.8), uv contributes at least $\frac{c}{2}$ to the loss of u . If $v \in V(C_i)$ for some i with $|C_i| \leq k$, then by (3.6), there is a gap between $\lfloor |C_i|/2 \rfloor (k-1)$ and $\frac{1}{2}|C_i|(|C_i| - 1)$, and for this gap, any edge uv' with $v' \in V(C_i)$ contributes

$$\frac{1}{|C_i|} \left(\left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1) - \frac{1}{2}|C_i|(|C_i| - 1) \right)$$

to the loss of u . On the other hand, because

$$\sum_{x \in N(uv) \cap C_i} w(uvx, u) \leq \frac{1}{2}(|C_i| - 1) = (|C_i| - 1) - \frac{1}{2}(|C_i| - 1),$$

this reduces the right hand of (3.6) by an additional $\frac{1}{4}(|C_i| - 1)$. Hence the total loss of u contributed by the edge uv is at least

$$\frac{1}{|C_i|} \left(\left\lfloor \frac{|C_i|}{2} \right\rfloor (k-1) - \frac{1}{2}|C_i|(|C_i| - 1) \right) + \frac{1}{4}(|C_i| - 1) \geq \frac{k}{4} - \frac{1}{2}.$$

Together with (3.3), $c \geq \frac{k}{12}$, it implies that the statements of the lemma are proved. ■

4. Proof of Theorem 4

Let G be an extremal graph of $\text{ex}(n, K_3, F_k)$.

If k is odd, then by [Theorem 5](#), we have

$$\mathcal{N}(K_3, G) = e(\mathcal{T}(G)) \leq \text{ex}_3(n, \mathcal{F}_k) = (n - 2k)k(k - 1) + 2\binom{k}{3},$$

and the unique extremal hypergraph is \mathcal{F}_k for which equality holds. Because

$$\mathcal{N}(K_3, \bar{K}_{n-2k} + 2K_k) = (n - 2k)k(k - 1) + 2\binom{k}{3},$$

and $\mathcal{T}(\bar{K}_{n-2k} + 2K_k) = \mathcal{F}_k$, we get

$$\mathcal{N}(K_3, G) = (n - 2k)k(k - 1) + 2\binom{k}{3},$$

where equality holds if and only if $G = \bar{K}_{n-2k} + 2K_k$.

The remaining part is devoted to the case when $k \geq 4$ is even. Because an edge not lying in a triangle makes no contribution to $\mathcal{N}(K_3, G)$, we may assume that each edge of G is covered by some triangles.

If $f(v) = k(k - \frac{3}{2})$, then we call v a *good* vertex. Let $U_1 = \{v : v \text{ is good}\}$. Since $n \geq 4k^3$ and $f(v) \leq k(k - \frac{3}{2}) - \frac{1}{2}$ for any $v \notin U_1$ by [Lemma 3](#), we have

$$\begin{aligned} nk\left(k - \frac{3}{2}\right) - \frac{1}{2}(n - |U_1|) &\geq \sum_{v \in V(G)} f(v) = \mathcal{N}(K_3, G) \\ &\geq (n - 2k + 1)k\left(k - \frac{3}{2}\right) + 2\binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2, \end{aligned}$$

which implies

$$|U_1| \geq n - 2(2k - 1)k\left(k - \frac{3}{2}\right) > 2k.$$

Moreover, if there exist $v, v' \in U_1$ such that $vv' \in E(G)$, then $vv' \in \mathcal{L}$ by [Lemma 3](#). Let $vv'x$ be a triangle. Applying [Lemma 3](#) to v , we have $v'x \in \mathcal{H}$ and using [Lemma 3](#) to v' , we have $v'x \in \mathcal{L}$, a contradiction. Therefore, U_1 is an independent set.

Let $u \in U_1$ be given, $G_u = G[N(u)]$ as before and $U_2 = V(G) - V(G_u) - U_1$. We will prove [Theorem 4](#) by showing G is an extremal graph only if $U_2 = \emptyset$. Since the proof is a little complicated and long, so we sketch it first in the following two paragraphs.

In the case when $N(u') = N(u)$ for any $u' \in U_1 - \{u\}$, our main idea for doing this is to partition the total weights of all vertices of G_u into two parts: One part comes from the triangles contained in G_u , which is exactly $\mathcal{N}(K_3, G_u)$, and another part is contributed by the triangles containing one or two vertices in U_2 . And then we use discharge method to transfer the latter part to the vertices in U_2 such that $f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$ after transferring. Using this method, we show that if $U_2 \neq \emptyset$, then the total weight of G is less than the expected number.

In the case when there is some $u' \in U_1 - \{u\}$ such that $N(u) \neq N(u')$, we transform G into a graph G' such that G' and G have the same good vertices, and all good vertices of G' have the same neighborhood as $N(u)$, through an operation as follows: Delete all the edges between u' and $G_{u'}$ and add new edges joining u' to all vertices in G_u . Repeat this operation until all vertices in U_1 have the same neighborhood $N(u)$. Let G' be the resulting graph, $U'_1 = \{v : v \text{ is good in } G'\}$ and $U'_2 = V(G') - V(G'_u) - U'_1$. We will see that $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$, G' is also \mathcal{F}_k -free and $U'_1 = U_1$.

Firstly, since u' is good, by [Lemma 3](#), we have $f(u') = e(G_{u'}) = k(k - \frac{3}{2})$, which implies we destroy $k(k - \frac{3}{2})$ triangles first and then add $k(k - \frac{3}{2})$ new triangles, and so $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$. Moreover, $G'_u = G_u$. Secondly, since G_u has no kK_2 and $\Delta(G_u) = k - 1$ by [Lemma 3](#), we can see that G' is also \mathcal{F}_k -free after an easy check. Finally, because $|U_1| > 2k$, we have $E(G'_u) \subseteq \mathcal{H}$, which implies $v \notin U'_1$ for any $v \in V(G'_u)$ by [Lemma 3](#). Furthermore, since $\Delta(G'_u) = k - 1$ and U'_1 is an independent set, $[U_1, G'_u] \subseteq \mathcal{L}$. Thus, we have $U_1 \subseteq U'_1$ by the definition of $w(T, \cdot)$. Suppose that there is some $v \in U_2$ in G such that $v \in U'_1$ in G' . Let $G'_v = G'[N(v)]$ and $X' \subseteq G'_v$ satisfy [\(3.1\)](#). Since $v \in U'_1$,

$E(G'_v) \subseteq \mathcal{H}$ and $[v, G'_v] \subseteq \mathcal{L}$ by Lemma 3. By the operation above, $E(G'_v) \subseteq \mathcal{H}$ in G . Since $v \in U_2$, there is some $v' \in G'_v$ such that $vv' \notin \mathcal{L}$ in G , which means there is some $u' \in U_1$ such that $u'vv'$ is a triangle in G . Note that $V(G'_v) \cup \{u'\} \subseteq N_G(v)$. If $v' \notin X'$, then by Lemma 3, it is easy to check that $G[\{u'\} \cup V(G'_v)]$ contains kK_2 , and so G has an F_k . Thus we have $v' \in X'$. In this case, by Lemma 3, $|\mathcal{H}(v')| = k - 1$ in G . Let $X'' \subseteq G_{v'}$ satisfy (3.1). By Lemma 1, $\mathcal{H}(v') \subseteq X''$ and hence $|X''| = k - 1$. Because $u'v$ is an edge in $G_{v'} - X''$, this contradicts (3.1). Thus, we have $U'_1 = U_1$.

Since $U'_1 = U_1$ implies $U'_2 = U_2$, and $U'_2 \neq \emptyset$ in this case, G' cannot be an extremal graph, and so does G since $\mathcal{N}(K_3, G') = \mathcal{N}(K_3, G)$. Therefore, it is sufficient to show G is an extremal graph only if $U_2 = \emptyset$ in the case when $N(u') = N(u)$ for any $u' \in U_1 - \{u\}$.

Let $X \subseteq G_u$ satisfy (3.1). By Lemma 3, $\Delta(G_u) = k - 1$. Moreover, G_u has the following structural properties.

Claim 1. Let $v \in V(C_i)$, where C_i is some component of $G_u - X$.

(1) If $d_{G_u}(v) = k - 1$, then $N_{U_2}(v)$ is an independent set and $[v, U_2] \subseteq \mathcal{L}$.

(2) If $d_{C_i}(v) = k - 2$, then $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $v_1 \in N_{U_2}(v)$ and $vv_1 \in \mathcal{H} \cup \mathcal{M}$, then v_1 is the center of the star with at least 3 vertices, or lies on the triangle.

Proof. (1) Since $d_{G_u}(v) = k - 1$ and $E(G_u) \subseteq \mathcal{H}$ by Lemma 3, we have $|\mathcal{H}(v)| \geq k - 1$. Let $X' \subseteq V(G_v)$ satisfy (3.1). By Lemma 1, $X' = \mathcal{H}(v) \subseteq V(G_u)$ which in turn implies $G_v - X'$ has no edges by (3.1), and $\mathcal{M}(v) = \emptyset$, and so $[v, U_2] \subseteq \mathcal{L}$.

(2) Since $d_{C_i}(v) = k - 2$ and $E(G_u) \subseteq \mathcal{H}$, we have $|\mathcal{H}(v)| \geq k - 2$. Let $X' \subseteq V(G_v)$ satisfy (3.1). By Lemma 1, $\mathcal{H}(v) \subseteq X'$ and so $|X' \cap V(G_u)| \geq k - 2$. Because $|X'| \leq k - 1$, we have $|X' \cap U_2| \leq 1$, which implies $G_v - \mathcal{H}(v)$ has no $2K_2$ by (3.1), that is, $G[N_{U_2}(v)]$ is a star or a triangle, together with some isolated vertices. Moreover, if $vv_1 \in \mathcal{H} \cup \mathcal{M}$, then we have $d_{G_u}(v) = k - 2$ by Lemma 1. Since $|N(vv_1)| \geq k$ and we have $|N(vv_1) \cap U_2| \geq 2$, that is, v_1 has at least 2 neighbors in $G[N_{U_2}(v)]$. Hence, v_1 is the center of the star or lies on a triangle in $G[N_{U_2}(v)]$. \square

Let C be the largest component of $G_u - X$. Then $\pi(C) = (k - 1, \dots, k - 1, k - 2)$ by Lemma 3. Let $z \in V(C)$ be the unique vertex with $d_C(z) = k - 2$ and $N_C(z) = \{z_1, \dots, z_{k-2}\}$. Since $\Delta(G_u) = k - 1$, we have $|[V(C), X]| \leq 1$ and $[V(C), X] \subseteq [z, X]$. In addition, we have the following.

Claim 2. $|\mathcal{H}(z) \cup \mathcal{M}(z) \cap U_2| \leq 1$.

Proof. Assume $v_1, v_2 \in N_{U_2}(z)$ with $zv_1, zv_2 \notin \mathcal{L}$. By Claim 1(2), v_1v_2v is a triangle in $N_{U_2}(z)$ for some v . Since $|N(zv_i)| \geq k$ for $i = 1, 2$, $\{z_1, \dots, z_{k-2}\} \subseteq N(v_1) \cap N(v_2)$ which contradicts Claim 1(1) since $v_1, v_2 \in N_{U_2}(z_1)$ and $v_1v_2 \in E(G)$. \square

If $|\mathcal{H}(z) \cup \mathcal{M}(z) \cap U_2| = 1$, we assume $zv_1 \in \mathcal{H} \cup \mathcal{M}$ and $N(zv_1) \cap U_2 = \{v_2, \dots, v_t\}$. By Claim 1(2), $G[\{v_1, \dots, v_t\}]$ is a K_3 or a star with center v_1 . If $N(zv_1) \cap V(C) \neq \emptyset$, let $N(zv_1) \cap V(C) = \{z_1, \dots, z_{t'}\}$, where $t' \leq k - 2$.

Let us consider the total weight in $\sum_{v \in V(C)} f(v)$ coming from the triangles not contained in C . Since $[U_1, V(C)] \subseteq \mathcal{L}$, U_1 is an independent set, $|[V(C), X]| \leq 1$ and $[X, U_2] \subseteq \mathcal{L}$ by Lemma 3 and Claim 1, the weight is contributed by the triangles intersecting only with U_2 . By Claims 1 and 2, only z is contained in some triangles with two vertices in U_2 , and so the weight coming from such triangles is $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$, where $\lambda(z) = w(zv_2v_3, z)$ if $v_1v_2v_3$ is a triangle and $\lambda(z) = 0$ otherwise. Furthermore, by Claims 1 and 2, for any triangle containing two vertices in C and one vertex in U_2 , only $w(zv_1z_i, z_i) \neq 0$ for $i \leq t'$ in the case $zv_1 \in \mathcal{H} \cup \mathcal{M}$. Therefore, the weight is

$$f_{U_2}(z) = \sum_{i \leq t} w(zv_1v_i, z) + \lambda(z) + \sum_{i=1}^{t'} w(zv_1z_i, z_i).$$

Claim 3. If $zv_1 \notin \mathcal{H}$, then $f_{U_2}(z) \leq k - 1$, and if $zv_1 \in \mathcal{H}$, then $f_{U_2}(z) \leq k - 1 + \frac{t'}{2} \leq \frac{3k}{2} - 2$ and zv_1 contributes at least $\frac{1}{2}(t' + 1)$ to the loss of v_1 .

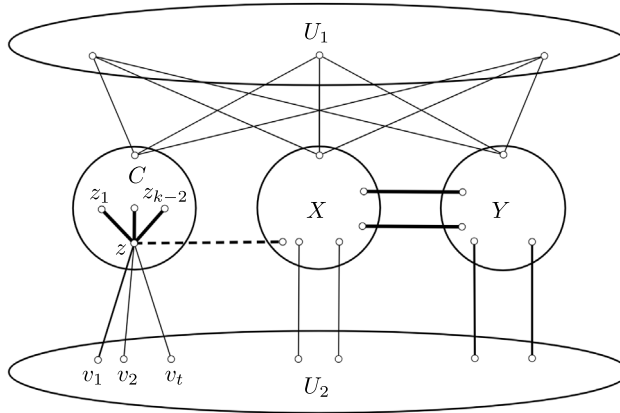


Fig. 2. $N(u') = N(u)$ for any $u' \in U_1$.

Proof. If $zv_1 \in \mathcal{L}$, then $\sum_{i \leq t'} w(z_i z v_1, z_i) = 0$ and hence $f_{U_2}(z) = \sum_{i \leq t} w(z v_1 v_i, z) + \lambda(z) \leq \max\{3, k-1\}$ by Lemma 2.

If $zv_1 \in \mathcal{M}$, then $|N(zv_1)| = (t-1) + t' \leq 2k-2$. By the definition of $w(T, \cdot)$ and Claim 1, we get $f_{U_2}(z) \leq \frac{1}{2}(t-1) + \lambda + \frac{t'}{2}$. Note that $\lambda(z) = 0$ if $(t-1) + t' = 2k-2$, we have $f_{U_2}(z) \leq k-1$.

If $zv_1 \in \mathcal{H}$, then $t > k$ and so $\lambda(z) = 0$. By Lemma 2, $\sum_{i \leq t} w(z v_1 v_i, z) \leq k-1$ and so $f_{U_2}(z) \leq k-1 + \frac{t'}{2} \leq \frac{3k}{2} - 2$. Moreover, since the triangle $z_i z v_1$ satisfies $z z_i \in \mathcal{H}$ for $i \leq t'$, by Observation 3, the edge $z v_1$ contributes at least $\frac{1}{2}(t'+1)$ to the loss of v_1 . \square

By Lemma 3, either $|C| = 2k-1-2|X| \geq k+1$ or $|C| = k+1$. Moreover, since each edge of G is covered by triangles, if $X = \emptyset$, then G_u has no isolated vertices. That is, $G_u = C$ or $C \cup K_{k-1}$ if $X = \emptyset$.

We distinguish the following two cases separately according to $|C|$.

Case 1. $|C| = 2k-1-2|X|$

In this case, the structure of G are shown in Fig. 2, where the thick edges are in \mathcal{H} and the thin edges are in \mathcal{L} .

Case 1.1 $X = \emptyset$.

In this case, $G_u = C$ and $|C| = 2k-1$. If $U_2 = \emptyset$, then since $[U_1, G_u] \subseteq \mathcal{L}$ and $E(G_u) \subseteq \mathcal{H}$, by the definition of $w(T, \cdot)$, we have

$$\begin{aligned} \sum_{v \in V(G)} f(v) &= \sum_{v \in U_1} f(v) + \sum_{v \in V(C)} f(v) \\ &= \sum_{v \in U_1} f(v) + \mathcal{N}(K_3, C) = (n-2k+1)k \left(k - \frac{3}{2} \right) + \mathcal{N}(K_3, C). \end{aligned}$$

Because $|C| = 2k-1$ and $\pi(C) = (k-1, \dots, k-1, k-2)$, by Theorem 8,

$$\sum_{v \in V(G)} f(v) \leq (n-2k+1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \binom{k/2}{2} + \binom{k/2-1}{2}$$

and equality holds if and only if $G = \bar{K}_{n-2k+1} + H_k$, and so the result follows. Therefore, we may assume that $U_2 \neq \emptyset$.

In this case, we will try to transfer the weight $f_{U_2}(z)$ to the vertices in U_2 such that $f(v) \leq k(k-\frac{3}{2})$ still valid after transferring.

(1) $zv_1 \in \mathcal{H}$.

By Claim 3, the edge zv_1 contributes at least $\frac{1}{2}(t' + 1)$ to the loss of v_1 . Transfer the weight $\sum_{i \leq t'} w(z_i zv_1, z_i)$ to v_1 to cover the loss of v_1 caused by the edge zv_1 and the weight $\sum_{i \leq t} w(zv_1 v_i, z)$ to v_2, \dots, v_t ($\lambda(z) = 0$ in this case). After transferring, $f_{U_2}(z) = 0, f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$ and $f(v_1) \leq k(k - \frac{3}{2}) - \frac{1}{2}$. Therefore,

$$\sum_{v \in V(G)} f(v) < (n - 2k + 1)k \left(k - \frac{3}{2} \right) + \mathcal{N}(K_3, C). \quad (4.1)$$

(2) $zv_1 \in \mathcal{M}$.

By the definition of $w(T, \cdot)$, we have $w(z_i zv_1, z_i) = \frac{1}{2}$ for $i \leq t'$, and $w(zv_1 v_i, z) \leq \frac{1}{2}$ for $i \leq t$. Let $|U_2| - t = t''$.

Suppose $t'' \geq t'$. Note that either $\lambda(z) = 1$, which implies $G[\{v_1, \dots, v_t\}]$ is a triangle and $v_2 v_3 \in \mathcal{H}$, or $\lambda(z) \leq \frac{1}{3}$. For the former case, $N(v_2 v_3) - \{v_1, z\} \subset U_2 - \{v_1, v_2, v_3\}$ and hence $|U_2 - \{v_1, v_2, v_3\}| \geq 2k - 3$ by Claim 1(1), which means $t'' > 2t'$. Thus we can transfer the weight $w(z_i zv_1, z_i)$ of z_i ($i \leq t'$) to the vertices in $U_2 - \{v_1, v_2, v_3\}$, the weight $\sum_{i \leq t} w(zv_1 v_i, z) + \lambda(z)$ to the vertices v_1, v_2, v_3 and some others in $U_2 - \{v_1, v_3, v_3\}$. For the latter case, we transfer the weights $w(z_i zv_1, z_i)$ to the vertices in $U_2 - \{v_1, \dots, v_t\}$ and the weights $\sum_{i \leq t} w(zv_1 v_i, z) + \lambda(z)$ to the vertices of $\{v_1, \dots, v_t\}$. After the transferring, $f_{U_2}(z) = 0, f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$, and (4.1) still holds.

Assume $t'' < t' \leq k - 2$. In this case, all edges of $G[\{v_1, \dots, v_t\}]$ are in \mathcal{L} for otherwise we have $t'' \geq k - 2$. Hence, $w(zv_1 v_i, z) = \frac{1}{3}$ for $i \leq t$ and $\lambda(z) \leq \frac{1}{3}$. Recalling $\{z_1, \dots, z_{t'}\} \subseteq N(zv_1)$, by Claim 1(1), we get $N(v_i) \cap \{z_1, \dots, z_{t'}\} = \emptyset$ for $2 \leq i \leq t$, and if $v_i z' v$ is a triangle such that $z' \in V(C)$ and $v \in U_2$, then $v_i z' v = v_i zv_1$, or $G[\{v_1, \dots, v_t\}] = K_3$ and $v_i z' v = v_2 zv_3$. Thus, for $2 \leq i \leq t$, we have

$$\begin{aligned} f(v_i) &= \sum_{z' z'' \in C} w(v_i z' z'', v_i) + \sum_{v', v'' \in U_2} w(v_i v' v'', v_i) + w(v_i v_1 z, v_i) + \eta \\ &\leq \frac{1}{2} \left((2k - 1 - t')(k - 1) - t'(k - t') \right) + \binom{t'' + 1}{2} + \frac{1}{3} + \frac{1}{3} \\ &= k \left(k - \frac{3}{2} \right) - \frac{1}{2} \left(t'(k + t' - 1) - (t'' + 1)t'' - 1 \right) + \frac{2}{3}, \end{aligned}$$

where $\eta = w(v_2 zv_3, v_2)$ or $w(v_2 zv_3, v_3)$ if $v_2 zv_3$ is a triangle, and $\eta = 0$ otherwise. Because the total loss of the vertices in U_2 is at least

$$\begin{aligned} &\left(\frac{1}{2} \left(t'(k + t' - 1) - (t'' + 1)t'' - 1 \right) - \frac{2}{3} \right) (t - 1) + \frac{1}{2} + \frac{t''}{2}, \\ &\geq \sum_{i \leq t} w(zv_1 v_i, z) + \lambda(z) + \sum_{i \leq t'} w(z_i zv_1, z_i) = \frac{t'}{2} + \frac{t}{3}, \end{aligned}$$

we can transfer these weights to vertices in U_2 and for $f_{U_2}(z) = 0, f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$, and (4.1) still holds.

(3) $zv_1 \in \mathcal{L}$.

In this case, we have $w(z_i zv_1, z_i) = 0$ for $i \leq t'$ by the definition of $w(T, \cdot)$. Since $zv_1 \in \mathcal{L}$ and $\{v_2, \dots, v_t\} \subseteq N(zv_1)$, we have $t \leq k - 1$. If $G[\{v_1, \dots, v_t\}]$ has an edge $v_i v_j \in \mathcal{H}$, then since $N(v_i v_j) - \{z, v_1\} \subseteq U_2 - \{v_1, \dots, v_t\}$ by Claim 1(1), we have $|U_2 - \{v_1, \dots, v_t\}| \geq 2k - 3 > k - 1 = t$, which implies $|U_2| > 2t$. If $G[\{v_1, \dots, v_t\}]$ has no edge in \mathcal{H} , then $w(zv_1 v_i, z) \leq \frac{1}{3}$ and $\lambda(z) \leq \frac{1}{3}$.

Thus, by Lemma 3, we can transfer the weight $\sum_{i \leq t} w(zv_1v_i, z) + \lambda(z)$ to the vertices of U_2 in the former case and to the vertices of $\{v_1, \dots, v_t\}$ in the latter case. So, $f_{U_2}(z) = 0$, $f(v) \leq k(k - \frac{3}{2})$ is still valid for any $v \in U_2$, and (4.1) still holds.

Thus, Theorem 8 and (4.1) hold.

Case 1.2 $X \neq \emptyset$.

Let $X = \{x_1, \dots, x_s\}$ and $Y = V(G_u) - V(C) - X$. By Lemma 3, both X and Y are independent sets. Moreover, since $|C| = 2k - 1 - 2|X| = 2k - 1 - 2s \geq k + 1$, we get that $s \leq \frac{k}{2} - 1$.

By Claim 3, $f_{U_2}(z) \leq \frac{3k}{2} - 2$. Moreover, if $|[V(C), X]| = 1$, then $d_{G_u}(z) = k - 1$. By Claim 1(1), $zv_1 \in \mathcal{L}$ and $N_{U_2}(z)$ is an independent set. Therefore,

$$f_{U_2}(z) = 0 \text{ if } |[V(C), X]| = 1. \quad (4.2)$$

Consider the total loss of all the vertices in Y contributed by the edges in $[X, Y]$. Let xy be any edge with $x \in X$ and $y \in Y$, and $X_y \subseteq V(G_y)$ satisfy (3.1). Because $xy \in \mathcal{H}$, by Lemma 1, $x \in \mathcal{H}(y) \subseteq X_y$. Since X and Y are independent sets, $N(xy) \subseteq U_1 \cup U_2$. Thus, if xyv is a triangle, then $xv \in \mathcal{L}$ by Claim 1(1), which implies $w(xyv, y) = 0$, and so $\sum_{v \in N(xy)} w(xyv, y) = 0$. By Observation 2, the edge xy contributes $k - 1$ to the loss of y . Therefore, the total loss of all the vertices in Y , contributed by the edges in $[X, Y]$, is at least $|[X, Y]| \cdot (k - 1)$.

Since X and Y are independent sets, $N_{U_2}(x)$ is an independent set by Claim 1(1), $|[V(C), X]| \leq 1$ and $[U_1, G_u] \subseteq \mathcal{L}$. So,

$$f(x) = \sum_{y \in N_Y(x)} \sum_{v \in N(xy) \cap U_2} w(xyv, x) \text{ for any } x \in X.$$

We try to transfer the weights $w(xyv, x)$ of $x \in X$ to the vertices y, v , such that the new weight of x is 0, and that of each other vertex remains no more than $k(k - \frac{3}{2})$.

Fix an edge yv and let $N(yv) \cap X = \{x_1, \dots, x_{s'}\}$. Then $x_i y \in \mathcal{H}$ and $x_i v \in \mathcal{L}$ for $1 \leq i \leq s'$ by the arguments above. If $yv \in \mathcal{L}$, then $w(x_i yv, x_i) = 0$, and so there is nothing to transfer. If $yv \notin \mathcal{L}$, then $w(x_i yv, x_i) = \frac{1}{2}$. Let $X' \subseteq V(G_v)$ satisfy (3.1).

If $yv \in \mathcal{H}$, then since $yv, x_i y \in \mathcal{H}$, by Observation 3, the edge yv contributes at least $\frac{1}{2}(s' + 1)$ to the loss of v , and so we can transfer the weight $\sum_{i=1}^{s'} w(x_i yv, x_i) = \frac{s'}{2}$ to v to cover the loss caused by the edge yv .

Suppose $yv \in \mathcal{M}$. Note that $\sum_{v' \in N(x_i v)} w(x_i v v', v) \leq (k - 1) - \frac{1}{2}$ by (3.4) because $w(x_i yv, v) = \frac{1}{2}$. If $x_i \in X'$, then by Observation 2, the edge $x_i v$ contributes $\frac{1}{2}$ to the loss of v . So we can transfer the weight $w(x_i yv, x_i)$ to v to cover the loss contributed by the edge $v x_i$. If $y \in X'$, then by (3.3) and Observation 2, the edge yv contributes $\frac{k}{6}$ to the loss of v , and $\frac{k}{12}$ to the loss of y by Observation 4, which means yv contributes at least $\frac{k}{4}$ to the total loss of y and v . Recalling $s' \leq s \leq \frac{k}{2} - 1$, we can transfer the weight $\sum_{i=1}^{s'} w(x_i yv, x_i) = \frac{s'}{2}$ to y, v to cover the loss contributed by the edge yv . If neither $x_i \in X'$ nor $y \in X'$, then the edge $x_i y$ lies in some component C' of $G_v - X'$. Remember $\sum_{v' \in N(x_i v)} w(x_i v v', v) \leq (k - 1) - \frac{1}{2}$, the edge $x_i v$ contributes at least $\frac{1}{4}$ to the loss of v . Since $yv \in \mathcal{M}$, by Observation 4, it contributes at least $\frac{k}{6}$ to the total loss of y, v . Thus, the total loss of y and v is at least $\frac{s'}{4} + \frac{k}{6} > \frac{s'}{2}$, and so we can transfer the weight $\sum_{i=1}^{s'} w(x_i yv, x_i) = \frac{s'}{2}$ to y, v such that the weights of y, v are still no more than $k(k - \frac{3}{2})$.

If $zv_1 \in \mathcal{H}$, then by Claim 3, we can transfer $\frac{s'}{2}$ from $f_{U_2}(z)$ to v_1 to cover the loss of v_1 caused by zv_1 such that $f_{U_2}(z) \leq k - 1$. After this possible transferring, we always have $f_{U_2}(z) \leq k - 1$. Thus, recalling the total loss of all the vertices in Y contributed by the edges in $[X, Y]$ is at least $|[X, Y]| \cdot (k - 1)$, (4.2) and $f(x) = 0$ for each $x \in X$ after transferring and Theorem 9, the total weight

of G is

$$\begin{aligned} \sum_{v \in V(G)} f(v) &\leq (n - 2k + 1 + s)k \left(k - \frac{3}{2} \right) + \mathcal{N}(K_3, C) + (k - 1) - s(k - 1)^2 \\ &\leq (n - 2k + 1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2 \\ &\quad + \frac{k^2}{4} + \left(-k^2 + \frac{7k}{2} - \frac{11}{3} \right) s + (2k - 2)s^2 - \frac{4}{3}s^3 \\ &= (n - 2k + 1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2 + \varphi(s, k). \end{aligned}$$

Because $1 \leq s \leq \frac{k}{2} - 1$, after an easy calculation, we get $\varphi(k, s) < 0$ except $\varphi(2, 6) = 1$, $\varphi(1, 4) = 3$ so the result holds if $(s, k) \neq (2, 6), (1, 4)$.

Now, consider the two exceptions. Let yv be any edge with $y \in Y$ and $v \in U_2$.

Suppose that $(s, k) = (2, 6)$. Then $X = \{x_1, x_2\}$. Let $U = N_{U_2}(x_1) \cup N_{U_2}(x_2)$. Assume that $yv \in \mathcal{H} \cup \mathcal{M}$. If both x_1yv and x_2yv are triangles, then since $N_{U_2}(x_1)$ and $N_{U_2}(x_2)$ are independent sets by Claim 1, $N(yv) - \{x_1, x_2\} \subseteq U_2 - U$. Clearly, $|U_2 - U| \geq |N(yv) - \{x_1, x_2\}| \geq k - 2$. If $|N(yv) \cap \{x_1, x_2\}| \leq 1$ for any $yv \in \mathcal{H} \cup \mathcal{M}$, then since $k = 6$, yv contributes at least $\frac{1}{2}$ to the loss of y by Observations 3 and 4, and so we can transfer $w(x_iyv, x_i)$ only to y to cover the loss of y contributed by yv if x_iyv is a triangle, such that $f(x_i) = 0$ for $i = 1, 2$ after the possible transferring. Moreover, since $|N(yv) \cap U_2| \geq k - 1$, $|U_2| \geq k$. Thus, note that no weights are transferred to the vertices in $U_2 - U$ in the former case and in U_2 in the latter case, U_2 has at least $k - 2$ vertices whose weights are at most $k(k - \frac{3}{2}) - \frac{1}{2}$ after transferring, which implies

$$\begin{aligned} \sum_{v \in V(G)} f(v) &\leq (n - 2k + 1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2 + \varphi(s, k) - \frac{1}{2}(k - 2) \\ &< (n - 2k + 1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2. \end{aligned}$$

If $[Y, U_2] \subseteq \mathcal{L}$, then $f(x_i) = 0$ for $i = 1, 2$ by Claim 1 and the definition of $w(T, \cdot)$. Since $\varphi(2, 6) = 1$, if $|U_2| \geq 3$, then replace $\frac{1}{2}(k - 2)$ with $\frac{1}{2} \cdot 3$ in the above inequality, we get the desired result. If $|U_2| \leq 2$, then $f(y) \leq 1$ for any $y \in Y$. It is easy to see the total weight of G is less than the expected number.

Suppose that $(s, k) = (1, 4)$. Then $X = \{x_1\}$. We will transfer the weight $f(x_1)$ and $f_{U_2}(z)$ to other vertices in a bit different way. Note that $f(x_1)$ comes from the triangles x_1yv with $yv \in \mathcal{H} \cup \mathcal{M}$. Since $x_1y \in \mathcal{H}$ for any $y \in Y$, by Lemma 1, $x_1 \in X_y$. If $yv \in \mathcal{H}$, then by Observation 3, the edge yv contributes $\frac{1}{2}$ to the loss of y . Assume that $yv \in \mathcal{M}$. If $v \in X_y$, then by (3.3) and Observation 2, yv contributes at least $\frac{k}{6} = \frac{2}{3} > \frac{1}{2}$ to the loss of y . If v lies in a component C' of $G_y - X_y$, then $x_1 \in X_y$ and (3.1) imply $|C'| \leq 2k - 3$, and so

$$\sum_{v' \in N(yv) \cap C'} w(yvv', y) \leq \frac{1}{2}(k - 2) + \frac{1}{3}(2k - 4 - (k - 2)) = (k - 1) - \left(\frac{k}{6} + \frac{2}{3} \right),$$

and so yv contributes $\frac{1}{2} \left(\frac{k}{6} + \frac{2}{3} \right) > \frac{1}{2}$ to the loss of y by Observation 4. Therefore, the edge yv contributes at least $\frac{1}{2}$ to the loss of y . Transfer the weight $w(x_1yv, x_1) = \frac{1}{2}$ to y such that the new weight of y is no more than $k(k - \frac{3}{2}) - (k - 1)$, where the loss $k - 1$ is contributed by the edge x_1y . Transfer the weight $f_{U_2}(z)$ to the vertices in U_2 in the same way used in Case 1.1. After the transferring, the weight of G satisfies

$$\sum_{v \in V(G)} f(v) < (n - 2k + 1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2,$$

and so the proof of Case 1 is complete.

Case 2. $|C| = k + 1$.

In this case, $X = \emptyset$ and $G_u = C \cup K_{k-1}$. Since $\pi(C) = (k - 1, \dots, k - 1, k - 2)$ by Lemma 3 and $|C| = k + 1$, C is the complement of $\frac{1}{2}(k - 2)K_2 \cup P_3$, and so we have

$$\mathcal{N}(K_3, G_u) = \binom{k-1}{3} + \binom{k+1}{3} - \frac{1}{2}(k-2)(k-1) - 2(k-2) - 1.$$

Set $V(K_{k-1}) = \{p_1, \dots, p_{k-1}\}$. We first discuss some properties of these vertices.

Claim 4. If $q_1, q_2 \in U_2$ such that $p_i q_1, p_i q_2 \in \mathcal{H} \cup \mathcal{M}$, then

- (1) $G[N_{U_2}(p_i)]$ consists of a triangle $q_1 q_2 q_3$ and some isolated vertices;
- (2) $|N(p_i q_1)| = |N(p_i q_2)| = k$ and $\{p_1, \dots, p_{k-1}\} \subseteq N(q_1) \cap N(q_2)$;
- (3) If $v \in U_2$ such that $p_j v \in \mathcal{H} \cup \mathcal{M}$, then $v \in \{q_1, q_2, q_3\}$. Moreover, if $v = q_3$, then $|N(p_j q_s)| = k$ for $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$.

Proof. (1) Since $d_{G_u}(p_i) = k - 2$ and $p_i q_1, p_i q_2 \in \mathcal{H} \cup \mathcal{M}$, by Claim 1(2), $G[N_{U_2}(p_i)]$ is a triangle containing q_1, q_2 , say $q_1 q_2 q_3$, together with some isolated vertices.

(2) Since $|N(p_i q_1)| \geq k$ and $N(p_i q_1) \cap U_2 = \{q_2, q_3\}$ by (1), $\{p_1, \dots, p_{k-1}\} \subseteq N(q_1)$. By the symmetry of q_1 and q_2 , $\{p_1, \dots, p_{k-1}\} \subseteq N(q_2)$, and so the result follows.

(3) Suppose $v \notin \{q_1, q_2, q_3\}$. By Claim 1(2) and (2), $G[N_{U_2}(p_j)]$ is a triangle $v q_1 q_2$ together with some isolated vertices. Because $|N(p_j v)| \geq k$ and $N(p_j v) \cap U_2 = \{q_1, q_2\}$, we have $\{p_1, \dots, p_{k-1}\} \subseteq N(v)$. Thus, $v \in N(p_i q_1)$ and hence $|N(p_i q_1)| \geq k + 1$ which contradicts (2). Therefore, $v \in \{q_1, q_2, q_3\}$. Moreover, if $v = q_3$, then since $|N(p_j q_3)| \geq k$ and $N(p_j q_3) \cap U_2 = \{q_1, q_2\}$, we can deduce $\{p_1, \dots, p_{k-1}\} \subseteq N(q_3)$. After an easy check, we get that $|N(p_j q_s)| = k$ for $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$. \square

By Lemma 1, we have $|(H(p_i) \cup M(p_i)) \cap U_2| \leq 3$ for all $1 \leq i \leq k - 1$. Suppose $|(H(p_i) \cup M(p_i)) \cap U_2| = 3$ for some i . By Claim 4, $G[N_{U_2}(p_i)]$ is a triangle $q_1 q_2 q_3$ together with some isolated vertices and $p_i q_1, p_i q_2, p_i q_3 \in \mathcal{M}$. Furthermore, we have $p_j q_s \in E(G)$ and $|N(p_j q_s)| = k$ for $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$, and $q_1 q_2, q_1 q_3, q_2 q_3 \in \mathcal{H} \cup \mathcal{M}$. Because

$$\sum_{v \in N(p_i q_s)} w(p_i q_s v, q_s) = \frac{k}{3} \leq \frac{k}{2} = k - 1 - \left(\frac{k}{2} - 1\right),$$

the edge $p_i q_s$ contributes at least $\frac{k}{4} - \frac{1}{2}$ to the loss of q_s by Observation 4. Hence, all the edges $p_j q_s$, $1 \leq j \leq k - 1$ and $1 \leq s \leq 3$, contribute at least $3(k - 1)\left(\frac{k}{4} - \frac{1}{2}\right)$ to the total loss of q_1, q_2 and q_3 . On the other hand, note that $G[N_{U_2}(p_j)]$ is the triangle $q_1 q_2 q_3$ together with some isolated vertices by Claim 4(1) and $[p_j, U_2 - \{q_1, q_2, q_3\}] \subseteq \mathcal{L}$ by Claim 4(3). So, the weight of p_j contributed by the triangles not in K_{k-1} is

$$\sum_{1 \leq r < s \leq 3} w(p_i q_r q_s, p_i) + \sum_{p_j \neq p_i} \sum_{1 \leq s \leq 3} w(p_i p_j q_s, p_i) = 3 \cdot \frac{1}{3} + 3(k - 2) \cdot \frac{1}{3} = k - 1.$$

By Claim 3, $f_{U_2}(z) \leq \frac{3k}{2} - 2$. Therefore, the total weight of G is at most

$$\begin{aligned} & (n - 2k)k \left(k - \frac{3}{2}\right) + \mathcal{N}(K_3, G_u) + \left(\frac{3k}{2} - 2\right) + (k - 1)^2 - 3(k - 1) \left(\frac{k}{4} - \frac{1}{2}\right) \\ & < (n - 2k + 1)k \left(k - \frac{3}{2}\right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2, \end{aligned}$$

a contradiction. So we assume that $|(H(p_i) \cup M(p_i)) \cap U_2| \leq 2$ for $1 \leq i \leq k - 1$.

Fix p_i and let $N_{U_2}(p_i) = \{q_1, \dots, q_r\}$. By Claim 1, we may assume that $G[N_{U_2}(p_i)]$ is a star with the center q_1 or a triangle $q_1 q_2 q_3$, and some isolated vertices. Let

$$f_{U_2}(p_i) = \sum_{v, v' \in U_2} w(p_i v v', p_i) + \sum_{p_j \neq p_i} \sum_{v \in N(p_i p_j) \cap U_2} w(p_i p_j v, p_j).$$

It is clear that $\sum_{i=1}^{k-1} f_{U_2}(p_i)$ is the total weight of $V(K_{k-1})$ contributed by the triangles not contained in K_{k-1} . We will complete the proof by showing that $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$ and $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$, after some appropriate weight transferring.

If $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| = 2$, say $p_i q_1, p_i q_2 \notin \mathcal{L}$, then by Claim 4, $N_{U_2}(p_i)$ is a triangle $q_1 q_2 q_3$ together with some isolated vertices, $|N(p_i q_1)| = |N(p_i q_2)| = k$ and so

$$f_{U_2}(p_i) = \sum_{v, v' \in \{q_1, q_2, q_3\}} w(p_i v v', p_i) + \sum_{p_j \neq p_i} w(p_i p_j q_1, p_j) + \sum_{p_j \neq p_i} w(p_i p_j q_2, p_j).$$

Since

$$\sum_{v \in N(p_i q_s)} w(p_i q_s v, q_s) \leq \frac{k}{2} = (k-1) - \left(\frac{k}{2} - 1\right) \text{ for } s = 1, 2,$$

the edge $p_i q_s$ contributes at least $\frac{k}{4} - \frac{1}{2}$ to the loss of q_s for $s = 1, 2$ by Observation 4. Because $\sum_{p_j \neq p_i} w(p_i p_j q_s, p_j) \leq \frac{1}{2}(k-2)$, transfer $\frac{k}{4} - \frac{1}{2}$ from $\sum_{p_j \neq p_i} w(p_i p_j q_s, p_j)$ to q_s for $s = 1, 2$. After transferring, we have

$$f_{U_2}(p_i) \leq \frac{3}{2} + 2 \cdot \frac{1}{2}(k-2) - 2 \left(\frac{k}{4} - \frac{1}{2}\right) = \frac{k+1}{2} \leq \frac{3k}{4} - \frac{1}{2}. \quad (4.3)$$

Now, let $|(\mathcal{H}(p_i) \cup \mathcal{M}(p_i)) \cap U_2| \leq 1$. Assume $\mathcal{H}(p_i) \cup \mathcal{M}(p_i) \subseteq \{q_1\}$ by Claim 1(2). Because $p_i p_j \in \mathcal{H}$ and $p_i q_s \in \mathcal{L}$, $w(p_i p_j q_s, p_j) = 0$ for $2 \leq s \leq r$ and hence

$$f_{U_2}(p_i) = \sum_{j=2}^r w(p_i q_1 q_j, p_i) + \lambda(p_i) + \sum_{p_j \neq p_i} w(p_i p_j q_1, p_j),$$

where $\lambda(p_i) = w(p_i q_2 q_3, p_i)$ if $q_1 q_2 q_3$ is a triangle and $\lambda(p_i) = 0$ otherwise. Using the same proof as that of Claim 3, we have

Claim 5. $f_{U_2}(p_i) \leq k-1$ if $p_i q_1 \notin \mathcal{H}$, and $f_{U_2}(p_i) \leq k-1 + \frac{\ell}{2} \leq \frac{3k}{2} - 2$ and $p_i q_1$ contributes at least $\frac{1}{2}(\ell+1)$ to the loss of q_1 if $p_i q_1 \in \mathcal{H}$, where $|N(p_i q_1) \cap V(K_{k-1})| = \ell$.

In order to show $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$ in this case and $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$, we need to consider the structure of $G[U_2]$.

If $vv' \in \mathcal{M}$ is an edge in $G[U_2]$, then by Observation 4, vv' contributes $\frac{k}{12}$ to the loss of v and v' , respectively, that is, vv' contributes $\frac{k}{6}$ to the total loss of vertices in U_2 . On the other hand, by Claims 3 and 5, we can transfer some weight from $f_{U_2}(z)$ and $f_{U_2}(p_i)$ to v_1 and q_1 , respectively, such that $f_{U_2}(z) \leq k-1$ and $f_{U_2}(p_i) \leq k-1$, and $f(v_1) \leq k(k - \frac{3}{2})$ and $f(q_1) \leq k(k - \frac{3}{2})$ still hold. This together with (4.3) implies that after transferring some weights to the vertices in U_2 , the total weight in $\sum_{v \in V(G_u)} f(v)$ coming from the triangles not in G_u is at most $k(k-1)$. Therefore, if $G[U_2]$ has $\frac{3k}{2} - 2$ edges in \mathcal{M} , then we have

$$\begin{aligned} \sum_{v \in V(G)} f(v) &\leq (n-2k)k \left(k - \frac{3}{2}\right) + \mathcal{N}(K_3, G_u) + k(k-1) - \left(\frac{3k}{2} - 2\right) \cdot \frac{k}{6} \\ &< (n-2k+1)k \left(k - \frac{3}{2}\right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1\right)^2, \end{aligned}$$

a contradiction. Hence, $G[U_2]$ contains at most $\frac{3k}{2} - 3$ edges in \mathcal{M} . Moreover, we have

Claim 6. Let $qq' \in \mathcal{H}$ be an edge in $G[U_2]$. If $\mathcal{M}(q) \cap \mathcal{M}(q') \cap \{p_1, \dots, p_{k-1}\} = \emptyset$, then qq' contributes $\frac{3k}{8}$ to the total loss of q and q' . Furthermore, $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $\mathcal{H} \cup \mathcal{M}$.

Proof. By Claim 2 and the assumption, $\mathcal{M}(q) \cap \mathcal{M}(q') \cap \{z, p_1, \dots, p_{k-1}\} = \emptyset$. Noting that $G[U_2]$ has at most $\frac{3k}{2} - 3$ edges in \mathcal{M} , we have $|\mathcal{M}(q)| + |\mathcal{M}(q')| \leq k + \frac{3k}{2} - 3$. Thus, by (3.2) and Lemma 1(ii),

we get

$$\begin{aligned} & \sum_{v \in N(qq')} w(qq'v, q) + \sum_{v \in N(qq')} w(q'qv, q') \\ & \leq \frac{1}{2} (|\mathcal{H}(q) - \{q'\}| + |\mathcal{M}(q)| + |\mathcal{H}(q') - \{q\}| + |\mathcal{M}(q')|) \leq 2(k-1) - \frac{3k}{8}, \end{aligned}$$

and so the conclusion follows by [Observation 2](#).

In addition, recall $|(\mathcal{H}(p_j) \cup \mathcal{M}(p_j)) \cap U_2| \leq 2$ for all $1 \leq j \leq k-1$, by [Claim 4](#), $G[U_2]$ has at most one edge $q_1'q_2'$ such that $\mathcal{M}(q_1') \cap \mathcal{M}(q_2') \cap \{p_1, \dots, p_{k-1}\} \neq \emptyset$. For any other \mathcal{H} -edge qq' in $G[U_2]$, qq' contributes at least $\frac{3k}{8} > \frac{k}{6}$ to the total loss of q and q' , which, together with the possible edge $q_1'q_2'$, implies $G[U_2]$ contains at most $\frac{3k}{2} - 2$ edges in $\mathcal{H} \cup \mathcal{M}$. \square

Now, let us re-consider $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$ based on [Claim 6](#). For convenience, let $a \in \{z, p_1, \dots, p_{k-1}\}$, $N_{U_2}(a) = \{b_1, \dots, b_m\}$ is a star with center b_1 or a triangle $b_1b_2b_3$ in $G[N_{U_2}(a)]$, $(\mathcal{H}(a) \cup \mathcal{M}(a)) \cap U_2 \subseteq \{b_1\}$ and $N(ab_1) \cap V(G_u) = \{a_1, \dots, a_\ell\}$. Recall the expressions of $f_{U_2}(z)$ and $f_i = f_{U_2}(p_i)$, we have

$$f_{U_2}(a) = \sum_{j=2}^m w(ab_1b_j, a) + \lambda(a) + \sum_{j=1}^{\ell} w(ab_1a_j, a_j).$$

If $ab_1 \in \mathcal{H}$, then since $aa_i \in \mathcal{H}$, we can transfer the weight $w(ab_1a_i, a_i)$ to b_1 to cover the loss caused by the edge ab_1 by [Observation 3](#). For the weight $w(ab_1b_j, a)$, we have $w(ab_1b_j, a) \leq \frac{1}{2}$ with equality only if $b_1b_j \in \mathcal{H} \cup \mathcal{M}$. Thus, by [Claim 6](#), after transferring, we have

$$f_{U_2}(a) = \sum_{j=2}^m w(ab_1b_j, a) + \lambda(a) \leq \max \left\{ 2, \frac{3k}{4} - 1 \right\} < \frac{3k}{4} - \frac{1}{2}.$$

Assume $ab_1 \in \mathcal{M}$. Consider $w(ab_1b_j, a)$. If $b_1b_j \in \mathcal{H}$, then $w(ab_1b_j, a) = \frac{1}{2}$, and b_1b_j contributes $\frac{3k}{8}$ to the total loss of b_1 and b_j by [Claim 6](#). Since $a \in \{z, p_1, \dots, p_{k-1}\}$, there are at most k such triangles, and so we can transfer $\frac{3}{8}$ of each $w(ab_1b_j, a)$ to b_1 and b_j to cover the total loss of b_1 and b_j caused by the edge b_1b_j . If $b_1b_j \in \mathcal{M} \cup \mathcal{L}$, then $w(ab_1b_j, a) = \frac{1}{3}$ and ab_1 contributes $\frac{k}{12}$ to the loss of b_1 by [Observation 4](#). Thus, we can transfer $\frac{1}{12}$ of each $w(ab_1b_j, a)$ to cover the loss of b_1 caused by the edge ab_1 . After transferring, we have $w(ab_1b_j, a) \leq \frac{1}{4}$ and so

$$f_{U_2}(a) \leq \frac{1}{4}|N(ab_1) \cap U_2| + \frac{1}{2}|N(ab_1) \cap V(G_u)| \leq \frac{3k}{4} - 1 < \frac{3k}{4} - \frac{1}{2}.$$

If $ab_1 \in \mathcal{L}$, then $w(ab_1a_j, a_j) = 0$ for $1 \leq i \leq \ell$. If $b_1b_j \in \mathcal{H}$, then $w(ab_1b_j, a) = 1$. By [Claim 6](#), we can transfer $\frac{3}{8}$ to cover the total loss of b_1 and b_j caused by the edge b_1b_j , with at most one exceptional edge in $G[U_2]$. If $b_1b_j \in \mathcal{M} \cup \mathcal{L}$, then $w(ab_1b_j, a) = \frac{1}{3}$. After transferring, we have $w(ab_1b_j, a) \leq \frac{5}{8}$ with at most one exception and so

$$f_{U_2}(a) = \sum_{j=2}^m w(ab_1b_j, a) + \lambda(a) \leq \frac{5}{8}(k-2) + 1 < \frac{3k}{4} - \frac{1}{2}.$$

By the three inequalities above, we have $f_{U_2}(z) < \frac{3k}{4} - \frac{1}{2}$. Moreover, combining the three inequalities with [\(4.3\)](#), we have $f_{U_2}(p_i) \leq \frac{3k}{4} - \frac{1}{2}$. Thus, after appropriate weight transferring, we have $f_{U_2}(z) + \sum_{i=1}^{k-1} f_i < k(\frac{3k}{4} - \frac{1}{2})$. Hence, the total weight of G is

$$\begin{aligned} \sum_{v \in V(G)} f(v) & < (n-2k)k \left(k - \frac{3}{2} \right) + \mathcal{N}(K_3, G_u) + k \left(\frac{3k}{4} - \frac{1}{2} \right) \\ & \leq (n-2k+1)k \left(k - \frac{3}{2} \right) + 2 \binom{k-1}{3} + \left(\frac{k}{2} - 1 \right)^2. \end{aligned}$$

The proof of [Theorem 4](#) is complete. \blacksquare

5. Concluding remarks

Theorem 4 is proved for $n \geq 4k^3$, but notice that the statement does not hold for small n . For example, take at most five disjoint copies of K_{2k} then the number of copies K_3 is more than the extremal number in the theorem. It would be nice to determine the sharp bound for n when this generalized Turán number is correct.

It is natural to ask what happens if we count larger cliques. The third author [13] showed that $\text{ex}(n, K_r, F_k) = O(n)$ for every k and r , but the constants in the upper bound are large. We conjecture that the extremal graph for $\text{ex}(n, K_r, F_k)$ is still $\bar{K}_{n-v(H)} + H$, where H is a graph with $V(H) = k - 1$, $\Delta(H) = k - 1$.

Let H^T be the graph obtained by replacing each edge of H with a triangle, e.g., the friendship graph can be considered as a S_k^T . So it is also interesting to ask what if we replace each edge of any other graph H with a triangle? For example, consider the extremal function $\text{ex}(n, K_3, P_k^T)$, $\text{ex}(n, K_3, C_k^T)$.

We have determined the largest number of triangles in F -free graphs when F is a friendship graph, but not when F is an extended friendship graph. Alon and Shikhelman [1] showed that in that case $c_1|V(F)|^2n \leq \text{ex}(n, K_3, F) \leq c_2|V(F)|^2n$ for absolute constants c_1 and c_2 . Better bounds were obtained for some forests, including exact results for stars [4], paths [19] and forests consisting only of path components of order different from 3 [23].

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References

- [1] N. Alon, C. Shikhelman, Many T copies in H -free graphs, *J. Combin. Theory Ser. B* 121 (2016) 146–172.
- [2] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Math. Acad. Sci. Paris* 247 (1958) 258–259.
- [3] B. Bollobás, E. Győri, Pentagons vs. triangles, *Discrete Math.* 308 (2008) 4332–4336.
- [4] Z. Chase, The maximum number of triangles in a graph of given maximum degree, *Adv. Comput.* (2020) paper (10) 5 pp.
- [5] F.R.K. Chung, Unavoidable stars in 3-graph, *J. Combin. Theory Ser. A* 3 (1983) 252–262.
- [6] F.R.K. Chung, P. Erdős, On unavoidable graphs, *Combinatorica* 3 (1983) 167–176.
- [7] F.R.K. Chung, P. Frankl, The maximum number of edges in a 3-graph not containing a given star, *Graphs Combin.* 3 (1987) 111–126.
- [8] R.A. Duke, P. Erdős, Systems of finite sets having a common intersection, in: *Proceedings, 8th S-E Conf. Combinatorics, Graph Theory and Computing*, 1977, pp. 247–252.
- [9] P. Erdős, Z. Füredi, R. Gould, D. Gunderson, Extremal graphs for intersecting triangles, *J. Combin. Theory Ser. B* 64 (1995) 89–100.
- [10] P. Erdős, M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* 1 (1966) 51–57.
- [11] P. Erdős, A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946) 1087–1091.
- [12] D. Gerbner, Generalized Turán problems for $K_{2,t}$, 2021, arXiv preprint arXiv:2107.10610.
- [13] D. Gerbner, A note on the uniformity threshold for Berge hypergraphs, *European J. Combin.* 105 (2022) 103561.
- [14] D. Gerbner, E. Győri, A. Methuku, M. Vizer, Generalized Turán problems for even cycles, *J. Combin. Theory Ser. B* 145 (2020) 169–213.
- [15] D. Gerbner, C. Palmer, Counting copies of a fixed subgraph of F -free graphs, *European J. Combin.* 82 (2019) 103001.
- [16] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, *J. Combin. Theory Ser. B* 102 (2012) 1061–1066.
- [17] H. Hatami, J. Hladký, D. Král, S. Norine, A. Razborov, On the number of Pentagons in triangle-free graphs, *J. Combin. Theory Ser. A* 120 (2013) 722–732.
- [18] E.L. Liu, J. Wang, The generalized Turán problem of two intersecting cliques, 2021, arXiv preprint arXiv:2101.08004.
- [19] R. Luo, The maximum number of cliques in graphs without long cycles, *J. Combin. Theory Ser. B* 128 (2017) 219–226.
- [20] J. Ma, Y. Qiu, Some sharp results on the generalized Turán numbers, *European J. Combin.* 84 (2020) 103026, 16 pages.
- [21] V.T. Sós, Remarks on the connection of graph theory, Finite geometry and block designs, *Teorie Combinatorie* 2 (1976) 223–233.
- [22] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok* 48 (1941) 436–452.
- [23] X. Zhu, Y. Chen, Generalized Turán number for linear forests, *Discrete Math.* 345 (10) (2022) 112997.
- [24] X. Zhu, E. Győri, Z. He, Z. Lv, N. Salia, C. Xiao, Stability version of dirac's theorem and its applications for generalized turán problems, *Bull. Lond. Math. Soc.* <http://dx.doi.org/10.1112/blms.12823>, Online.
- [25] A. Zykov, On some properties of linear complexes, *Mat. Sbornik N. S.* 24 (1949) 163–188.